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# On Fliess operators driven by $L_2$-Itô processes

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Fliess operators, which are a type of functional series expansion, have been used to describe a broad class of nonlinear input–output maps driven by deterministic inputs. But in most applications, a system's inputs have noise components. This paper has three objectives. The first objective is to show that the notion of a Fliess operator can be generalized to admit a class of $L_2$-Itô stochastic input processes. The next objective is to show that they converge absolutely over an arbitrarily large but finite time interval when a certain coefficient growth condition is met. However, a significant number of systems fail to meet this condition. Thus, the final objective is to consider an interval of convergence having a random length so that a Fliess operator might converge under less restrictive growth conditions.

**Keywords:** formal power series; nonlinear functionals; Itô stochastic processes; stochastic integration

**AMS Subject Classification**: 47H30; 60G15; 60H05; 93C10



## 1. Introduction

Fliess operators provide a general framework under which analytic nonlinear input–output systems can be characterized [7–12,14,15,24–26]. When the inputs involved are deterministic, these operators are described by an infinite summation of Lebesgue iterated integrals codified using the theory of non-commutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and $X^*$ the free monoid comprised all words over $X$ (including the empty word $\emptyset$) under the catenation product. A formal power series in $X$ is any mapping of the form

$$c : X^* \to \mathbb{R}^\ell$$
$$\eta \mapsto (c, \eta),$$

and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For a measurable function $u : [a, b] \to \mathbb{R}^m$ define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \le i \le m\}$, where $\|u_i\|_{L_p}$ is the usual $L_p$-norm for a measurable real-valued component function $u_i$. Define iteratively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t) = \int_{t_0}^{t} u_i(\tau)E_{\eta'}[u](\tau)\mathrm{d}\tau, \tag{1}$$

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where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$. The $m$-input, $\ell$-output operator corresponding to any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is then

$$F_c[u](t) \triangleq \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t), \tag{2}$$

which is called a *Fliess operator*. All Volterra operators with analytic kernels, for example, are Fliess operators [8]. The most general results regarding the convergence of Fliess operators appear in [15]. There it was shown that if the generating series $c$ is *globally convergent*, i.e. satisfies the growth condition

$$|(c, \eta)| \leq KM^{|\eta|}, \quad \forall \eta \in X^*,$$

where $|\eta|$ denotes the number of letters in $\eta$ and $K, M > 0$ are fixed, then $F_c[u]$ converges absolutely on $[t_0, \infty)$ for any $u \in L_{p,e}(t_0) \triangleq \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t]} \in L_p^m[t_0, t], \forall t \in (t_0, \infty)\}$, where $u_{[t_0,t]}$ is the restriction of $u$ to $[t_0, t]$. On the other hand, if $c$ is *locally convergent*, i.e. satisfies the growth condition

$$|(c, \eta)| \leq KM^{|\eta|}|\eta|!, \quad \forall \eta \in X^*,$$

then $F_c[u]$ converges absolutely on $[t_0, t_0 + T]$ provided $T$ and $\|u\|_{L_p}$ are sufficiently small.

Noise is always present in real system inputs. In the most simple setting, it can be modelled by a Wiener process. From the Fliess operator point of view, various approaches for Wiener process inputs have been developed [1,7,9,12,20]. Unfortunately, such inputs are not suitable in certain contexts, such as those involving system interconnection [3]. For example, in order to cascade two systems as illustrated in Figure 1, the output of the first system, $y_1$, needs to be a Wiener process, which is generally not the case even if its input $u_1$ is a Wiener process. Therefore, a broader class of stochastic processes known as $L_2$-Itô processes should be considered as potential inputs [2,18]. Such processes appear naturally in a large number of important applications [17]. But to provide such a generalization, (1) and (2) need to be redefined in terms of Lebesgue and Stratonovich integrals. Then the issue of convergence needs to be addressed.

This paper has three objectives. The first objective is to provide a theoretical framework under which a Fliess operator can be driven by a class of $L_2$-Itô stochastic processes. The second objective is to show that for any suitable $L_2$-Itô input process such a Fliess operator is absolutely mean square convergent over an arbitrarily large but finite interval of time provided its generating series is globally convergent. To motivate the final objective of this paper, consider for instance a system in Stratonovich form

$$\begin{aligned} \mathrm{d}z(t) &= Mz(t)\mathrm{d}W(t), \quad z(0) = 1, \\ y(t) &= Kz(t), \end{aligned} \tag{3}$$

where $W$ is a Wiener process and $K, M > 0$ are fixed. The generating series for (3) can be specified using the alphabet $Y = \{y_0\}$ as $c = \sum_{k=0}^{\infty} KM^k y_0^k$, which is clearly globally convergent. The corresponding output process is then

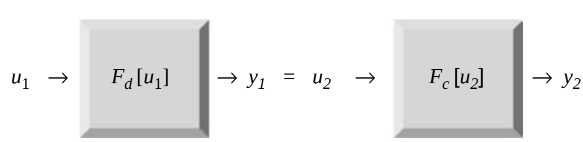


Figure 1. Cascade of two Fliess operators with overall output $y_2 = (F_c \circ F_d)[u_1]$.

$$y(t) = \sum_{k=0}^{\infty} KM^k \oint_0^t \cdots \oint_0^{t_2} \mathrm{d}W(t_1)\cdots \mathrm{d}W(t_k).$$

Since Stratonovich integrals follow the rules of standard integral calculus,

$$y(t) = F_c[0](t) = \sum_{k=0}^{\infty} KM^k \frac{W^k(t)}{k!} = K\mathrm{e}^{MW(t)}.$$

Hence, the output $y$ is well defined for all $t \geq 0$. However, globally convergent input–output systems are fairly rare in applications. For example, consider the system

$$\begin{aligned} \mathrm{d}z(t) &= Mz^2(t)\mathrm{d}W(t), \quad z(0) = 1, \\ y(t) &= Kz(t). \end{aligned} \tag{4}$$

The generating series for (4) is $c = \sum_{k=0}^{\infty} KM^k k! y_0^k$, which is only locally convergent. The corresponding output is

$$y(t) = F_c[0](t) = \sum_{k=0}^{\infty} KM^k W^k(t).$$

At first glance, $y$ appears not to be convergent. Nevertheless, if for a fixed $0 < R < 1$ the *first passage time*

$$\tau_R \triangleq \inf\{t > 0 : |MW(t)| = R\}$$

is positive, then $y(t)$ will be a well-defined random variable $Kz(t) = K/(1 - MW(t))$ for any $t \in [0, \tau_R]$, where $[0, \tau_R]$ is a non-zero interval of time. Clearly, the interval of convergence now has a random nature, that is,

$$[0, \tau_R] = \{0 \leq t \leq \tau_R(\omega) : (\tau_R(\omega), \omega) \in [0, \infty) \times \Omega\},$$

where $\Omega$ is the sample space. Hence, the final objective is to describe a theoretical framework under which a Fliess operator driven by an $L_2$-Itô random process and having a locally convergent generating series converges over a time interval of random positive length. It should be noted that Arous studied the series expansions of the solutions of stochastic differential equations over stochastic time intervals when the coefficients of the series grow at a locally convergent rate [1]. But his approach was based on *a priori* knowledge of a state equation for $F_c$ and certain conditions on the corresponding vector fields. No such assumptions will be utilized here.

This paper is organized as follows. Section 2 introduces the stochastic tools and definitions used throughout the paper. In the subsequent section, the definition of a Fliess operator driven by an $L_2$-Itô random input process is given. Then, in preparation for the global convergence analysis of Fliess operators, $L_2$ upper bounds for stochastic iterated integrals are determined. Section 4 presents the global convergence results, and Section 5 develops the local convergence theory. Conclusions and suggestions for future research are summarized in the final section.

## 2. Stochastic setting

Consider a Wiener process, $W$, defined over a complete probability space $(\Omega, \mathcal{F}, P)$. For a predictable function $u : \Omega \times [t_0, t_0 + T] \to \mathbb{R}^m$ let $\|u\|_p = \max\{\|u_i\|_{L_p} : 1 \leq i \leq m\}$, where $\|\cdot\|_{L_p}$ is the usual norm on $L_p(\Omega \times [t_0, t_0 + T], \mathcal{P}, P \otimes \lambda)$, the set of all predictable functions defined on $[t_0, t_0+T]$ having finite $\|\cdot\|_{L_p}$-norm, $\mathcal{P}$ is the predictable algebra and $\lambda$ is the Lebesgue measure.

Definition 2.1 [18]. Let $T>0$ and $t_0$ be fixed. An $m$-dimensional stochastic process $w$ over $[t_0, t_0+T]$ is called an *$L_2$-Itô process* if it can be written as

$$w(t) = w(0) + \int_{t_0}^{t} a(s)\mathrm{d}s + \int_{t_0}^{t} b(s)\mathrm{d}W(s),$$

where $a, b \in L_2^m(\Omega \times [t_0, t_0+T], \mathcal{P}, P \otimes \lambda)$, $w(0)$ is a constant and $\int_{t_0}^{t} \cdot \mathrm{d}W(s)$ denotes Itô integration.

The set of all $L_2$-Itô processes will be denoted by $\mathscr{I} \subset L_2^m(\Omega \times [t_0, t_0+T], \mathcal{P}, P \otimes \lambda)$. The Stratonovich integral can be defined in terms of an Itô integral as follows.

Definition 2.2 [17,18]. For a stochastic process $v \in L_2(\Omega \times [t_0, t_0+T], \mathcal{P}, P \otimes \lambda)$, the *Stratonovich integral* of $v$ is defined by

$$\oint_{t_0}^{t} v(s)\mathrm{d}W(s) = \int_{t_0}^{t} v(s)\mathrm{d}W(s) + \frac{1}{2}\langle v, W\rangle_{[t_0,t]}, \tag{5}$$

where the quadratic covariation is

$$\langle v, W\rangle_{[t_0,t]} \triangleq \lim_{\|\Pi\|\to 0} \sum_{k=0}^{n-1} (v(t_{k+1}) + v(t_k))(W(t_{k+1}) - W(t_k)),$$

and $\|\Pi\| = \max_{k=0,\dots,n-1}(t_{k+1} - t_k)$ is the measure of the partition $\Pi$.

A well known property of Stratonovich integrals is that they obey the usual integration by parts formula. The next definition introduces the set of admissible stochastic processes that will drive a Fliess operator.

Definition 2.3. Let $T>0$ and $t_0$ be fixed. Consider the set of all $m$-dimensional stochastic processes over $[t_0, t_0+T]$, denoted by $\widetilde{\mathcal{UV}}^m[t_0, t_0+T]$, which can be written as

$$w(t) = \int_{t_0}^{t} u(s)\mathrm{d}s + \oint_{t_0}^{t} v(s)\mathrm{d}W(s)$$

for some $u, v \in \mathscr{I}$. The latter are called the *drift* and *diffusion* inputs, respectively. Moreover, the subset $\mathcal{UV}^m[t_0, t_0+T]$ of $\widetilde{\mathcal{UV}}^m[t_0, t_0+T]$ will refer to all processes satisfying:

(a) Each integrand consists of $m$ components such that $\mathbf{E}[u_i(t)] < \infty$, $\mathbf{E}[v_i(t)] < \infty$, $t \in [t_0, t_0+T]$.
(b) The integrands $u$ and $v$ are such that

$$\|u\|_{L_2}, \|v\|_{L_2}, \|b\|_{L_2}, \|v\|_{L_4} \le R \in \mathbb{R}^+,$$

where $b$ is the integrand of the Itô integral in $v$.
(c) The random variables $u_i(t_1)$, $u_i(t_2)$, $v_i(t_1)$ and $v_i(t_2)$ are independent for $1 \le i \le m$ and $t_1 \ne t_2$.

Observe that since $u, v \in \mathscr{I}$, then by Definition 2.2 any $w \in \widetilde{\mathcal{UV}}^m[t_0, t_0+T]$ is also an $L_2$-Itô process. Without loss of generality, it is assumed hereafter that $t_0 = 0$.

The following setting will be employed in Section 5, where time intervals of random length will play a central role in the local convergence analysis.

DEFINITION 2.4 [21]. Let $\mathbf{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ be a filtration with $\mathcal{T} = [0, \infty]$. A random variable $\tau : \Omega \to \mathcal{T}$ is a *stopping time* with respect to $\mathbf{F}$ if the event $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \mathcal{T}$.

DEFINITION 2.5 [21]. Let $X$ be a stochastic process, and let $D$ be a Borel set in $\mathbb{R}$. Define the *hitting time* of $D$ for $X$ as $\tau = \inf\{t > 0 : X(t) \in D\}$ or $\tau = +\infty$ if $X(t) \notin D$ for all $t \in \mathcal{T}$.

THEOREM 2.6 [21]. Let $X$ be an adapted càdlàg stochastic process, and let $D$ be either an open or closed Borel set. Then the hitting time, $\tau$, of $D$ is a stopping time.

*Example 2.7*. Consider an almost sure (a.s.) continuous time stochastic process $X$. A special case of a hitting time is $\tau_R = \inf\{t > 0 : X(t) = R\}$, where $R \in \mathbb{R}^+$. It is usually called the *first passage time* for the barrier $R$. When $X$ is a Wiener process, the probability density function for $\tau_R$ is

$$P(t) = \frac{|R|}{\sqrt{2\pi}} t^{-\frac{3}{2}} \mathrm{e}^{-\frac{R^2}{2t}},$$

which is an inverse gamma density function with parameters $\alpha = 1/2$ and $\beta = R^2/2$ [17]. In Figure 2, a Monte Carlo generated estimate of the probability density function for $\tau_R$ is shown when $R = 1$.

DEFINITION 2.8. Let $X$ be a stochastic process on $\mathcal{T}$, and let $\tau$ be a stopping time. A *stopped* or *truncated process* is any process of the form

$$X^\tau(t, \omega) \triangleq X(t \wedge \tau(\omega), \omega) = X(t, \omega) 1_{[0, \tau(\omega))}(t) + X(\tau(\omega), \omega) 1_{[\tau(\omega), \infty)}(t),$$

where $t \wedge \tau \triangleq \min(t, \tau)$ for $t \geq 0$, and $1_A(t)$ indicates whether or not $t \in A$. The random variable $X_\tau(\omega) \triangleq X(\tau(\omega), \omega)$ is called a *stopped random variable*.

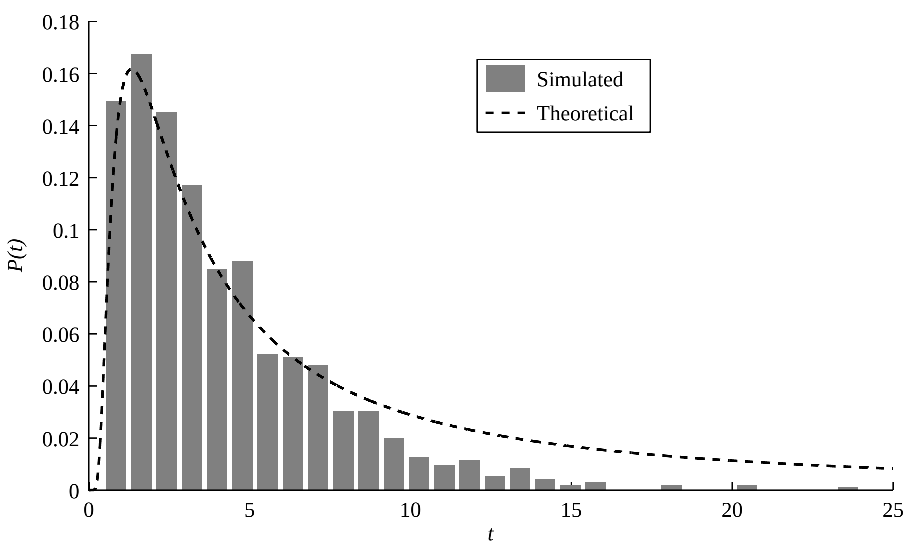


Figure 2. Estimated probability density function for $\tau_1$.

Observe that if the stopped process $X^{\tau}(t,\omega)$ is restricted to the stochastic interval

$$[0,\tau] \triangleq \{(t,\omega) \in [0,\infty) \times \Omega : 0 \leq t \leq \tau(\omega)\},$$

then $X^{\tau}(t,\omega) = X(t,\omega)1_{[0,\tau(\omega)]}$. Usually, a path of this process is denoted simply by $X(t \wedge \tau)$. Also, if $X(t) = \int_0^t v(s)\mathrm{d}W(s)$ then the stopped random variable $X_\tau$ satisfies $X_\tau = \int_0^{\tau} v(s)\mathrm{d}W(s) = \int_{\mathcal{T}} v(s)1_{[0,\tau]}(s)\mathrm{d}W(s)$.

THEOREM 2.9 [22,23]. Let $\tau$ be a stopping time and $v \in L_2(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda)$. Define $X(t) = \int_0^t v(s)\mathrm{d}W(s)$. Then $X$ stopped at $\tau$ satisfies

$$X^{\tau}(t) = X(t \wedge \tau) = \int_0^{t\wedge\tau} v(s)\mathrm{d}W(s) = \int_0^t v(s)1_{[0,\tau]}(s)\mathrm{d}W(s).$$

When $v$ is an $L_2$-Itô process, a similar result for Stratonovich integrals follows directly from Theorem 2.9.

COROLLARY 2.10. Let $\tau$ be a stopping time and $v \in \mathscr{I}$. If $X(t) = \oint_0^t v(s)\mathrm{d}W(s)$ then the stopped process $X^{\tau}(t) = \oint_0^{t\wedge\tau} v(s)\mathrm{d}W(s)$ satisfies

$$X^{\tau}(t) = \oint_0^t v(s)1_{[0,\tau]}(s)\mathrm{d}W(s).$$

*Proof*. Since $v$ can be written as

$$v(t) = \int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}W(s)$$

for $a, b \in L_2(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda)$, then $X$ can be written as

$$X(t) = \int_0^t v(s)\mathrm{d}W(s) + \int_0^t \frac{b(s)}{2}\mathrm{d}s.$$

It follows from Theorem 2.9 that

$$\begin{aligned} X^{\tau}(t) &= X(t)1_{[0,\tau]}(t) \\ &= \int_0^{t\wedge\tau} v(s)\mathrm{d}W(s) + \int_0^{t\wedge\tau} \frac{b(s)}{2}\mathrm{d}s \\ &= \int_0^t v(s)1_{[0,\tau]}(s)\mathrm{d}W(s) + \int_0^t \frac{b(s)}{2}1_{[0,\tau]}(s)\mathrm{d}s \\ &= \int_0^t v(s)1_{[0,\tau]}(s)\mathrm{d}W(s) + \frac{1}{2}\left\langle \int_0^{\cdot} a(s)1_{[0,\tau]}(s)\mathrm{d}s + \int_0^{\cdot} b(s)1_{[0,\tau]}(s)\mathrm{d}W(s), W \right\rangle_{[0,t]} \\ &= \int_0^t v(s)1_{[0,\tau]}(s)\mathrm{d}W(s) + \frac{1}{2}\langle v1_{[0,\tau]}, W\rangle_{[0,t]} = \oint_0^t v(s)1_{[0,\tau]}(s)\mathrm{d}W(s). \end{aligned}$$

This completes the proof. □

THEOREM 2.11. Let $X(t) = \oint_0^t v(s)\mathrm{d}W(s)$, where $v \in \mathscr{I}$ and $t \in [0,T]$. Then:

(a) For any finite $R > 0$, there exists a positive stopping time $\tau_R = \inf\{t > 0 : |X(t)| = R\} \wedge T$.
(b) $X^{\tau_R}(t,\omega)$ restricted to $[0,\tau_R]$ is a well-defined $L_2$-bounded, a.s. continuous and adapted $L_2$-Itô process.

*Proof*. For part (a), since $v \in \mathscr{I}$ with diffusion $b$, $X$ can be written as

$$X(t) = \int_0^t \frac{b(s)}{2} \mathrm{d}s + \int_0^t v(s) \mathrm{d}W(s).$$

Therefore, $\tau_R$ is a positive well-defined stopping time since the Lebesgue and Itô integrals produce a.s. continuous processes, and the absolute value function is continuous. Moreover, since $v$ is defined over $[0, T]$, it is clear that if $|X(t)| < R$ for all $t \in [0, T']$ then $\tau_R = T$ when $T' > T$. Regarding part (b), it is well known that since $X$ is adapted and a.s. continuous, the stopped process is also adapted and a.s. continuous [21]. Now evaluate $\int_0^T \mathbf{E}[X^2(s)]\mathrm{d}s$. By Itô's formula,

$$X^2(t) = 2\int_0^t X(s)\bar{b}(s)\mathrm{d}s + 2\int_0^t X(s)v(s)\mathrm{d}W(s) + \int_0^t v^2(s)\mathrm{d}s,$$

where $\bar{b}(s) = b(s)/2$. Since $2k_1k_2 \leq k_1^2 + k_2^2$ for all $k_1, k_2 \in \mathbb{R}$, it follows from the properties of the Itô integral that

$$\mathbf{E}[X^2(t)] \leq \mathbf{E}\left[\int_0^t (X^2(s) + \bar{b}(s))\mathrm{d}s + \int_0^t v^2(s)\mathrm{d}s\right].$$

The $L_2$ bound for $X$ restricted to $[0, \tau_R]$ can be calculated from the previous expression as

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq \mathbf{E}\left[\int_0^t (X^2(s \wedge \tau_R) + \bar{b}^2(s \wedge \tau_R))\mathrm{d}s + \int_0^t v^2(s \wedge \tau_R)\mathrm{d}s\right].$$

Given that $\bar{b}, v \in L_2(\Omega \times [0, T], \mathcal{P}, P \otimes \lambda)$, define a real number $M \geq \mathbf{E}\left[\int_0^t \bar{b}^2(s)\mathrm{d}s\right] + \mathbf{E}\left[\int_0^t v^2(s)\mathrm{d}s\right]$. Then by Fubini's theorem

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq M + \int_0^t \mathbf{E}[X^2(s \wedge \tau_R)]\mathrm{d}s. \tag{6}$$

If (6) is applied recursively to the integrand of (6), then

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq M + Mt + \int_0^t \int_0^s \mathbf{E}[X^2(r \wedge \tau_R)]\mathrm{d}r\mathrm{d}s.$$

Repeating this procedure infinitely many times and noting that $\mathbf{E}[X^2(r \wedge \tau_R)] \leq R^2$, it follows that

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq \lim_{p \to \infty} M \sum_{n=0}^{p} \frac{t^n}{n!} + R^2 \frac{t^p}{p!} = M\mathrm{e}^t$$

for any $t \in [0, T]$. This implies that $\int_0^T \mathbf{E}[X^2(t \wedge \tau_R)]\mathrm{d}t < \infty$, and thus, $X^{\tau_R}(t, \omega) \in L_2(\Omega \times [0, T], \mathcal{P}, P \otimes \lambda)$. This completes the proof. □

## 3. Fliess operators and $L_2$ upper bounds for iterated integrals

The purpose of this section is to introduce a definition for a Fliess operator acting on a process in $\mathcal{UV}^n[0,T]$, $T > 0$ and to develop $L_2$ upper bounds for the associated stochastic iterated integrals. These upper bounds are used directly in the proof of global convergence of such Fliess operators in Section 4.

Consider the following alphabets: $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_m\}$ and $XY = X \cup Y$. For each $\eta \in XY^*$, define iteratively the mapping

$E_\eta : L_2^m(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda) \to \mathcal{C}_{\text{a.s.}}[0,T]$ by first setting $E_\emptyset = 1$ and then letting

$$E_{x_i\eta'}[w](t) \triangleq \int_0^{t-} u_i(s) E_{\eta'}[w](s)\mathrm{d}s, \quad x_i \in X, \tag{7}$$

$$E_{y_i\eta'}[w](t) \triangleq \oint_0^{t-} v_i(s) E_{\eta'}[w](s)\mathrm{d}W(s), \quad y_i \in Y, \tag{8}$$

where $\eta' \in XY^*$, $u_0 = v_0 = 1$, and the notation $t-$ (suppressed in subsequent sections) indicates that the integration is over $[0,t)$. The iterated integral defined in (7) and (8) can be extended linearly to a polynomial $p \in \mathbb{R}\langle XY\rangle$ as

$$E_p[w](t) = \sum_{\eta \in \text{supp}(p)} (p,\eta) E_\eta[w](t),$$

where $\text{supp}(p) = \{\eta \in XY^* : (p,\eta) \neq 0\}$. The set of all such integrals forms an $\mathbb{R}$-vector space denoted as $\mathcal{E}(\mathbb{R}\langle XY\rangle)$. A Fliess operator is defined over $\mathcal{UV}^m[0,T]$ as follows.

Definition 3.1 [3–5]. A causal $m$-input, $\ell$-output *Fliess operator* $F_c$, $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$, driven by a random process $w \in \mathcal{UV}^m[0,T]$ is formally defined as

$$F_c[w](t) = \sum_{\eta \in XY^*} (c,\eta) E_\eta[w](t), \tag{9}$$

where each $E_\eta$ is given in (7) and (8).

The primary objective of this paper is to ensure that $F_c[w]$ is mean square convergent. That is,

$$\sum_{\eta \in XY^*} |(c,\eta)| \|E_\eta[w](t)\|_2 < \infty$$

for any $t \in [0,T]$, where $\|\cdot\|_2$ denotes the $L_2$ random variable norm. Given that each $E_\eta[w](t)$ in (9) involves both Stratonovich and Lebesgue integrals in its definition, a special approach has to be employed to calculate an upper bound for $\|E_\eta[w](t)\|_2$. It is known that Stratonovich integrals lack certain important properties such as isometry [18]. But if a Stratonovich integral is written in terms of Itô integrals, then all the properties associated with Itô integrals are available [16,19]. A convenient formula for the iterated integral $E_\eta[w]$, $\eta \in X^kY^n$, can be obtained by a successive application of (5). To develop this expression, some terminology needs to be introduced. Let $\mathbb{N}^{m+1}$ be the set of all vectors with components in $\mathbb{N} = \{0,1,\ldots\}$. For any $A \subset XY$, let $|\eta|_A$ denote the number of letters in $\eta$ that belongs to $A$. Define the language $X^kY^n = \{\eta \in XY^*, |\eta|_X = k, |\eta|_Y = n\}$ formed by words having $k$ letters in $X$ and $n$ letters in $Y$. For a fixed word $\eta \in X^kY^n$, define the vectors $\boldsymbol{\alpha} = (\alpha_m,\cdots,\alpha_0) \in \mathbb{N}^{m+1}$ and $\boldsymbol{\beta} = (\beta_m,\cdots,\beta_0) \in \mathbb{N}^{m+1}$, where $\alpha_i = |\eta|_{x_i}$, $\beta_i = |\eta|_{y_i}$, $k = \sum_{i=0}^m \alpha_i$ and $n = \sum_{i=0}^m \beta_i$. The summations over all possible $\alpha_i$'s that sum to $k$ and all possible $\beta_i$'s that sum to $n$ are denoted, respectively, by $\sum_{\|\alpha\|=k}$ and $\sum_{\|\beta\|=n}$. Since one is often interested in working with arbitrary letters in the alphabet $XY$, hereafter $q_i^l$ will denote an element of $XY$ with $q_i^l = x_i$ if $l=1$ and $q_i^l = y_i$ if $l=2$. Similarly, $w_{q_i^l}$ will denote either a drift or a diffussion input, and $\mathrm{d}q_i^l$ will signify either Lebesgue or Stratonovich integration according to the value of $l$. Finally, define $J(\eta) = (j_n,\ldots,j_1) \in \mathbb{N}^{|\eta|_Y}$ to be those places in $\eta$ where all the letters belonging to $Y$ are located. For example, if $\eta = x_{i_5}y_{i_4}x_{i_3}y_{i_2}y_{i_1}$ then $J(\eta) = (j_3,j_2,j_1) = (4,2,1)$.

Theorem 3.2. Let $\eta \in X^k Y^n$ and $w \in \mathcal{UV}^m[0,T]$ be arbitrary. Then

$$E_\eta[w](t) = \sum_{r_1=0,r_2=0}^{n,\lfloor n/2 \rfloor} \frac{1}{2^{r_1}2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{nr_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{nr_2}}} \mathbf{I}_\eta^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t), \tag{10}$$

where

$$\bar{A}_{nr_2} \triangleq \left\{ \bar{\mathbf{s}}_{\mathbf{r_2}} = (\bar{s}_{r_2}, \ldots, \bar{s}_1) \in \mathbb{N}^{r_2} : \bar{s}_{l_2} + 1 < \bar{s}_{l_2+1},\, l_{\bar{s}_{l_2}+1} = 2, 1 \le l_2 \le r_2 - 1, \bar{s}_{l_2} \in J(\eta) \right\}$$

for $1 \le r_2 \le \lfloor n/2 \rfloor$, $\bar{A}_{n0} = \emptyset$,

$$A_{nr_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \triangleq \Big\{ \mathbf{s_{r_1}} = (s_{r_1}, \ldots, s_1) \in \mathbb{N}^{r_1} : s_{l_1} < s_{l_1+1}, 1 \le l_1 \le r_1 - 1, s_{l_1} \ne \bar{s}_{l_2} \quad \text{or}$$
$$\bar{s}_{l_2} + 1, \bar{s}_{l_2} \in \bar{\mathbf{s}}_{\mathbf{r_2}}, s_{l_1} \in J(\eta) \Big\}$$

for $1 \le r_1 \le n$, $A_{n0}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} = \emptyset$, and $\lfloor \cdot \rfloor$ is the floor function. In addition, if $\eta = q_{i_{k+n+1}}^{l_{k+n+1}} \eta'$ with $\eta' \in X^k Y^n$ then

$$\mathbf{I}_\eta[w](t) \triangleq \int_0^t w_{q_{i_{k+n+1}}^{l_{k+n+1}}}(s) \mathbf{I}_{\eta'}[w](t_{k+n+1}) \mathrm{d}q_{i_{k+n+1}}^{l_{k+n+1}}(s), \tag{11}$$

$$\mathbf{I}_\eta^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t) \triangleq \mathbf{I}_\eta[w](t) \Bigg|_{\substack{\int v_{i_{\bar{s}_{l_2}+1}} \int v_{i_{\bar{s}_{l_2}}} \mathrm{d}W(t')\mathrm{d}W(t) \to \int v_{i_{\bar{s}_{l_2}+1}} v_{i_{\bar{s}_{l_2}}} \mathrm{d}t \\ \int v_{i_{s_{l_1}}} \mathrm{d}W(t) \to \int b_{i_{s_{l_1}}} \mathrm{d}t}} \tag{12}$$

with $b_{i_{s_l}} \in L_2(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda)$, $1 \le l_1 \le r_1$, $1 \le l_2 \le r_2$, and $i_{s_{l_1}}, i_{\bar{s}_{l_2}} \in \{0, \ldots, m\}$ are the indices of the $s_{l_1}$th and $\bar{s}_{l_2}$th elements of $J(\eta)$, respectively.

*Proof*. The proof is by induction on the length of the word $\eta \in X^k Y^n$. The $k+n = 0, 1$ cases are trivial. For $k+n=2$, one has $\eta \in \{x_{i_2}x_{i_1}, x_{i_2}y_{i_1}, y_{i_2}x_{i_1}, y_{i_2}y_{i_1}\}$. Evaluating $E_\eta[w](t)$ for each of these four cases, the claim follows by direct application of (5). For example, when $\eta = y_{i_2}y_{i_1}$, it is not difficult to obtain

$$E_\eta[w](t) = \mathbf{I}_\eta[w](t) + \frac{1}{2}\mathbf{I}_\eta^{\substack{(1) \\ \emptyset}}[w](t) + \frac{1}{2}\mathbf{I}_\eta^{\substack{(2) \\ \emptyset}}[w](t) + \frac{1}{2}\mathbf{I}_\eta^{\substack{\emptyset \\ (1)}}[w](t) + \frac{1}{4}\mathbf{I}_\eta^{\substack{(2,1) \\ \emptyset}}[w](t)$$
$$= \sum_{r_1=0,r_2=0}^{2,1} \frac{1}{2^{r_1}2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{2r_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{2r_2}}} \mathbf{I}_\eta^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t).$$

Now assume that (10) holds up to some $n+k \ge 0$ and let $\eta = q_{i_{k+n+1}}^{l_{k+n+1}} \eta'$, where $\eta' \in X^k Y^n$. If $l_{k+n+1} = 1$ then a drift term has been added and (10) holds. If $l_{k+n+1} = 2$ then a diffusion term has been added and

$$E_\eta[w](t) = \oint_0^t v_{i_{k+n+1}}(t_{k+n+1}) \times \left( \sum_{r_1=0, r_2=0}^{n, \lfloor n/2 \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{nr_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{nr_2}}} \mathbf{I}_{\eta'}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t_{k+n+1}) \right) \mathrm{d}W(t_{k+n+1}). \tag{13}$$

Observe that $s_{l_1} \neq \bar{s}_{l_2}, \bar{s}_{l_2}+1$ for $0 \le l_2 \le \lfloor n/2 \rfloor$. Using formula (5), if $s_{r_1} < k+n, \bar{s}_{r_2} < k+n-1$ and $q_{i_{n+k}}^{l_{n+k}} \in Y$ then

$$\left\langle v_{i_{k+n+1}} \left( \mathbf{I}_{\eta'}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w] \right), W \right\rangle_{[0,t]} = \mathbf{I}_{\eta}^{\substack{\mathbf{s'_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t) + \mathbf{I}_{\eta}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}'_{\mathbf{r_2}}}}[w](t).$$

But if $s_{r_1} = k+n$ or $\bar{s}_{r_2} = k+n-1$ or $q_{i_{n+k}}^{l_{n+k}} \in X$ then

$$\left\langle v_{i_{k+n+1}} \left( \mathbf{I}_{\eta'}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w] \right), W \right\rangle_{[0,t]} = \mathbf{I}_{\eta}^{\substack{\mathbf{s'_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t),$$

where $\mathbf{s'_{r_1}} = (k+n+1, s_{r_1}, \ldots, s_1)$ and $\bar{\mathbf{s}}'_{\mathbf{r_2}} = (k+n, \bar{s}_{r_2}, \ldots, \bar{s}_1)$. Substituting the quadratic covariation above into (13) and regrouping gives

$$E_\eta[w](t) = \sum_{r_1=0, r_2=0}^{n, \lfloor n/2 \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{nr_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{nr_2}}} \mathbf{I}_{\eta}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t) + \frac{1}{2} \left\langle v_{i_{k+n+1}} \left( \mathbf{I}_{\eta'}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w] \right), W \right\rangle_{[0,t]}.$$

For $n$ even,

$$E_\eta[w](t) = \sum_{r_1=0, r_2=0}^{n+1, \lfloor n/2 \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{(n+1)r_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{(n+1)r_2}}} \mathbf{I}_{\eta}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t), \tag{14}$$

and for $n = n'+1$ odd,

$$E_\eta[w](t) = \sum_{r_1=0, r_2=0}^{n'+1, \lfloor n/2 \rfloor + 1} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{(n'+1)r_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{(n'+1)r_2}}} \mathbf{I}_{\eta}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t). \tag{15}$$

Together (14) and (15) become

$$E_\eta[w](t) = \sum_{r_1=0, r_2=0}^{n+1, \lfloor (n+1)/2 \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\mathbf{s_{r_1}} \in A_{(n+1)r_1}^{\bar{\mathbf{s}}_{\mathbf{r_2}}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}} \in \bar{A}_{(n+1)r_2}}} \mathbf{I}_{\eta}^{\substack{\mathbf{s_{r_1}} \\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t).$$

Thus, the induction is complete. □

Given that $E_\eta$ can be expanded in terms of Itô integrals (11) and (12), the goal now is to compute an $L_2$ upper bound for each of these integrals.

THEOREM 3.3. Let $\eta \in X^k Y^n$ and $w \in \mathcal{UV}^m[0,T]$ be arbitrary. An $L_2$ upper bound for the iterated Itô integral (11) evaluated at a fixed $t \in [0,T]$ is

$$\|\mathbf{I}_\eta[w](t)\|_2^2 \le t^k \prod_{i=0}^{m} \frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!} \frac{V_i^{\beta_i}(t)}{\beta_i!}, \tag{16}$$

where $\tilde{U}_i(t) = \int_0^t \mathbf{E}[u_i^2(s)]\mathrm{d}s$ and $V_i(t) = \int_0^t \mathbf{E}[v_i^2(s)]\mathrm{d}s$.

*Proof.* The inequality is proved by induction over the total number of $k+n$ integrals comprising (11). For $k+n=0$, the claim follows trivially. If $k+n=1$, then there are two cases to consider. First, if $\eta = x_{i_1}$ then by Hölder's inequality

$$\|\mathbf{I}_{x_{i_1}}[w](t)\|_2^2 \le t \int_0^t \mathbf{E}[u_{i_1}^2(t_1)]\mathrm{d}t_1 = t\tilde{U}_{i_1}(t).$$

The second case is when $\eta = y_{i_1}$. By the isometry property

$$\|\mathbf{I}_{y_{i_1}}[w](t)\|_2^2 = \int_0^t \mathbf{E}[v_{i_1}^2(t_1)]\mathrm{d}t_1 = V_{i_1}(t).$$

Now calculate the bound for $n+k+1$ integrals assuming that (16) holds up to some fixed $n+k \ge 0$. Set $\eta = q_{i_{k+n+1}}^{l_{k+n+1}}\eta'$ with $\eta' \in X^k Y^n$. If $l_{k+n+1} = 1$, then the independence property (c) in Definition 2.3 gives

$$\begin{aligned}
\|\mathbf{I}_\eta[w](t)\|_2^2 &= \mathbf{E}\left[\left(\int_0^t u_{i_{k+n+1}}(t_{k+n+1})\mathbf{I}_{\eta'}[w](t_{k+n+1})\mathrm{d}t_{k+n+1}\right)^2\right] \\
&\le t\int_0^t \mathbf{E}[u_{i_{k+n+1}}^2(t_{k+n+1})]\mathbf{E}[(\mathbf{I}_{\eta'}[w](t_{k+n+1}))^2]\mathrm{d}t_{k+n+1} \\
&= t\int_0^t \mathbf{E}[u_{i_{k+n+1}}^2(t_{k+n+1})]\left[t_{k+n+1}^k \prod_{i=0}^m \frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{k+n+1})}{\bar{\alpha}_i!}\frac{V_i^{\beta_i}(t_{k+n+1})}{\beta_i!}\right]\mathrm{d}t_{k+n+1} \\
&\le t^{k+1} \prod_{\substack{i=0\\ i\neq i_{k+n+1}}}^m \frac{\tilde{U}_i^{\bar{\alpha}_i}(t)}{\bar{\alpha}_i!}\prod_{i=0}^m \frac{V_i^{\beta_i}(t)}{\beta_i!} \underbrace{\int_0^t \mathbf{E}[u_{i_{k+n+1}}^2(t_{k+n+1})]\frac{\tilde{U}_{i_{k+n+1}}^{\bar{\alpha}_{i_{k+n+1}}}(t_{k+n+1})}{\bar{\alpha}_{i_{k+n+1}}!}\mathrm{d}t_{k+n+1}}_{=\frac{\tilde{U}_i^{\bar{\alpha}_{i_{k+n+1}}+1}(t)}{(\bar{\alpha}_{i_{k+n+1}}+1)!}} \\
&\le t^{k+1}\prod_{i=0}^m \frac{\tilde{U}_i^{\bar{\alpha}_i}(t)}{\bar{\alpha}_i!}\frac{V_i^{\beta_i}(t)}{\beta_i!},
\end{aligned}$$

where $\sum_{i=0}^m \bar{\alpha}_i = k$. Letting $\alpha_i = \bar{\alpha}_i + \delta_{i(i_{k+n+1})}$ (here $\delta_{ij}$ denotes the Kronecker delta function), then

$$\|\mathbf{I}_\eta[w](t)\|_2^2 \le t^{k+1}\prod_{i=0}^m \frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\beta_i}(t)}{\beta_i!},$$

where $\sum_{i=0}^{m}\alpha_i = k+1$. Now, for the case when $l_{k+n+1}=2$, the isometry property gives

$$\begin{aligned}\|\mathbf{I}_\eta[w](t)\|_2^2 &= \mathbf{E}\left[\left(\int_0^t v_{i_{k+n+1}}(t_{k+n+1})\mathbf{I}_{\eta'}[w](t_{k+n+1})\mathrm{d}W(t_{k+n+1})\right)^2\right]\\ &= \int_0^t \mathbf{E}[v^2_{i_{k+n+1}}(t_{k+n+1})]\mathbf{E}\left[\left(\mathbf{I}_{\eta'}[w](t_{k+n+1})\right)^2\right]\mathrm{d}t_{k+n+1}\\ &= \int_0^t \mathbf{E}[v^2_{i_{k+n+1}}(t_{k+n+1})]\left[t^k_{k+n+1}\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t_{k+n+1})}{\alpha_i!}\prod_{i=0}^{m}\frac{V_i^{\bar{\beta}_i}(t_{k+n+1})}{\bar{\beta}_i!}\right]\mathrm{d}t_{k+n+1}\\ &\le t^k\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\prod_{\substack{i=0\\ i\neq i_{k+n+1}}}^{m}\frac{V_i^{\bar{\beta}_i}(t)}{\bar{\beta}_i!}\frac{V_i^{\bar{\beta}_{i_{k+n+1}}+1}(t)}{(\bar{\beta}_{i_{k+n+1}}+1)!},\end{aligned}$$

where $\sum_{i=0}^{m}\bar{\beta}_i = n$. Similarly, let $\beta_i = \bar{\beta}_i + \delta_{i(i_{k+n+1})}$, then

$$\|\mathbf{I}_\eta[w](t)\|_2^2 \le t^k\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\beta_i}(t)}{\beta_i!},$$

where $\sum_{i=0}^{m}\beta_i = n+1$. Hence, the proof is complete. □

THEOREM 3.4. Let $\eta \in X^kY^n$ and $w \in \mathcal{UV}^m[0,T]$ be arbitrary. An $L_2$ upper bound for the iterated Itô integral (12) evaluated at a fixed $t\in[0,T]$ is

$$\left\|\mathbf{I}_\eta^{\substack{\mathbf{s_{r_1}}\\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t)\right\|_2^2 \le 2^{r_2}t^{k+r_1+r_2}\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\bar{\beta}_i}(t)}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}}\frac{B_i^{\gamma_i}(t)}{\gamma_i!},$$

where $\bar{\gamma}_i = \sum_{l_2=1}^{r_2}(\delta_{ii_{\bar{s}_{l_2}}} + \delta_{i(i_{\bar{s}_{l_2}}+1)})$, $\bar{s}_{l_2}\in\bar{\mathbf{s}}_{\mathbf{r_2}}$; $\gamma_i = \sum_{l_1=1}^{r_1}\delta_{ii_{s_{l_1}}}$, $s_{l_1}\in\mathbf{s_{r_1}}$; $\bar{\beta}_i = \beta_i - \bar{\gamma}_i - \gamma_i$; $\tilde{U}_i(t) = \int_0^t\mathbf{E}[u_i^2(s)]\mathrm{d}s$; $V_i(t) = \int_0^t\mathbf{E}[v_i^2(s)]\mathrm{d}s$; $\bar{V}_i^2(t) = \int_0^t\mathbf{E}[v_i^4(s)]\mathrm{d}s$ and $B_i(t) = \int_0^t\mathbf{E}[b_i^2(s)]\mathrm{d}s$.

*Proof*. Use induction over $r = r_1 + r_2$. Without loss of generality, set $\eta = q_{i_{k+n}}^{l_{k+n}}\cdots q_{i_{s_r+1}}^{l_{s_r+1}}y_{i_{s_r}}\eta' \in X^kY^n$, $\eta'\in X^{k-k_1}Y^{s_r-1}$ with $s_r = s_{r_1}$ or $\bar{s}_{r_2}+1$, and $k_1\le k$. For $s_{r_1} > \bar{s}_{r_2}$, applying the isometry property $n - s_{r_1}$ times and Hölder's inequality $k_1$ times gives

$$\begin{aligned}\left\|\mathbf{I}_\eta^{\substack{\mathbf{s_{r_1}}\\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t)\right\|_2^2 \le t^{k_1}\int_0^t\mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}^{l_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\\ \times\mathbf{E}\left[\left(\int_0^{t_{s_{r_1}+1}} b_{i_{s_{r_1}}}(t_{s_{r_1}})\mathbf{I}_{\eta'}^{\substack{\mathbf{s_{r_1-1}}\\ \bar{\mathbf{s}}_{\mathbf{r_2}}}}[w](t_{s_{r_1}})\mathrm{d}t_{s_{r_1}}\right)^2\right]\mathrm{d}t_{s_{r_1}+1}\cdots\mathrm{d}t_{k+n}.\end{aligned}$$

Using Hölder's inequality once more and applying the induction hypothesis with $r-1 = r_1+r_2-1$,

$$\begin{aligned}
\left\|\mathbf{I}_{\eta}^{\substack{s_{r_1}\\ \bar{s}_{r_2}}}[w](t)\right\|_2^2 &\le t^{k_1}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}^{l_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\\
&\quad\times t_{s_{r_1}+1}\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b^2_{i_{s_{r_1}}}(t_{s_{r_1}})\right]\left\|\mathbf{I}_{\eta'}^{\substack{s_{r_1}-1\\ \bar{s}_{r_2}}}[w](t_{s_r})\right\|_2^2 \mathrm{d}t_{s_{r_1}}\,\mathrm{d}t_{s_{r_1}+1}\cdots\mathrm{d}t_{k+n}\\
&\le 2^{r_2}t^{k+r_1+r_2}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}^{l_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\\
&\quad\times\prod_{i=0}^{m}\frac{\tilde{U}^{\bar{\alpha}_i}(t_{s_{r_1}})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}_i}(t_{s_{r_1}+1})}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t_{s_{r_1}+1})}{\sqrt{\bar{\gamma}_i!}}\prod_{\substack{l=0\\ l\neq i_{s_{r_1}}}}^{m}\frac{B_l^{\gamma_l}(t_{s_{r_1}+1})}{\gamma_l!}\\
&\quad\times\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b^2_{i_{s_{r_1}}}(t_{s_{r_1}})\right]\frac{B_{i_{s_{r_1}}}^{\gamma_{i_{s_{r_1}}}}(t_{s_{r_1}})}{\gamma_{i_{s_{r_1}}}!}\mathrm{d}t_{s_{r_1}}\,\mathrm{d}t_{s_{r_1}+1}\cdots\mathrm{d}t_{k+n}\\
&\le 2^{r_2}t^{k+r_1+r_2}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\\
&\quad\times\prod_{i=0}^{m}\frac{\tilde{U}^{\bar{\alpha}_i}(t_{s_{r_1}})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}_i}(t_{s_{r_1}+1})}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t_{s_{r_1}+1})}{\sqrt{\bar{\gamma}_i!}}\frac{B_l^{\gamma'_i}(t_{s_{r_1}+1})}{\gamma'_i!}\mathrm{d}t_{s_{r_1}+1}\cdots\mathrm{d}t_{k+n},
\end{aligned}$$

where $\gamma'_i=\gamma_i+\delta_{ii_{s_{r_1}}}$. Observe that $\sum_{i=0}^m\gamma'_i=r_1$. The remaining $(n+k_1-s_{r_1})$ integrals are evaluated exactly as in the proof of Theorem 3.3. Thus, the $\bar{\alpha}_i$'s and $\bar{\beta}_i$'s increase rather than the $\gamma_i$'s, and

$$\left\|\mathbf{I}_{\eta}^{\substack{s_{r_1}\\ \bar{s}_{r_2}}}[w](t)\right\|_2^2 \le 2^{r_2}t^{k+r_1+r_2}\prod_{i=0}^{m}\frac{\tilde{U}^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\bar{\bar{\beta}}_i}(t)}{\bar{\bar{\beta}}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}}\frac{B_l^{\gamma'_i}(t)}{\gamma'_i!},$$

where $\sum_{i=0}^m\alpha_i=k$, $\alpha_i=\bar{\alpha}_i+|q_{i_{k+n}}^{l_{k+n}}\cdots q_{i_{s_{r+1}}}^{l_{s_{r_1}+1}}|_{x_i}$, $\sum_{i=0}^m(\bar{\bar{\beta}}_i+\bar{\gamma}_i+\gamma'_i)=n$, $\bar{\bar{\beta}}_i=\bar{\beta}_i+|q_{i_{k+n}}^{l_{k+n}}\cdots\ q_{i_{s_{r+1}}}^{l_{s_{r_1}+1}}|_{y_i}$, and $\beta_i=\bar{\bar{\beta}}_i+\bar{\gamma}_i+\gamma'_i$. For the case where $\bar{s}_{r_2}>s_{r_1}$, one instead applies the isometry property $n-(\bar{s}_{r_2}-1)$ times and Hölder's inequality $k_1$ times. There are two cases to consider: $i_{\bar{s}_{r_2}+1}\neq i_{\bar{s}_{r_2}}$ and $i_{\bar{s}_{r_2}+1}=i_{\bar{s}_{r_2}}$. In the first case, it follows by Hölder's inequality that

$$\begin{aligned}
\left\|\mathbf{I}_{\eta}^{\substack{s_{r_1}\\ \bar{s}_{r_2}}}[w](t)\right\|_2^2 &\le t^{k_1}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}}\mathbf{E}\left[w^2_{q_{i_{\bar{s}_{r_2}+2}}^{l_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]t_{\bar{s}_{r_2}+1}\\
&\quad\times\int_0^{t_{\bar{s}_{r_2}+1}}\mathbf{E}\left[v^2_{i_{\bar{s}_{r_2}}+1}(t_{\bar{s}_{r_2}})v^2_{i_{\bar{s}_{r_2}}}(t_{\bar{s}_{r_2}})\right]\left\|\mathbf{I}_{\eta'}^{\substack{s_{r_1}\\ \bar{s}_{r_2}-1}}[w](t_{s_r})\right\|_2^2\\
&\quad\times\mathrm{d}t_{\bar{s}_{r_2}}\,\mathrm{d}t_{\bar{s}_{r_2}+1}\cdots\mathrm{d}t_{k+n}\\
&\le 2^{r_2-1}t^{k+r_1+r_2}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}}\mathbf{E}\left[w^2_{q_{i_{\bar{s}_{r_2}+2}}^{l_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]\\
&\quad\times\prod_{i=0}^{m}\frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\prod_{\substack{l=0\\ l\neq i_{\bar{s}_{r_2}},i_{\bar{s}_{r_2}+1}}}^{m}\frac{\bar{V}_l^{\bar{\gamma}'_l}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}'_l!}}
\end{aligned}$$

$$\times\left(\int_0^{t_{\bar{s}_{r_2}+1}} \mathbf{E}\left[v^2_{i_{\bar{s}_{r_2}}+1}(t_{\bar{s}_{r_2}})\right] \frac{\bar{V}_{i_{\bar{s}_{r_2}+1}}^{2\bar{\gamma}'_{i_{\bar{s}_{r_2}+1}}}(t_{\bar{s}_{r_2}})}{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}!} \mathrm{d}t_{\bar{s}_{r_2}}\right)^{1/2}$$

$$\times\left(\int_0^{t_{\bar{s}_{r_2}+1}} \mathbf{E}\left[v^2_{i_{\bar{s}_{r_2}}}(t_{\bar{s}_{r_2}})\right] \frac{\bar{V}_{i_{\bar{s}_{r_2}}}^{2\bar{\gamma}'_{i_{\bar{s}_{r_2}}}}(t_{\bar{s}_{r_2}})}{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}!} \mathrm{d}t_{\bar{s}_{r_2}}\right)^{1/2} \mathrm{d}t_{\bar{s}_{r_2}+1}\cdots \mathrm{d}t_{k+n}$$

$$\leq 2^{r_2-1}t^{k+r_1+r_2}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[w^2_{q_{i_{\bar{s}_{r_2}+2}}^{l_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]$$

$$\times\prod_{i=0}^m \frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}$$

$$\times\prod_{\substack{l=0\\ l\neq i_{\bar{s}_{r_2}},i_{\bar{s}_{r_2}+1}}}^m \frac{\bar{V}_l^{\bar{\gamma}_l}(t_{\bar{s}_{r_2}}+1)}{\sqrt{\bar{\gamma}_l!}}\frac{\bar{V}_{i_{\bar{s}_{r_2+1}}}^{\bar{\gamma}'_{i_{\bar{s}_{r_2}+1}}+1}(t_{\bar{s}_{r_2}+1})}{\sqrt{(\bar{\gamma}'_{i_{\bar{s}_{r_2+1}}}+1)!}}\frac{\bar{V}_{i_{\bar{s}_{r_2}}}^{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}+1}(t_{\bar{s}_{r_2}+1})}{\sqrt{(\bar{\gamma}'_{i_{\bar{s}_{r_2}}}+1)!}}$$

$$\times\,\mathrm{d}t_{\bar{s}_{r_2}+1}\cdots\mathrm{d}t_{k+n}$$

$$\leq 2^{r_2-1}t^{k+r_1+r_2}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[w^2_{q_{i_{\bar{s}_{r_2}+2}}^{l_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]$$

$$\times\prod_{i=0}^m \frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_i!}}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\mathrm{d}t_{\bar{s}_{r_2+1}}\cdots\mathrm{d}t_n,$$

where $\bar{\gamma}_i = \bar{\gamma}'_i + \delta_{i_{\bar{s}_{r_2}}} + \delta_{i(\bar{s}_{r_2}+1)}$ and $\sum_{i=0}^m \bar{\gamma}_i = 2r_2$. Thus, the remaining $(n + k_1 - (s_{r_2} + 1))$ integrals are calculated as in the case when $s_{r_1} > \bar{s}_{r_2}$, and again the $\bar{\alpha}_i$'s and $\bar{\beta}_i$'s increase instead of the $\bar{\gamma}'_i$'s. Therefore,

$$\left\|\mathbf{I}_\eta^{\substack{s_{r_1}\\ \bar{s}_{r_2}}}[w](t)\right\|_2^2 \leq 2\left(2^{r_2-1}t^{k+r_1+r_2}\prod_{i=0}^m \frac{\tilde{U}^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\bar{\bar{\beta}}_i}(t)}{\bar{\bar{\beta}}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}}\frac{B_l^{\gamma_i}(t)}{\gamma_i!}\right),$$

where $\sum_{i=0}^m \alpha_i = k$, $\alpha_i = \bar{\alpha}_i + |q_{i_{k+n}}^{l_{k+n}}\cdots q_{i_{s_{r_2}+2}}^{l_{s_{r_2}+2}}|_{x_i}$, $\sum_{i=0}^m\left(\bar{\bar{\beta}}_i + \bar{\gamma}_i + \gamma_i\right) = n$, $\bar{\bar{\beta}}_i = \bar{\beta}_i + |q_{i_{k+n}}^{l_{k+n}}\cdots q_{i_{s_{r_2}+2}}^{l_{s_{r_2}+2}}|_{y_i}$, and $\beta_i = \bar{\bar{\beta}}_i + \bar{\gamma}_i + \gamma_i$. In the second case, since $(\bar{\gamma}'_i + 2) \geq \sqrt{\bar{\gamma}'_i + 1}\sqrt{\bar{\gamma}'_i + 2}$, it follows that

$$\left\|\mathbf{I}_\eta^{\substack{s_{r_1}\\ \bar{s}_{r_2}}}[w](t)\right\|_2^2 \leq 2^{r_2-1}t^{k+r_1+r_2}\int_0^t \mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[w^2_{q_{i_{\bar{s}_{r_2}+2}}^{l_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]$$

$$\times\prod_{i=0}^m \frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\prod_{\substack{l=0\\ l\neq i_{\bar{s}_{r_2}},i_{\bar{s}_{r_2}+1}}}^m \frac{\bar{V}_l^{\bar{\gamma}'_l}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}'_l!}}$$

$$\times \int_0^{t_{\bar{s}_{r_2}+1}} \mathbf{E}\left[v^4_{i_{\bar{s}_{r_2}}}(t_{\bar{s}_{r_2}})\right] \frac{\bar{V}_{i_{\bar{s}_{r_2}}}^{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}}(t_{\bar{s}_{r_2}})}{\sqrt{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}!}} \mathrm{d}t_{\bar{s}_{r_2}} \mathrm{d}t_{\bar{s}_{r_2}+1} \cdots \mathrm{d}t_{k+n}$$

$$\leq 2^{r_2} t^{k+r_1+r_2} \int_0^t \mathbf{E}\left[w^2_{q^{l_{k+n}}_{i_{k+n}}}(t_{k+n})\right] \cdots \int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[w^2_{q^{l_{\bar{s}_{r_2}+2}}_{i_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]$$

$$\times \prod_{i=0}^{m} \frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\alpha}_i!} \frac{V_i^{\bar{\beta}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!} \frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}$$

$$\times \prod_{\substack{l=0 \\ l \neq i_{\bar{s}_{r_2}}, i_{\bar{s}_{r_2}+1}}}^{m} \frac{\bar{V}_l^{\bar{\gamma}'_l}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}'_l!}} \frac{\bar{V}_{i_{\bar{s}_{r_2}}}^{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}+2}(t_{\bar{s}_{r_2}})}{\sqrt{\bar{\gamma}'_{i_{\bar{s}_{r_2}}}!(\bar{\gamma}'_{i_{\bar{s}_{r_2}}}+2)}} \mathrm{d}t_{\bar{s}_{r_2}+1} \cdots \mathrm{d}t_{k+n}$$

$$\leq 2^{r_2} t^{k+r_1+r_2} \int_0^t \mathbf{E}\left[w^2_{q^{l_{k+n}}_{i_{k+n}}}(t_{k+n})\right] \cdots \int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[w^2_{q^{l_{\bar{s}_{r_2}+2}}_{i_{\bar{s}_{r_2}+2}}}(t_{\bar{s}_{r_2}+1})\right]$$

$$\times \prod_{i=0}^{m} \frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\alpha}_i!} \frac{V_i^{\bar{\beta}_i}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!} \frac{\bar{V}_i^{\bar{\gamma}_i}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_i!}} \frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!} \mathrm{d}t_{\bar{s}_{r_2}+1} \cdots \mathrm{d}t_n,$$

where $\bar{\gamma}_i = \bar{\gamma}'_i + 2\delta_{ii_{\bar{s}_{r_2}}}$ and $\sum_{i=0}^m \bar{\gamma}'_i = 2r_2$. Similarly, the remaining $n + k_1 - (\bar{s}_{r_2} - 1)$ integrals are calculated as in the case when $s_{r_1} > \bar{s}_{r_2}$. Thus,

$$\left\| \mathbf{I}_\eta^{\substack{s_{r_1} \\ \bar{s}_{r_2}}}[w](t) \right\|_2^2 \leq 2^{r_2} t^{k+r_1+r_2} \prod_{i=0}^{m} \frac{\tilde{U}^{\alpha_i}(t)}{\alpha_i!} \frac{V_i^{\bar{\bar{\beta}}_i}(t)}{\bar{\bar{\beta}}_i!} \frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}} \frac{B_l^{\gamma_i}(t)}{\gamma_i!},$$

where $\sum_{i=0}^m \alpha_i = k$, $\alpha_i = \bar{\alpha}_i + |q^{l_{k+n}}_{i_{k+n}} \cdots q^{l_{s_{r_2}+2}}_{i_{s_{r_2}+2}}|_{x_i}$, $\sum_{i=0}^m (\bar{\bar{\beta}}_i + \bar{\gamma}_i + \gamma_i) = n$, $\beta_i = \bar{\bar{\beta}}_i + \bar{\gamma}_i + \gamma_i$ and $\bar{\bar{\beta}}_i = \bar{\beta}_i + |q^{l_{k+n}}_{i_{k+n}} \cdots q^{l_{s_{r_2}+2}}_{i_{s_{r_2}+2}}|_{y_i}$. This completes the proof. □

Theorems 3.2, 3.3 and 3.4 together with the binomial theorem are used below to calculate an $L_2$ upper bound for $E_\eta[w]$.

THEOREM 3.5. Let $\eta \in X^k Y^n$ and $w \in \mathcal{UV}^m[0, T]$ be arbitrary. Then for a fixed $t \in [0, T]$

$$\left\| E_\eta[w](t) \right\|_2 < \frac{(R\sqrt{t})^k (\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\alpha!)^{1/2}(\beta!)^{1/4}},$$

where $\alpha! \triangleq \alpha_0! \cdots \alpha_m!$, $\beta! \triangleq \beta_0! \cdots \beta_m!$ and $\max\{\|u\|_{L_2}, \|v\|_{L_2}, \|v_0\|_{L_2}, \|v\|_{L_4}\} \leq R$.

*Proof*. Let $\#(\cdot)$ denote the cardinality of a set. Note that $\#(\bar{A}_{nr_2}) \leq \binom{n - r_2}{r_2} < \binom{n}{r_2}$ and $\#(A_{nr_1}^{\bar{s}_{r_2}}) = \binom{n - 2r_2}{r_1} \leq \binom{n}{r_1}$. Using the triangle inequality, (10), Theorem 3.4, and the binomial theorem, observe that

$$
\begin{aligned}
\|E_\eta[w](t)\|_2 &\le \sum_{r_1=0,r_2=0}^{n,\lfloor n/2\rfloor} \frac{1}{2^{r_1}2^{r_2}} \sum_{\substack{\mathbf{s}_{r_1}\in A_{nr_1}^{\bar{\mathbf{s}}_{r_2}}\\ \bar{\mathbf{s}}_{r_2}\in \bar{A}_{nr_2}}} \left\| \mathbf{I}_\eta^{\substack{\mathbf{s}_{r_1}\\ \bar{\mathbf{s}}_{r_2}}}[w](t)\right\|_2 \\
&\le \left(\frac{R^{k+n}}{(\alpha!)^{1/2}}\right) \sum_{r_1=0,r_2=0}^{n,\lfloor n/2\rfloor} \frac{t^{(k+r_1+r_2)/2}}{2^{r_1}2^{r_2/2}} \sum_{\substack{\mathbf{s}_{r_1}\in A_{nr_1}^{\bar{\mathbf{s}}_{r_2}}\\ \bar{\mathbf{s}}_{r_2}\in \bar{A}_{nr_2}}} \prod_{i=0}^{m} \frac{1}{(\bar\beta_i!)^{1/2}(\bar\gamma_i!)^{1/4}(\gamma_i!)^{1/2}} \\
&\le \left(\frac{R^{k+n}t^{k/2}}{(\alpha!)^{1/2}(\beta!)^{1/4}}\right) \sum_{r_1=0,r_2=0}^{n,\lfloor n/2\rfloor} \frac{t^{(r_1+r_2)/2}}{2^{r_1}2^{r_2/2}} \sum_{\substack{\mathbf{s}_{r_1}\in A_{nr_1}^{\bar{\mathbf{s}}_{r_2}}\\ \bar{\mathbf{s}}_{r_2}\in \bar{A}_{nr_2}}} \prod_{i=0}^{m} \frac{\beta!}{\bar\beta_i!\bar\gamma_i!\gamma_i!} \\
&\le \left(\frac{R^{k+n}t^{k/2}}{(\alpha!)^{1/2}(\beta!)^{1/4}}\right) 3^n \sum_{r_1=0}^{n} \frac{t^{r_1/2}}{2^{r_1}} \binom{n}{r_1} \sum_{r_2=0}^{n} \frac{t^{r_2/2}}{2^{r_2/2}} \binom{n}{r_2} \\
&\le \frac{(R\sqrt{t})^k(3\sqrt{2}R(\sqrt{t}+2)(\sqrt{t}+\sqrt{2}))^n}{4^n(\alpha!)^{1/2}(\beta!)^{1/4}} < \frac{(R\sqrt{t})^k(\sqrt{2}R(\sqrt{t}+2))^{2n}}{(\alpha!)^{1/2}(\beta!)^{1/4}}.
\end{aligned}
$$

□

## 4. Global convergence of Fliess operators

For a series $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ and any $w \in \mathcal{UV}^m[0,T]$, the general goal of this section is to show that $F_c[w]$ defines a random process over an arbitrarily large but finite interval of time provided that $c$ is a globally convergent series. Since stochastic integrals are involved, it is appropriate to consider convergence in the mean square sense. Consider the following definition.

Definition 4.1 [22]. For a fixed $t \in [0,T]$, the series $F_c[w](t)$ in (9) is said to be a *Cauchy series* if for any $\varepsilon > 0$ there exists an $N > 0$ such that

$$
\left\| \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta\in X^kY^{j-k}} (c,\eta)E_\eta[w](t) \right\|_2 < \epsilon,
$$

when $N_2 > N_1 > N$.

Two $m$-dimensional random vectors $x, y$ are called *equivalent* if $P(\omega \in \Omega : x(\omega) = y(\omega)) = 1$. It is well known that $L_2^m(\Omega, \mathcal{F}, P)$ with its usual inner product is a Hilbert space modulo this equivalence relation. $F_c[w]$ will be shown to be convergent in the mean square sense by showing that it defines a Cauchy series.

Theorem 4.2. Let $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ be a globally convergent series, i.e., there exist $K, M > 0$ such that $|(c,\eta)| \le KM^{|\eta|}, \forall \eta \in XY^*$. Then for any finite $T > 0$ and $w \in \mathcal{UV}^m[0,T]$, the series (9) converges absolutely in the mean square sense to a well-defined random vector $y(t) = F_c[w](t)$, $t \in [0,T]$.

*Proof*. For simplicity, assume that $\ell = 1$. Pick a $t \in [0, T]$ and any $w \in \mathcal{UV}^m[0, T]$. Let $R = \max\{\|u\|_{L_2}, \|v\|_{L_2}, \|b\|_{L_2}, \|v\|_{L_4}, T\}$. Define

$$a_{k,n}(t) = \sum_{\eta \in X^k Y^n} (c, \eta) E_\eta[w](t). \tag{17}$$

Note that the language $L_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \{\eta \in X^k Y^n : |\eta|_{x_i} = \alpha_i, |\eta|_{y_i} = \beta_i, i = 0, \ldots, m\}$ consists of $(k+n)!/(\alpha!\beta!)$ words. Applying Theorem 3.5,

$$\begin{aligned}\|a_{k,n}(t)\|_2 &\leq \sum_{\eta \in X^k Y^n} |(c, \eta)| \left\| E_\eta[w](t) \right\|_2 \\ &\leq KM^{n+k} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{(R\sqrt{t})^k (\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\alpha!)^{1/2}(\beta!)^{1/4}} \frac{(k+n)!}{\alpha!\beta!}.\end{aligned}$$

Without loss of generality, it is assumed that $R \geq 1$. If $R' \triangleq 4R(R+4)$, then from the multinomial theorem,

$$\begin{aligned}\|a_{k,n}(t)\|_2 &\leq K(MR')^{k+n} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{(k+n)!}{(\alpha!)^{3/2}(\beta!)^{5/4}} \\ &\leq K(2MR')^{k+n} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{k!n!}{(\alpha!)^{3/2}(\beta!)^{5/4}} \\ &\leq \frac{K(2MR')^{k+n}}{(k!)^{1/2}(n!)^{1/4}} \sum_{\|\alpha\|=k} \frac{(k!)^{3/2}}{(\alpha!)^{3/2}} \sum_{\|\beta\|=n} \frac{(n!)^{5/4}}{(\beta!)^{5/4}} \\ &\leq \frac{K(2MR')^{k+n}}{(k!)^{1/2}(n!)^{1/4}} \left(\sum_{\|\alpha\|=k} \frac{k!}{\alpha!}\right)^2 \left(\sum_{\|\beta\|=n} \frac{n!}{\beta!}\right)^2 \\ &\leq \frac{K(2MR'(m+1)^2)^{k+n}}{(k!)^{1/2}(n!)^{1/4}}.\end{aligned} \tag{18}$$

Since $|\eta| = |\eta|_X + |\eta|_Y = k + n \triangleq j$, it follows immediately from the triangle inequality that

$$\left\| \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta \in X^k Y^{j-k}} (c, \eta) E_\eta[w](t) \right\|_2 \leq \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \|a_{k,j-k}(t)\|_2$$

for any $N_2 > N_1 \geq 0$. Now for any $\epsilon > 0$ there exists an $N > 0$ such that from equation (18) it follows that

$$\begin{aligned}\sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \left\|a_{k,j-k}(t)\right\|_2 &\leq \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta \in X^k Y^{j-k}} \frac{K(2MR'(m+1)^2)^j}{(k!)^{1/2}((j-k)!)^{1/4}} \\ &\leq K \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(2MR'(m+1)^2)^k (2MR'(m+1)^2)^{j-k}}{(k!)^{1/2}((j-k)!)^{1/4}} \\ &= K \sum_{k=0}^{\infty} \frac{(M'')^k}{(k!)^{1/2}} \sum_{n=0}^{\infty} \frac{(M'')^n}{(n!)^{1/4}} < \epsilon\end{aligned}$$

for all $N_2 > N_1 > N$, where $M'' \triangleq 2MR'(m+1)^2$. Note that $(M'')^k/(k!)^{1/2}$ and

$(M'')^n/(n!)^{1/4}$ are the $k$th and $n$th terms of an absolutely convergent series, respectively. Hence, by the ratio test, the series (9) is Cauchy. This implies that $\sum_{\eta \in XY^*} |(c,\eta)| \|E_\eta[w]\|_2 < \infty$, and thus, it is an absolutely convergent series in the mean-square sense. □

*Example 4.3*. Consider the following integral system

$$\begin{aligned} z(t) &= 1 - a\int_0^t z(s)\mathrm{d}s + b\left(\int_0^t z(t)u(s)\mathrm{d}s + \int_0^t v(s)\mathrm{d}W(s)\right), \\ y(t) &= z(t), \end{aligned} \tag{19}$$

where $a, b \in \mathbb{R}$, and $w \in \mathcal{UV}[0,T]$. Assume there exists a series $c \in \mathbb{R}\langle\langle XY \rangle\rangle$ such that $y(t) = F_c[w](t)$. Then (19) is equivalent to

$$c - 1 = -ax_0c + b(x_1 + y_1)c.$$

Solving for $c$ gives

$$c = (1 + ax_0 - b(x_1 + y_1))^{-1}.$$

Observe that $c$ is a rational series, and therefore, it is easy to prove that $c$ is globally convergent. From Theorem 4.2, it follows that $y(t) = F_c[w](t)$ converges to a well-defined random variable for all $t \in [0,T]$. The operator $F_c$ can also be computed directly as

$$F_c[w](t) = F_{(1+ax_0-b(x_1+y_1))^{-1}}[w](t) = \sum_{k=0}^{\infty} F_{(-ax_0+b(x_1+y_1))^k}[w](t).$$

Let $⧢ : \mathbb{R}\langle\langle XY \rangle\rangle \times \mathbb{R}\langle\langle XY \rangle\rangle \to \mathbb{R}\langle\langle XY \rangle\rangle$ represent the *shuffle product* [3,13]. As a consequence of the integration by parts formula for the Stratonovich integral, the shuffle product and the scalar product $E_{\eta_1}[w](t)E_{\eta_2}[w](t)$ when $\eta_1, \eta_2 \in XY^*$ are related by

$$E_{\eta_1}[u](t)E_{\eta_2}[u](t) = E_{\eta_1 ⧢ \eta_2}[u](t). \tag{20}$$

Riccomagno in [22] proved the identity

$$(ax_0 + by_0)^{⧢ k} \triangleq \underbrace{(ax_0 + by_0) ⧢ \cdots ⧢ (ax_0 + by_0)}_{k \text{ times}} = k!(ax_0 + by_0)^k,$$

for any $a, b \in \mathbb{R}$. Using the minor extension,

$$(ax_0 + bx_i + dy_i)^{⧢ k} = k!(ax_0 + bx_i + dy_i)^k$$

for $a, b, d \in \mathbb{R}$, observe that

$$\begin{aligned} F_c[w](t) &= \sum_{k=0}^{\infty} \frac{1}{k!} F_{(-ax_0+b(x_1+y_1))^{⧢ k}}[w](t) = \sum_{k=0}^{\infty} \frac{1}{k!} F^k_{(-ax_0+b(x_1+y_1))}[w](t) \\ &= \sum_{k=0}^{\infty} \frac{(-at + bw(t))^k}{k!} = \mathrm{e}^{-at+bw(t)}. \end{aligned}$$

Clearly, $F_c[w](t)$ is a well-defined random variable for all $t \in [0,T]$, where $T$ can be arbitrarily large but finite. This is consistent with Theorem 4.2. □

## 5. Local convergence of Fliess operators

In this section, a stochastic notion of local convergence is introduced using the stopping time concept introduced in Section 2. Then a corresponding sufficient condition for local convergence is presented.

DEFINITION 5.1. Let $X_i(t) = \oint_0^t v_i(s)\mathrm{d}W(s)$, where $v_i \in \mathscr{I}$ and $i = \{0, 1, \ldots, m\}$. The set $\mathcal{UV}^m[0, \tau_R]$ is defined as the set of processes $w \in \mathcal{UV}^m[0, T]$ stopped at

$$\tau_R = \min_{i \in \{0,1,\cdots,m\}} \inf \{t > 0 : |X_i(t)| = R\} \wedge T. \tag{21}$$

The main result regarding convergence of Fliess operators over a time interval of random length is given in the theorem below.

THEOREM 5.2. Let $c \in \mathbb{R}^\ell \langle\langle XY \rangle\rangle$ be a locally convergent series, i.e., there exist $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!, \forall \eta \in XY^*$. Then for a sufficiently small $R > 0$ and any random process $w \in \mathcal{UV}^m[0, \tau_R]$ with $\tau_R$ defined as in (21), it follows that the series

$$F_c[w](t) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{\eta \in X^k Y^{j-k}} (c, \eta) E_\eta[w](t) \tag{22}$$

converges in the mean square sense to a random vector $y(t)$ for all $t \in [0, \tau_R]$. Here $R$ is called the *radius of* convergence and $[0, \tau_R]$ the *interval* of convergence.

Note that in (22) there is an implied order of summation over $XY^*$. Thus, the current proof for the convergence of $F_c$ is strictly speaking addressing *conditional* local convergence.

Before presenting the proof of Theorem 5.2, some facts and terminology are required. For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ define the polynomials $p_{\boldsymbol{\alpha}} = x_0^{\alpha_0} ⧢ \cdots ⧢ x_m^{\alpha_m}$ and $p_{\boldsymbol{\beta}} = y_0^{\beta_0} ⧢ \cdots ⧢ y_m^{\beta_m}$, respectively. The next lemma shows that the characteristic series of the language $X^k Y^n$ can be decomposed in terms of $p_{\boldsymbol{\alpha}}$'s and $p_{\boldsymbol{\beta}}$'s.

LEMMA 5.3. [3] The characteristic series of the language $X^k Y^n$ is

$$\boldsymbol{X^k Y^n} \triangleq \sum_{\eta \in X^k Y^n} \eta = \sum_{\|\alpha\|=k, \|\beta\|=n} p_{\boldsymbol{\alpha}} ⧢ p_{\boldsymbol{\beta}}.$$

For fixed $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$, $w \in \mathcal{UV}^m[0, T]$ and $t \in [0, T]$, define the following sum of iterated integrals

$$\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t) = E_{p_{\boldsymbol{\alpha}} ⧢ p_{\boldsymbol{\beta}}}[w](t) = E_{p_{\boldsymbol{\alpha}}}[w](t) E_{p_{\boldsymbol{\beta}}}[w](t).$$

The importance of $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w]$ comes from the fact that, using the commutativity of the shuffle product and (20), the Lebesgue integrals and Stratonovich integrals are separable, and thus, an $L_2$ upper bound for $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)$ can be obtained by calculating individual $L_2$ upper bounds for the random variables $E_{p_{\boldsymbol{\alpha}}}[w](t)$ and $E_{p_{\boldsymbol{\beta}}}[w](t)$. Then from the independence assumptions in Definition 2.3,

$$\|\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)\|_2^2 = \|E_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \|E_{p_{\boldsymbol{\beta}}}[w](t)\|_2^2. \tag{23}$$

Now consider the next lemma.

LEMMA 5.4 [3,4]. Let $u$ be the drift input of $w \in \mathcal{UV}^m[0,T]$. Then for $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_m) \in \mathbb{N}^{m+1}$ and any real numbers $0 \le s < t \le T$

$$|E_{p_{\boldsymbol{\alpha}}}[w](t)| \le \prod_{i=0}^{m} \frac{U_i^{\alpha_i}(t)}{\alpha_i!}$$

and

$$\mathbf{E}\left[\prod_{i=0}^{m} (U_i(t) - U_i(s))^{\alpha_i}\right] \le \prod_{i=0}^{m} \bar{U}_i^{\alpha_i}(t),$$

where $U_i(t) \triangleq \int_0^t |u_i(s)| \mathrm{d}s$ and $\bar{U}_i(t) \triangleq \int_0^t \mathbf{E}[|u_i(s)|] \mathrm{d}s$.

Using the above lemma, an upper bound for $\|E_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2$ can be easily calculated as

$$\|E_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \le \prod_{i=0}^{m} \frac{\bar{U}_i^{2\alpha_i}(t)}{(\alpha_i!)^2} \le \frac{R^{2k}}{(\alpha!)^2}, \tag{24}$$

where $\|u\|_{L_1} \le R$. Since $E_{p_{\boldsymbol{\beta}}}[w](t)$ is comprised exclusively of Stratonovich integrals, a different approach has to be employed to determine a bound for $\|E_{p_{\boldsymbol{\beta}}}[w](t)\|_2^2$ as described below.

*Proof of Theorem 5.2*. Without loss of generality, it is assumed that $\ell = 1$. Select $R$ as in (24) and pick any $w \in \mathcal{UV}^m[0, \tau_R]$ and $t \in [0, \tau_R]$. By definition, $\tau_R \le T$, thus, $R$ also bounds $\|u\|_{L_1}$ when computed over $[0, \tau_R]$. Furthermore, recall that $\tau_R$ is the first time in which the fastest $X_i(t) = ⨏_0^t v_i(s) \mathrm{d}W(s)$ hits the barrier $(-R, R)$, and, therefore, $|X_i(t \wedge \tau_R)| \le R$, $i = 0, \ldots, m$. Since every $X_i$ is an a.s. continuous process, and the absolute value function is a continuous function, one can always choose, without loss of generality, a continuous version of the process $X_i$. Then by Theorem 2.11 the random variable $\tau_R$ is a positive stopping time. Thus, the stopped process $X_i^{\tau_R}$ is a well-defined $L_2$-bounded, a.s. continuous and adapted $L_2$-Itô process. Now, using (20) observe

$$E_{p_{\boldsymbol{\beta}}}[w](t) = \prod_{i=0}^{m} E_{y_i^{\beta_i}}[w](t) = \prod_{i=0}^{m} \frac{E_{y_i^{\sqcup\sqcup \beta_i}}[w](t)}{\beta_i!} = \prod_{i=0}^{m} \frac{(E_{y_i}[w](t))^{\beta_i}}{\beta_i!}.$$

Given that $E_{y_i}[w](t) = ⨏_0^t v_i(s) \mathrm{d}W(s)$, the $L_2$-norm for $E_{p_{\boldsymbol{\beta}}}[w](t)$ truncated at the stopping time $\tau_R$ is

$$\|E_{p_{\boldsymbol{\beta}}}[w](t \wedge \tau_R)\|_2^2 = \frac{1}{(\beta!)^2} \mathbf{E}\left[\prod_{i=0}^{m} \left(\left| ⨏_0^{t\wedge\tau_R} v_i(s) \mathrm{d}W(s)\right|\right)^{2\beta_i}\right] \le \frac{R^{2n}}{(\beta!)^2}.$$

Define

$$a_{k,n}(t) = KM^{k+n}(k+n)! \sum_{\|\alpha\|=k, \|\beta\|=n} \mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t).$$

Unlike (17), here $a_{k,n}$ is defined in terms of $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, which implies a particular order when computing its norm. Using (23), (24) and the multinomial theorem, the following bound is obtained

$$\|a_{k,n}(t \wedge \tau_R)\|_2 \le KM^{k+n}(k+n)! \sum_{\|\alpha\|=k,\|\beta\|=n} \frac{R^k}{\alpha!}\frac{R^n}{\beta!}$$

$$= K(MR)^{k+n}(k+n)!\frac{(m+1)^k}{k!}\frac{(m+1)^n}{n!} = K(MR(m+1))^{k+n}\binom{k+n}{n}.$$

To show that (22) is mean square convergent, it is sufficient to show that it is a Cauchy series. In light of Lemma 5.3 and the triangle inequality,

$$\left\| \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\sum_{\eta\in X^kY^{j-k}}(c,\eta)E_\eta[w](t)\right\|_2 \le \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\|a_{k,j-k}(t)\|_2$$

for $N_2 > N_1 \ge 0$. For any $\epsilon > 0$ there exists an $N > 0$ such that

$$\sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\|a_{k,j-k}(t\wedge\tau_R)\|_2 \le \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}K(MR(m+1))^j\frac{j!}{k!(j-k)!}$$

$$= \sum_{j=N_1}^{N_2}K(MR(m+1))^j\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}$$

$$= \sum_{j=N_1}^{N_2}K(2MR(m+1))^j < \epsilon,$$

provided that $2MR(m+1) < 1$ and $N_2 > N_1 > N$. Note that if

$$R < \frac{1}{2M(m+1)} \tag{25}$$

then the series (22) is Cauchy on $[0, \tau_R]$, and thus, the theorem is proved. □

One can remove the conditionality in Theorem 5.2 if an extra condition on the generating series of a Fliess operator is introduced.

Definition 5.5 [6]. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ and define

$$\mathbf{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \{\eta \in XY^* : |\eta|_{x_i} = \alpha_i, |\eta|_{y_i} = \beta_i, i = 0, \ldots, m\}.$$

A series $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ is called *exchangeable* if for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ the words in $\boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ have the same coefficient.

Corollary 5.6. Let $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ be an exchangeable and locally convergent series. Then for a sufficiently small $R > 0$ and any random process $w \in \mathcal{UV}^m[0, \tau_R]$ with $\tau_R$ defined as in (21), the series

$$F_c[w](t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\eta\in X^kY^{j-k}}(c,\eta)E_\eta[w](t)$$

converges absolutely in the mean square sense for all $t \in [0, \tau_R]$.

*Proof*. Since $c$ is exchangeable, one can group all the iterated integrals associated with words having the same $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, i.e.,

$$\begin{aligned} F_c[w](t) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\eta\in X^k Y^{j-k}}(c,\eta)E_\eta[w](t) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t), \end{aligned} \tag{26}$$

where $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} = (c,\eta)$ for all $\eta \in \boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Here the series (26) is Cauchy if for any $\epsilon > 0$ there exists an $N > 0$ such that

$$\left\| \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)\right\|_2 < \epsilon,$$

when $N_2 > N_1 > N$. Using (20), $F_c[w]$ can be uniquely written as

$$F_c[w](t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\prod_{i=0}^{m}\frac{E_{x_i}^{\alpha_i}[w](t)}{\alpha_i!}\frac{E_{y_i}^{\beta_i}[w](t)}{\beta_i!}. \tag{27}$$

Therefore, following the procedure in the proof of Theorem 5.2, (27) is Cauchy, and thus,

$$\|F_c[w](t)\|_2 \leq \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} |c_{\boldsymbol{\alpha},\boldsymbol{\beta}}|\prod_{i=0}^{m}\left\|\frac{E_{x_i}^{\alpha_i}[w](t)}{\alpha_i!}\frac{E_{y_i}^{\beta_i}[w](t)}{\beta_i!}\right\|_2$$

is finite for any $t \in [0, \tau_R]$. This completes the proof. □

*Example 5.7*. Consider the formal power series

$$c = \sum_{k=0}^{\infty} k!(x_1 + y_1)^k = \sum_{k=0}^{\infty}(x_1 + y_1)^{⧢ k}.$$

This series is both locally convergent and exchangeable. Observe

$$\begin{aligned} y(t) = F_c[w](t) &= \sum_{k=0}^{\infty} E_{(x_1+y_1)^{⧢ k}}[w](t) \\ &= \sum_{k=0}^{\infty} E^k_{(x_1+y_1)}[w](t) = (1 - E_{(x_1+y_1)}[w](t))^{-1} \end{aligned} \tag{28}$$

for all $t \in [0, \tau_R]$, where $\tau_R$ is defined in (21) and $R < 1/4$ from (25). Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{(x_1+y_1)}[w](t) = \frac{\mathrm{d}}{\mathrm{d}t}w(t) = \dot{w}(t).$$

Here $\dot{w}$ is the formal derivative of $w \in \mathcal{UV}[0, T]$ for a fixed $T > 0$, i.e., if

$$w(t) = \int_0^t u(s)\mathrm{d}s + \oint_0^t v(s)\mathrm{d}W(s),$$

then $\dot{w}(t) = u(t) + v(t)\bar{w}(t)$, where $\bar{w}$ denotes white Gaussian noise and $t \leq T$. Given that the Stratonovich integral obeys the usual differential chain rule, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_c[w](t) = \frac{\mathrm{d}}{\mathrm{d}t}(1 - E_{(x_1+y_1)}[w])^{-1} = (1 - E_{(x_1+y_1)}[w])^{-2}\,\dot{w} = (F_c[w](t))^2\,\dot{w}.$$

Thus, $y = F_c[w]$ has a state-space realization

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}z(t) &= z^2(t)\dot{w}, \quad z(0) = 1, \\ y(t) &= z(t). \end{aligned}$$

A sample path of the output for this system when the input $\dot{w} = \bar{w}$ is shown in Figure 3. The corresponding sample value of the random variable $\tau_R = 1.626$. However, from (28) it also follows directly that $F_c[\bar{w}](t)$ converges on $[0, \tau_{\bar{R}}]$ with $\tau_{\bar{R}} = \inf\{t > 0 : |W(t)| = \bar{R}\} \wedge T$ and $\bar{R} < 1$. Recall that $R$ as defined in the proof of Theorem 5.2 is the maximum of certain norm bounds for $u$ and $v$ individually. On the other hand, $\bar{R}$ involves a bound on the sum of the integrals of $u$ and $v$. Here the sample value of the random variable $\tau_{\bar{R}} = 2.386$. Observe that $\bar{R}$ can always be chosen to be larger than $R$ since $R < 1/4$ and, therefore, the interval of convergence $[0, \tau_{\bar{R}}]$ is almost surely longer than $[0, \tau_R]$. In other words, the generic interval of convergence found in Theorem 5.2 can often be improved upon in specific cases. □

*Example 5.8*. Consider the following state-space system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}z(t) &= z^3(t)\dot{w}, \quad z(0) = 1, \\ y(t) &= z(t), \end{aligned} \tag{29}$$

where $w \in \mathcal{UV}[0, \tau_R]$ for a sufficiently small $R$. Strictly speaking, (29) is only meaningful in its integral form

$$z(t) = \int_0^t z^3(s)u(s)\mathrm{d}s + \mathcal{S}\!\!\int_0^t z^3(s)v(s)\mathrm{d}W(s). \tag{30}$$

Given that the shuffle product represents the product of iterated integrals, the $n$th power of $z(t)$ can be uniquely associated with the series $c^{\sqcup\!\sqcup n}$ such that (30) can be written

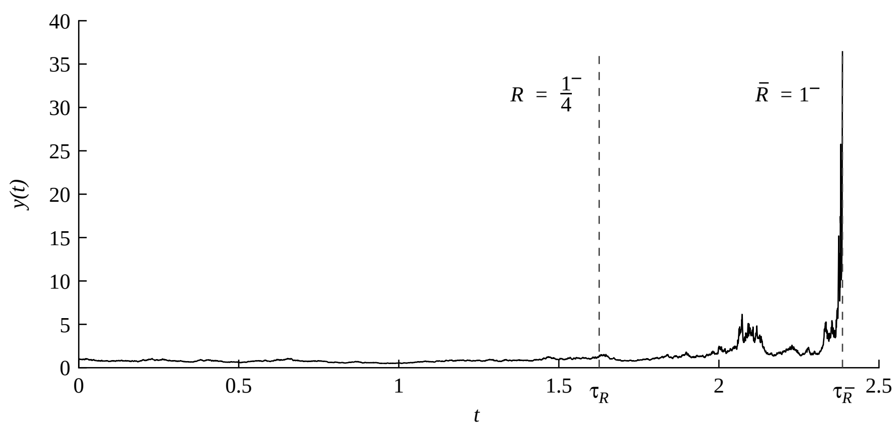


Figure 3. Sample path of the output in Example 5.7.

algebraically as

$$c = (x_1 + y_1)c^{\sqcup\!\sqcup 3}.$$

Thus, $c$ can be calculated as $c = \sum_{k=0}^{\infty} c_k$, where

$$\begin{aligned} c_0 &= 1 \\ c_k &= (x_1 + y_1) \sum_{\substack{i_1+i_2+i_3=k \\ i_1,i_2,i_3<k}} c_{i_1} \sqcup\!\sqcup\, c_{i_2} \sqcup\!\sqcup\, c_{i_3} \end{aligned}$$

[12,20,22]. The next three $c_k$'s are given below:

$$\begin{aligned} c_1 &= (x_1 + y_1)(c_0 \sqcup\!\sqcup\, c_0 \sqcup\!\sqcup\, c_0) = (x_1 + y_1) \\ c_2 &= (x_1 + y_1)(c_0 \sqcup\!\sqcup\, c_0 \sqcup\!\sqcup\, c_1 + c_0 \sqcup\!\sqcup\, c_1 \sqcup\!\sqcup\, c_0 + c_1 \sqcup\!\sqcup\, c_0 \sqcup\!\sqcup\, c_0) \\ &= (x_1 + y_1)((x_1 + y_1) + (x_1 + y_1) + (x_1 + y_1)) \\ &= 3(x_1 + y_1)^2 \\ c_3 &= (x_1 + y_1)(c_0 \sqcup\!\sqcup\, c_0 \sqcup\!\sqcup\, c_2 + c_0 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup\, c_0 + c_2 \sqcup\!\sqcup\, c_0 \sqcup\!\sqcup\, c_0 \\ &\quad + c_0 \sqcup\!\sqcup\, c_1 \sqcup\!\sqcup\, c_1 + c_1 \sqcup\!\sqcup\, c_0 \sqcup\!\sqcup\, c_1 + c_1 \sqcup\!\sqcup\, c_1 \sqcup\!\sqcup\, c_0) \\ &= (x_1 + y_1)(9(x_1 + y_1)^2 + 3(x_1 + y_1)^{\sqcup\!\sqcup 2}) \\ &= 15(x_1 + y_1)^3. \end{aligned}$$

Inductively, one can show that $c_k = (2k-1)!!(x_1 + y_1)^k$, $k \geq 0$. (The double factorial is defined as $n!! = n(n-2)\cdots 1$ when $n$ is odd and likewise when $n$ is even.) It is easy to verify that

$$(2k-1)!! = \frac{(2k)!}{2^k k!} = \frac{(2k)!}{2^k k! k!} k! = \frac{C_k^{2k}}{2^k} k! \leq 2^k k!.$$

Therefore, $c$ is locally convergent and exchangeable, and Theorem 5.2 and Corollary 5.6 are applicable. This implies from (25) that $y(t) = F_c[w](t)$ converges for all $t \in [0, \tau_R]$ when $R < 1/(2M(m+1)) = 1/8$. However, applying properties of the shuffle product gives

$$c = \sum_{k=0}^{\infty} (2k+1)!!(x_1 + y_1)^k = \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k} (x_1 + y_1)^{\sqcup\!\sqcup k}.$$

Observe that the series expansion of

$$\frac{1}{\sqrt{1-2z}} = 1 + z + \frac{3}{2}z^2 + \frac{5}{2}z^3 + \frac{35}{8}z^4 + O(z^5) = \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k} z^k.$$

Hence, $c$ is the generating series of the input–output operator

$$F_c[w](t) = \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k} \underbrace{E_{(x_1+y_1)}^k[w](t)}_{w^k(t)} = \frac{1}{\sqrt{1-2w(t)}}, \tag{31}$$

where $t \in [0, \tau_{\bar{R}}]$ with $\tau_{\bar{R}} = \inf\{t > 0 : |2w(t)| = \bar{R}\}$ and $\bar{R} < 1$. Let $t'$ be a sample value

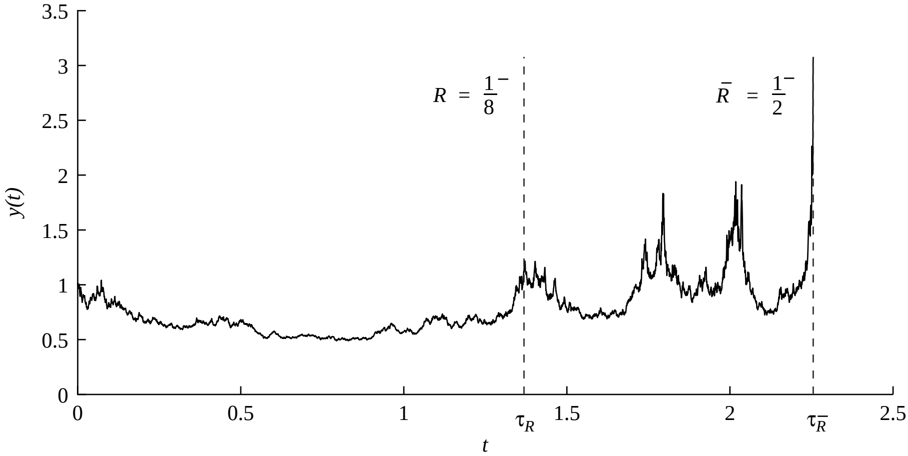


Figure 4. Sample path of the output in Example 5.8.

of $\tau_R$. Then here the radii of convergence $R$ and $\bar{R}$ can be related as follows:

$$\bar{R} = |2w(t')| \leq 2\left(\left|\int_0^{t'} u(s)\mathrm{d}s\right| + \left|\oint_0^{t'} v(s)\mathrm{d}W(s)\right|\right) \leq 2(R+R) \leq \frac{1}{2}.$$

Hence, it is evident that one can always choose the radius of convergence $\bar{R}$ to be larger than $R$. This implies that Theorem 5.2 almost surely gives a more conservative interval of convergence than the one found directly from (31). A sample path of the output for this system confirms this when $\dot{w} = \bar{w}$ as shown in Figure 4. □

## 6. Conclusions and future research

In this paper, a definition for a Fliess operator driven by a class of $L_2$-Itô stochastic processes was presented. It was then shown that a Fliess operator with inputs from the particular subclass $\mathcal{UV}^m[0,T]$, $T > 0$ is absolutely globally convergent when its generating series is globally convergent. Furthermore, a Fliess operator over $\mathcal{UV}^m[0,T]$ was shown to be conditionally locally convergent over a finite interval of time having random length when its generating series is locally convergent. Finally, if the generating series is also exchangeable, then the operator is absolutely locally convergent.

Many directions for future research are possible. First, it is likely that the exchangeability condition for the local convergence case can be relaxed somehow. In [12,20,22], Volterra series analysis was developed for white Gaussian noise inputs via Fliess operator methods. It is plausible that the convergence of Volterra series with $L_2$-Itô input processes could be addressed using the tools presented in this paper. Also, an extension of the material presented here seems possible for Poisson input processes, which are useful in the analysis of switched dynamical systems.

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