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# On the Absolute Global Convergence of Fliess Operators Driven by $L_2$-Itô Processes†

Luis A. Duffaut Espinosa   W. Steven Gray   Oscar R. González

***Abstract*— Fliess operators have been an object of study in connection with nonlinear systems acting on deterministic inputs since the early 1970's. They describe a broad class of nonlinear input-output maps using a type of functional series expansion. But in most applications, a system's inputs have noise components. It has been shown that the notion of a Fliess operator can be generalized to admit a class of $L_2$-Itô stochastic input processes, and that they converge conditionally over an arbitrarily large but finite interval of time. The purpose of this paper is to prove the more difficult proposition that Fliess operators driven by $L_2$-Itô stochastic processes converge absolutely under the same conditions.**

## I. Introduction

Fliess operators have been a frequent object of study in the theory of nonlinear input-output systems [4]–[9]. They are described by an infinite summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and $X^*$ the free monoid comprised of all words over $X$ (including the empty word $\emptyset$) under the catenation product. A formal power series in $X$ is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For a measurable function $u : [a, b] \to \mathbb{R}^m$ define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \le i \le m\}$, where $\|u_i\|_{L_p}$ is the usual $L_p$-norm for a measurable real-valued component function $u_i$. Define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0+T] \to \mathcal{C}[t_0, t_0+T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\eta'}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$. The input-output operator corresponding to $c$ is then

$$F_c[u](t) \triangleq \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t),$$

which is called a *Fliess operator*. If the generating series $c$ satisfies the Cauchy growth condition $|(c, \eta)| \le KM^{|\eta|}|\eta|!$ for all $\eta \in X^*$, where $|\eta|$ denotes the number of symbols in $\eta$ and $K, M > 0$, then $F_c[u]$ converges absolutely on $[t_0, t_0 + T]$ if $T$ and $\|u\|_{L_p}$ are sufficiently small [9].

Noise is usually considered to be present in real system inputs. In the most simple setting, it can be modeled by a Wiener process. From the Fliess operator theory point of view, various approaches for Wiener process inputs have been developed [1], [4], [5], [7], [13]. Unfortunately, such inputs are not suitable in certain contexts, such as those involving system interconnection [2]. For example, in order to cascade two systems, the output of the first system needs to be a Wiener process, which is generally not the case. For this reason, a larger set of admissible inputs was defined in [2], [3], and the notion of a Fliess operator was generalized to admit a class of $L_2$-Itô stochastic input processes. This class of processes was chosen since a large number of interesting and important phenomena can be described by Itô processes. Specifically, it was shown that for any such input process $w$, $F_c[w]$ is *conditionally globally* mean square convergent if $c$ is a *globally convergent series*, i.e., it satisfies the more restrictive growth bound $|(c, \eta)| \le KM^{|\eta|}$. The main objective of this paper is to prove the more difficult proposition that this is a sufficient condition for *absolute global* convergence of $F_c[w]$ in the mean square sense.

The paper is organized as follows. Section II introduces the basic stochastic tools and definitions used throughout the paper. In Section III, $L_2$ bounds for Itô and Stratonovich stochastic iterated integrals are developed. Section IV presents the proof of absolute global convergence for Fliess operators.

## II. Preliminaries

Consider the one-dimensional Wiener process, $W(t)$, defined over a complete probability space $(\Omega, \mathcal{F}, P)$. For a predictable function $u : \Omega \times [t_0, t_0 + T] \to \mathbb{R}^m$ define $\|u\|_{L_2} = \max\{\|u_i\|_{L_2} : 1 \le i \le m\}$, where $\|\cdot\|_{L_2}$ is the usual norm on $L_2(\Omega \times [t_0, t_0+T], \mathcal{P}, P \otimes \lambda)$, the set of all predictable functions defined on $\mathcal{T} = [t_0, t_0 + T]$ having finite $\|\cdot\|_{L_2}$-norm, $\mathcal{P}$ is the predictable algebra, and $\lambda$ is the Lebesgue measure.

*Definition 1:* [11] Let $T > 0$ and $t_0$ be fixed. An $m$-dimensional stochastic process $w$ over $[t_0, t_0 + T]$ is called an *$L_2$-Itô process* if it can be written as

$$w(t) = w(0) + \int_{t_0}^{t} a(s)\, ds + \int_{t_0}^{t} b(s)\, dW(s),$$

where $a, b \in L_2^m(\Omega \times [t_0, t_0 + T], \mathcal{P}, P \otimes \lambda)$, $w(0)$ is a constant and $\int_{t_0}^{t} \cdot\, dW(s)$ denotes Itô integration.
The set of all $L_2$-Itô processes is denoted by $\mathscr{I}$, and $\mathscr{I} \subset L_2^m(\Omega \times [t_0, t_0 + T], \mathcal{P}, P \otimes \lambda)$.

*Definition 2:* For a stochastic process $v \in L_2(\Omega\times[t_0, t_0+T], \mathcal{P}, P \otimes \lambda)$, the *Stratonovich integral* of $v$ is

$$\oint_{t_0}^{t} v(s)\, dW(s) \triangleq \int_{t_0}^{t} v(s)\, dW(s) + \frac{1}{2}\langle v, W\rangle_{[t_0,t]}\,, \qquad (1)$$

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The authors are affiliated with the Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, Virginia 23529-0246, USA. lduff004@odu.edu, {gray,gonzalez}@ece.odu.edu

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where $\langle v, W\rangle_{[t_0,t]}$ denotes the quadratic covariation of $v$ and $W$ over $[t_0, t]$.

*Definition 3:* For any $T > 0$ and $t_0 \in \mathbb{R}$, consider the set of all $m$-dimensional stochastic processes over $[t_0, t_0 + T]$, denoted by $\widetilde{\mathcal{UV}}^m[t_0, t_0 + T]$, which can be written as

$$w(t) = \int_{t_0}^{t} u(s)\, ds + \oint_{t_0}^{t} v(s)\, dW(s)$$

for some $u, v \in \mathscr{I}$. The latter are called the *drift* and *diffusion* inputs, respectively. Moreover, the subset $\mathcal{UV}^m[t_0, t_0 + T] \subset \widetilde{\mathcal{UV}}^m[t_0, t_0 + T]$ will refer to all processes satisfying:

a) Each integrand consists of $m$ components such that $\mathbf{E}[u_i(t)] < \infty$, $\mathbf{E}[v_i(t)] < \infty$, $t \in [t_0, t_0 + T]$.

b) The integrands $u$ and $v$ are such that

$$\|u\|_{L_2}, \|v\|_{L_2}, \|b\|_{L_2}, \|v\|_{L_4} \leq R \in \mathbb{R}^+,$$

where $b$ is the diffusion integrand of $v$.

c) The random variables $u_i(t_1)$, $u_i(t_2)$, $v_i(t_1)$ and $v_i(t_2)$ are independent for $1 \leq i \leq m$ and $t_1 \neq t_2$.

To describe an iterated integral over $\mathcal{UV}^m$, consider the following alphabets: $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_m\}$ and $XY = X \cup Y$. For each $\eta \in XY^*$, define recursively the mapping $E_\eta : L_2^m(\Omega \times [t_0, t_0 + T], \mathcal{P}, P \otimes \lambda) \to \mathcal{C}_{a.s.}[t_0, t_0 + T]$ by first setting $E_\emptyset = 1$ and then letting

$$E_{x_i\eta'}[w](t) = \int_{t_0}^{t-} u_i(s) E_{\eta'}[w](s)\, ds,\ x_i \in X \qquad (2)$$

$$E_{y_i\eta'}[w](t) = \oint_{t_0}^{t-} v_i(s) E_{\eta'}[w](s)\, dW(s),\ y_i \in Y, \qquad (3)$$

where $\eta' \in XY^*$, $u_0 = v_0 = 1$, and the notation $t-$ indicates that the integration is over $[t_0, t)$. The notation $t-$ will be suppressed in subsequent sections. Also, without loss of generality, it is assumed hereafter that $t_0 = 0$.

A Fliess operator can now be defined over the set of admissible inputs $\mathcal{UV}^m[0, T]$.

*Definition 4:* [2], [3] A causal $m$-input, $\ell$-output *Fliess operator* $F_c$, $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$, driven by a random process in $\mathcal{UV}^m[0, T]$ is formally defined as

$$F_c[w](t) = \sum_{\eta \in XY^*} (c, \eta)\, E_\eta[w](t), \qquad (4)$$

where each $E_\eta$ is given in (2)-(3).

In this setting, the specific goal of the paper is to show that $F_c[w]$ with $w \in \mathcal{UV}^m[0, T]$, $T > 0$ is absolutely mean square convergent when $c$ is globally convergent. That is,

$$\sum_{\eta \in XY^*} |(c, \eta)|\, \|E_\eta[w](t)\|_2 < \infty$$

for any $t \in [0, T]$, where $\|\cdot\|$ denotes the $L_2$ random variable norm. This represents a major generalization of the theorems developed in [3].

The following terminology will be used throughout the rest of the paper. Let $\mathbb{N}^{m+1}$ be the set of all vectors with components in $\mathbb{N} = \{0, 1, \ldots\}$. Define the language $X^kY^n \triangleq \{\eta \in XY^*, |\eta|_X = k, |\eta|_Y = n\}$ formed by words having $k$ letters in $X$ and $n$ letters in $Y$. For a fixed word $\eta \in X^kY^n$, define the vectors $\boldsymbol{\alpha} = (\alpha_m, \cdots, \alpha_0) \in \mathbb{N}^{m+1}$ and $\boldsymbol{\beta} = (\beta_m, \cdots, \beta_0) \in \mathbb{N}^{m+1}$, where $\alpha_i = |\eta|_{x_i}$, $\beta_i = |\eta|_{y_i}$, $k = \sum_{i=0}^m \alpha_i$ and $n = \sum_{i=0}^m \beta_i$. The summations over all possible $\alpha_i$'s that sum to $k$ and all possible $\beta_i$'s that sum to $n$ are denoted, respectively, by $\sum_{\|\alpha\|=k}$ and $\sum_{\|\beta\|=n}$. Since one is interested in working with arbitrary letters in the alphabet $XY$, hereafter $q_i^l$ will denote an element of $XY$ with $q_i^l = x_i$ if $l = 1$ and $q_i^l = y_i$ if $l = 2$. Similarly, $w_{q_i^l}$ will denote either a drift or a diffussion input, and $dq_i^l$ will signify either Lebesgue or Stratonovich integration according to the value of $l$.

## III. Iterated Stochastic Integrals and their $L_2$ Upper Bounds

Given that each $E_\eta[w](t)$ in (4) involves both Stratonovich and Lebesgue integrals in its definition, a special approach has to be employed to calculate $\|E_\eta[w](t)\|_2$. It is known that Stratonovich integrals lack certain important properties such as isometry [11]. But if a Stratonovich integral is written in terms of Itô integrals, then all the properties associated with Itô integrals are available [10], [12]. A specific formula for the iterated integral $E_\eta[w]$, $\eta \in X^kY^n$, can be obtained by a successive application of (1). To develop this formula, first define $J(\eta) = (j_n, \ldots, j_1) \in \mathbb{N}^{|\eta|_Y}$ to be those places in $\eta$ where all the letters belonging to $Y$ are located. For example, if $\eta = x_{i_5}y_{i_4}x_{i_3}y_{i_2}y_{i_1}$ then $J(\eta) = (j_3, j_2, j_1) = (4, 2, 1)$. The following theorems are essential to proving the main result.

*Theorem 1:* Let $\eta \in X^kY^n$ and $w \in \mathcal{UV}^m[0, T]$ be arbitrary. Then

$$E_\eta[w](t) = \sum_{r_1=0, r_2=0}^{n, \lfloor \frac{n}{2} \rfloor} \frac{1}{2^{r_1}2^{r_2}} \sum_{\substack{\boldsymbol{s_{r_1}} \in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}} \in \bar{A}_{nr_2}}} \mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t)\ , \qquad (5)$$

where

$$\bar{A}_{nr_2} = \Big\{ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}} = (\bar{s}_{r_2}, \ldots, \bar{s}_1) \in \mathbb{N}^{r_2} : \bar{s}_{l_2} + 1 < \bar{s}_{l_2+1}, \; l_{\bar{s}_{l_2}+1} = 2,\ 1 \leq l_2 \leq r_2 - 1,\ \bar{s}_{l_2} \in J(\eta) \Big\}$$

for $1 \leq r_2 \leq \lfloor \frac{n}{2} \rfloor$, $\bar{A}_{n0} = \emptyset$,

$$A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} = \Big\{ \boldsymbol{s_{r_1}} = (s_{r_1}, \ldots, s_1) \in \mathbb{N}^{r_1} : s_{l_1} < s_{l_1+1}, \; 1 \leq l_1 \leq r_1 - 1,\ s_{l_1} \neq \bar{s}_{l_2} \text{ or } \bar{s}_{l_2} + 1,\ \bar{s}_{l_2} \in \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}, \; s_{l_1} \in J(\eta) \Big\}$$

for $1 \leq r_1 \leq n$, $A_{n0}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} = \emptyset$, and $\lfloor \cdot \rfloor$ is the floor function. In addition, if $\eta = q_{i_{k+n}}^{l_{k+n}}\eta' \in X^kY^n$ then

$$\mathbf{I}_\eta[w](t) \triangleq \int_0^t w_{q_{i_{k+n}}^{l_{k+n}}}(s)\mathbf{I}_{\eta'}[w](t_{k+n})\, dq_{i_{k+n}}^{l_{k+n}}(s) \qquad (6)$$

$$\mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w](t) \triangleq \mathbf{I}_\eta[w](t)\Bigg|_{\substack{\int v_{i_{\bar{s}_{l_2}+1}}\int v_{i_{\bar{s}_{l_2}}}dW(t')dW(t)\to\int v_{i_{\bar{s}_{l_2}+1}}v_{i_{\bar{s}_{l_2}}}dt\\ \int v_{i_{s_{l_1}}}dW(t)\to\int b_{i_{s_{l_1}}}dt.}} \qquad (7)$$

with $b_{i_{s_l}} \in L_2(\Omega\times[0,T],\mathcal{P},P\otimes\lambda)$, $1\le l_1\le r_1$, $1\le l_2\le r_2$, and $i_{s_{l_1}}, i_{\bar{s}_{l_2}} \in \{0,\dots,m\}$ are the indices of the $s_{l_1}$-th and $\bar{s}_{l_2}$-th elements of $J(\eta)$, respectively.

*Proof:* The proof is by induction on the length of the word $\eta\in X^kY^n$. The $k+n=0,1$ cases are trivial. For $k+n=2$, one has $\eta\in\{x_{i_2}x_{i_1}, x_{i_2}y_{i_1}, y_{i_2}x_{i_1}, y_{i_2}y_{i_1}\}$. Evaluating $E_\eta[w](t)$ for each of these four cases, the claim follows by direct application of (1). For example, when $\eta = y_{i_2}y_{i_1}$, it is not difficult to obtain

$$\begin{aligned} E_\eta[w](t) &= \mathbf{I}_\eta[w](t) + \frac{1}{2}\mathbf{I}_\eta^{\substack{(1)\\ \emptyset}}[w](t) + \frac{1}{2}\mathbf{I}_\eta^{\substack{(2)\\ \emptyset}}[w](t) \\ &\quad + \frac{1}{2}\mathbf{I}_\eta^{\substack{\emptyset\\ (1)}}[w](t) + \frac{1}{4}\mathbf{I}_\eta^{\substack{(2,1)\\ \emptyset}}[w](t) \\ &= \sum_{r_1=0,r_2=0}^{2,1}\frac{1}{2^{r_1}2^{r_2}}\sum_{\substack{\boldsymbol{s_{r_1}}\in A_{2r_1}^{\boldsymbol{\bar{s}_{r_2}}}\\ \boldsymbol{\bar{s}_{r_2}}\in\bar{A}_{2r_2}}}\mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w](t). \end{aligned}$$

Now assume (5) holds up to $n+k$ and let $\eta = q_{i_{k+n+1}}^{l_{k+n+1}}\eta'$, where $\eta'\in X^kY^n$. If $l_{k+n+1}=1$ then a drift term has been added and (5) holds. If $l_{k+n+1}=2$ then a diffusion term is added and

$$E_\eta[w](t) = \oint_0^t v_{i_{k+n+1}}(t_{k+n+1})\left(\sum_{r_1=0,r_2=0}^{n,\lfloor\frac{n}{2}\rfloor}\frac{1}{2^{r_1}2^{r_2}}\cdot \sum_{\substack{\boldsymbol{s_{r_1}}\in A_{nr_1}^{\boldsymbol{\bar{s}_{r_2}}}\\ \boldsymbol{\bar{s}_{r_2}}\in\bar{A}_{nr_2}}}\mathbf{I}_{\eta'}^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w](t_{k+n+1})\right)dW(t_{k+n+1}). \qquad (8)$$

Observe that $s_{l_1}\ne\bar{s}_{l_2},\bar{s}_{l_2}+1$ for $0\le l_2\le\lfloor\frac{n}{2}\rfloor$. Using formula (1), if $s_{r_1}<k+n$, $\bar{s}_{r_2}<k+n-1$ and $q_{i_{n+k}}^{l_{n+k}}\in Y$ then

$$\left\langle v_{i_{k+n+1}}\left(\mathbf{I}_{\eta'}^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w]\right),W\right\rangle_{[0,t]} = \mathbf{I}_\eta^{\substack{\boldsymbol{s'_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w](t) + \mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}'_{r_2}}}}[w](t).$$

But if $s_{r_1}=k+n$ or $\bar{s}_{r_2}=k+n-1$ or $q_{i_{n+k}}^{l_{n+k}}\in X$ then

$$\left\langle v_{i_{k+n+1}}\left(\mathbf{I}_{\eta'}^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w]\right),W\right\rangle_{[0,t]} = \mathbf{I}_\eta^{\substack{\boldsymbol{s'_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w](t),$$

where $\boldsymbol{s'_{r_1}} = (k+n+1, s_{r_1},\dots,s_1)$ and $\boldsymbol{\bar{s}'_{r_2}} = (k+n, \bar{s}_{r_2},\dots,\bar{s}_1)$. Substituting the quadratic covariation above into (8) and regrouping gives

$$E_\eta[w](t) = \sum_{r_1=0,r_2=0}^{n+1,\lfloor\frac{n+1}{2}\rfloor}\frac{1}{2^{r_1}2^{r_2}}\sum_{\substack{\boldsymbol{s_{r_1}}\in A_{(n+1)r_1}^{\boldsymbol{\bar{s}_{r_2}}}\\ \boldsymbol{\bar{s}_{r_2}}\in\bar{A}_{(n+1)r_2}}}\mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}}\\ \boldsymbol{\bar{s}_{r_2}}}}[w](t).$$

Thus, the induction is completed. ■

*Theorem 2:* Let $\eta\in X^kY^n$ and $w\in\mathcal{UV}^m[0,T]$ be arbitrary. An $L_2$ upper bound for the iterated Itô integral (6) at a fixed $t\in[0,T]$ is

$$\|\mathbf{I}_\eta[w](t)\|_2^2 \le t^k\prod_{i=0}^m\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\beta_i}(t)}{\beta_i!}, \qquad (9)$$

where $\tilde{U}_i(t)=\int_0^t\mathbf{E}\left[u_i^2(s)\right]ds$ and $V_i(t)=\int_0^t\mathbf{E}\left[v_i^2(s)\right]ds$.

*Proof:* The inequality is proved by induction over the total number of $k+n$ integrals. For $k+n=0$, the claim follows trivially. If $k+n=1$, then there are two cases to consider. First, if $\eta=x_{i_1}$ then by Hölder's inequality

$$\left\|\mathbf{I}_{x_{i_1}}[w](t)\right\|_2^2 \le t\int_0^t\mathbf{E}\left[u_{i_1}^2(t_1)\right]dt_1 = t\,\tilde{U}_{i_1}(t).$$

The second case is when $\eta=y_{i_1}$. By the isometry property

$$\left\|\mathbf{I}_{y_{i_1}}[w](t)\right\|_2^2 = \int_0^t\mathbf{E}\left[v_{i_1}^2(t_1)\right]dt_1 = V_{i_1}(t).$$

Now calculate the bound for $n+k+1$ integrals assuming that (9) holds up to some fixed $n+k\ge0$. Set $\eta=q_{i_{k+n+1}}^{l_{k+n+1}}\eta'$ with $\eta'\in X^kY^n$. If $l_{k+n+1}=1$, then the independence property $a)$ in Definition 3 gives

$$\begin{aligned} &\|\mathbf{I}_\eta[w](t)\|_2^2 \\ &= \mathbf{E}\left[\left(\int_0^t u_{i_{k+n+1}}(t_{k+n+1})\mathbf{I}_{\eta'}[w](t_{k+n+1})\,dt_{k+n+1}\right)^2\right] \\ &\le t\int_0^t\mathbf{E}\left[u_{i_{k+n+1}}^2(t_{k+n+1})\right]\mathbf{E}\left[(\mathbf{I}_{\eta'}[w](t_{k+n+1}))^2\right]\,dt_{k+n+1} \\ &= t\int_0^t\mathbf{E}\left[u_{i_{k+n+1}}^2(t_{k+n+1})\right]\cdot \\ &\qquad\left[t_{k+n+1}^k\prod_{i=0}^m\frac{\tilde{U}_i^{\bar{\alpha}_i}(t_{k+n+1})}{\bar{\alpha}_i!}\frac{V_i^{\beta_i}(t_{k+n+1})}{\beta_i!}\right]dt_{k+n+1} \\ &\le t^{k+1}\prod_{\substack{i=0\\ i\ne i_{k+n+1}}}^m\frac{\tilde{U}_i^{\bar{\alpha}_i}(t)}{\bar{\alpha}_i!}\prod_{i=0}^m\frac{V_i^{\beta_i}(t)}{\beta_i!}\cdot \\ &\qquad\underbrace{\int_0^t\mathbf{E}\left[u_{i_{k+n+1}}^2(t_{k+n+1})\right]\frac{\tilde{U}_{i_{k+n+1}}^{\bar{\alpha}_{i_{k+n+1}}}(t_{k+n+1})}{\bar{\alpha}_{i_{k+n+1}}!}dt_{k+n+1}}_{=\frac{\tilde{U}_i^{\bar{\alpha}_{i_{k+n+1}}+1}(t)}{(\bar{\alpha}_{i_{k+n+1}}+1)!}} \end{aligned}$$

$$\leq\ t^{k+1}\prod_{i=0}^{m}\frac{\tilde{U}_i^{\bar{\alpha}_i}(t)}{\bar{\alpha}_i!}\frac{V_i^{\beta_i}(t)}{(\beta_i)!},$$

where $\sum_{i=0}^{m}\bar{\alpha}_i = k$. Letting $\alpha_i = \bar{\alpha}_i + \delta_{i(i_{k+n+1})}$ (here $\delta_{ij}$ denotes the Kronecker delta function), then

$$\|\mathbf{I}_\eta[w](t)\|_2^2 \leq t^{k+1}\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\beta_i}(t)}{\beta_i!},$$

where $\sum_{i=0}^{m}\alpha_i = k+1$. Now, for the case when $l_{k+n+1}=2$, the isometry property gives

$$\begin{aligned}
\|\mathbf{I}_\eta[w](t)\|_2^2 &= \mathbf{E}\left[\left(\int_0^t v_{i_{k+n+1}}(t_{k+n+1})\mathbf{I}_{\eta'}[w](t_{k+n+1})\cdot\right.\right.\\
&\qquad\left.\left. dW(t_{k+n+1})\right)^2\right]\\
&= \int_0^t \mathbf{E}\left[v_{i_{k+n+1}}^2(t_{k+n+1})\right]\mathbf{E}\left[(\mathbf{I}_{\eta'}[w](t_{k+n+1}))^2\right]\ dt_{k+n+1}\\
&= \int_0^t \mathbf{E}\left[v_{i_{k+n+1}}^2(t_{k+n+1})\right]\left[t_{k+n+1}^k\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t_{k+n+1})}{\alpha_i!}\cdot\right.\\
&\qquad\left.\prod_{i=0}^{m}\frac{V_i^{\bar{\beta}_i}(t_{k+n+1})}{\bar{\beta}_i!}\right]\ dt_{k+n+1}\\
&\leq t^k\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\prod_{\substack{i=0\\ i\neq i_{k+n+1}}}^{m}\frac{V_i^{\bar{\beta}_i}(t)}{\bar{\beta}_i!}\frac{V_i^{\bar{\beta}_{i_{k+n+1}}+1}(t)}{(\bar{\beta}_{i_{k+n+1}}+1)!},
\end{aligned}$$

where $\sum_{i=0}^{m}\bar{\beta}_i = n$. Similarly, let $\beta_i = \bar{\beta}_i + \delta_{i(i_{k+n+1})}$, then

$$\|\mathbf{I}_\eta[w](t)\|_2^2 \leq t^{k}\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\beta_i}(t)}{\beta_i!},$$

where $\sum_{i=0}^{m}\beta_i = n+1$. Hence, the proof is complete. ■

*Theorem 3:* Let $\eta \in X^kY^n$ and $w \in \mathcal{UV}^m[0,T]$ be arbitrary. An $L_2$ upper bound for the iterated Itô integral (7) evaluated at a fixed $t\in[0,T]$ is

$$\left\|\mathbf{I}_\eta^{\substack{\boldsymbol{s}_{r_1}\\ \bar{\boldsymbol{s}}_{r_2}}}[w](t)\right\|_2^2 \leq 2^{r_2}t^{k+r_1+r_2}\prod_{i=0}^{m}\frac{\tilde{U}_i^{\alpha_i}(t)}{\alpha_i!}\frac{V_i^{\bar{\beta}_i}(t)}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}}\frac{B_i^{\gamma_i}(t)}{\gamma_i!},$$

where $\bar{\gamma}_i = \sum_{l_2=1}^{r_2}(\delta_{ii_{\bar{s}_{l_2}}}+\delta_{i(i_{\bar{s}_{l_2}}+1)})$, $\bar{s}_{l_2}\in\bar{\boldsymbol{s}}_{r_2}$; $\gamma_i = \sum_{l_1=1}^{r_1}\delta_{ii_{s_{l_1}}}$, $s_{l_1}\in\boldsymbol{s}_{r_1}$; $\bar{\beta}_i = \beta_i - \bar{\gamma}_i - \gamma_i$; $\tilde{U}_i(t) = \int_0^t\mathbf{E}\left[u_i^2(s)\right]ds$; $V_i(t)=\int_0^t\mathbf{E}\left[v_i^2(s)\right]ds$; $\bar{V}_i^2(t) = \int_0^t\mathbf{E}\left[v_i^4(s)\right]ds$ and $B_i(t)=\int_0^t\mathbf{E}\left[b_i^2(s)\right]ds$.

*Proof:* Use induction over $r=r_1+r_2$. Without loss of generality, set $\eta = q_{i_{k+n}}^{l_{k+n}}\cdots q_{i_{s_r+1}}^{l_{s_r+1}}y_{i_{s_r}}\eta' \in X^kY^n$, $\eta'\in X^{k-k_1}Y^{s_r-1}$ with $s_r = s_{r_1}$ or $\bar{s}_{r_2}+1$, and $k_1\leq k$. For $s_{r_1} > \bar{s}_{r_2}$, applying the isometry property $n-s_{r_1}$ times and Hölder's inequality $k_1$ times gives

$$\begin{aligned}
&\left\|\mathbf{I}_\eta^{\substack{\boldsymbol{s}_{r_1}\\ \bar{\boldsymbol{s}}_{r_2}}}[w](t)\right\|_2^2\\
&\leq\ t^{k_1}\int_0^t\mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}^{l_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\cdot\\
&\qquad\mathbf{E}\left[\left(\int_0^{t_{s_{r_1}+1}}b_{i_{s_{r_1}}}(t_{s_{r_1}})\mathbf{I}_{\eta'}^{\substack{\boldsymbol{s}_{r_1-1}\\ \bar{\boldsymbol{s}}_{r_2}}}[w](t_{s_{r_1}})\ dt_{s_{r_1}}\right)^2\right]\cdot\\
&\qquad dt_{s_{r_1}+1}\cdots dt_{k+n}.
\end{aligned}$$

Using Hölder's inequality once more and applying the inductive step with $r-1 = r_1+r_2-1$,

$$\begin{aligned}
&\left\|\mathbf{I}_\eta^{\substack{\boldsymbol{s}_{r_1}\\ \bar{\boldsymbol{s}}_{r_2}}}[w](t)\right\|_2^2\\
&\leq\ t^{k_1}\int_0^t\mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}^{l_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\cdot\\
&\qquad t_{s_{r_1}+1}\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b^2_{i_{s_{r_1}}}(t_{s_{r_1}})\right]\left\|\mathbf{I}_{\eta'}^{\substack{\boldsymbol{s}_{r_1-1}\\ \bar{\boldsymbol{s}}_{r_2}}}[w](t_{s_r})\right\|_2^2\cdot\\
&\qquad dt_{s_{r_1}}dt_{s_{r_1}+1}\cdots dt_{k+n}\\
&\leq\ 2^{r_2}t^{k+r_1+r_2}\int_0^t\mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\\
&\qquad\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}^{l_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\prod_{i=0}^{m}\frac{\tilde{U}^{\bar{\alpha}_i}(t_{s_{r_1}})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}}(t_{s_{r_1}+1})}{\bar{\beta}_i!}\cdot\\
&\qquad\frac{\bar{V}_i^{\bar{\gamma}_i}(t_{s_{r_1}+1})}{\sqrt{\bar{\gamma}_i!}}\prod_{\substack{l=0\\ l\neq i_{s_{r_1}}}}^{m}\frac{B_l^{\gamma_l}(t_{s_{r_1}+1})}{\gamma_l!}\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b^2_{i_{s_{r_1}}}(t_{s_{r_1}})\right]\cdot\\
&\qquad\frac{B_{i_{s_{r_1}}}^{\gamma_{i_{s_{r_1}}}}(t_{s_{r_1}})}{\gamma_{i_{s_{r_1}}}!}dt_{s_{r_1}}dt_{s_{r_1}+1}\cdots dt_{k+n}\\
&\leq\ 2^{r_2}t^{k+r_1+r_2}\int_0^t\mathbf{E}\left[w^2_{q_{i_{k+n}}^{l_{k+n}}}(t_{k+n})\right]\cdots\\
&\qquad\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[w^2_{q_{i_{s_{r_1}+1}}}(t_{s_{r_1}+1})\right]\prod_{i=0}^{m}\frac{\tilde{U}^{\bar{\alpha}_i}(t_{s_{r_1}})}{\bar{\alpha}_i!}\frac{V_i^{\bar{\beta}}(t_{s_{r_1}+1})}{\bar{\beta}_i!}\cdot\\
&\qquad\frac{\bar{V}_i^{\bar{\gamma}_i}(t_{s_{r_1}+1})}{\sqrt{\bar{\gamma}_i!}}\frac{B_l^{\gamma_i'}(t_{s_{r_1}+1})}{\gamma_i'!}dt_{s_{r_1}+1}\cdots dt_{k+n},
\end{aligned}$$

where $\gamma_i' = \gamma_i + \delta_{ii_{s_{r_1}}}$. Observe that $\sum_{i=0}^{m}\gamma_i' = r_1$. The remaining $(n+k_1-s_{r_1})$ integrals are evaluated exactly as in the proof of Theorem 2. Thus, the $\bar{\alpha}_i$'s and $\bar{\beta}_i$'s increase rather than the $\gamma_i$'s, and

$$\left\| \mathbf{I}_{\eta}^{\substack{\boldsymbol{s_{r_1}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t) \right\|_2^2 \leq 2^{r_2} t^{k+r_1+r_2} \prod_{i=0}^{m} \frac{\tilde{U}^{\alpha_i}(t)}{\alpha_i!} \frac{V_i^{\bar{\bar{\beta}}}(t)}{\bar{\bar{\beta}}_i!} \frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}} \frac{B_l^{\gamma_i'}(t)}{\gamma_i'!},$$

where $\sum_{i=0}^m \alpha_i = k$, $\alpha_i = \bar{\alpha}_i + \left| q_{i_{k+n}}^{l_{k+n}} \cdots q_{i_{s_r+1}}^{l_{s_{r_1}+1}} \right|_{x_i}$, $\sum_{i=0}^m \left( \bar{\bar{\beta}}_i + \bar{\gamma}_i + \gamma_i' \right) = n$, $\bar{\bar{\beta}}_i = \bar{\beta}_i + \left| q_{i_{k+n}}^{l_{k+n}} \cdots q_{i_{s_r+1}}^{l_{s_{r_1}+1}} \right|_{y_i}$, and $\beta_i = \bar{\bar{\beta}}_i + \bar{\gamma}_i + \gamma_i'$. Similarly for $\bar{s}_{r_2} > s_{r_1}$, one instead applies the isometry property $n-(\bar{s}_{r_2}-1)$ times and Hölder's inequality $k_1$ times. There are two cases: $\bar{s}_{r_2}+1 \neq \bar{s}_{r_2}$ and $\bar{s}_{r_2}+1 = \bar{s}_{r_2}$. For brevity, and since the procedure is similar to the previous procedure, the inductive step for the remaining cases is illustrated by the following calculations. To compute the squared norm of $\int_0^t v_{i_2}(t_1) v_{i_1}(t_1)\, dt_1$, one must consider two subcases. If $i_1 \neq i_2$ then

$$\begin{aligned} \left\| \int_0^t v_{i_2}(t_1) v_{i_1}(t_1) dt_1 \right\|_2^2 &\leq t \int_0^t \mathbf{E}\left[ v_{i_2}^2(t_1) v_{i_1}^2(t_1) \right] dt_1 \\ &\leq t \left( \int_0^t \mathbf{E}\left[ v_{i_2}^4(t_1) \right] dt_1 \right)^{\frac{1}{2}} \left( \int_0^t \mathbf{E}\left[ v_{i_1}^4(t_1) \right] dt_1 \right)^{\frac{1}{2}} \\ &= t \ \bar{V}_{i_1}(t) \bar{V}_{i_2}(t) \leq 2t \ \bar{V}_{i_1}(t) \bar{V}_{i_2}(t). \end{aligned}$$

If $i_1 = i_2$ then

$$\left\| \int_0^t v_{i_1}^2(t_1) dt_1 \right\|_2^2 \leq t \int_0^t \mathbf{E}\left[ v_{i_1}^4(t_1) \right] dt_1 \leq 2t \ \frac{\left( \bar{V}_{i_1}(t) \right)^2}{\sqrt{2!}}.$$

See [2, Theorem IV.1.1] for the complete proof. ■

Theorems 1, 2 and 3 together with the binomial theorem are used to calculate an $L_2$ upper bound for $E_\eta[w]$.

*Theorem 4:* Let $\eta \in X^k Y^n$ and $w \in \mathcal{UV}^m$ be arbitrary. Then for a fixed $t \in [0,T]$

$$\|E_\eta[w](t)\|_2 < \frac{(R\sqrt{t})^k (\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\alpha!)^{\frac{1}{2}} (\beta!)^{\frac{1}{4}}},$$

where $\alpha! \triangleq \alpha_0! \cdots \alpha_m!$, $\beta! \triangleq \beta_0! \cdots \beta_m!$ and $\max\{\|u\|_{L_2}, \|v\|_{L_2}, \|v_0\|_{L_2}, \|v\|_{L_4}\} \leq R$.

*Proof:* Note that $\#(\bar{A}_{nr_2}) \leq \binom{n-r_2}{r_2} < \binom{n}{r_2}$ and $\#(A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}) = \binom{n-2r_2}{r_1} \leq \binom{n}{r_1}$. Using the triangle inequality, equation (5), Theorem 3 and the binomial theorem, observe

$$\begin{aligned} \|E_\eta[w](t)\|_2 &\leq \sum_{r_1=0, r_2=0}^{n, \lfloor \frac{n}{2} \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\boldsymbol{s_{r_1}} \in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}} \in \bar{A}_{nr_2}}} \left\| \mathbf{I}_{\eta}^{\substack{\boldsymbol{s_{r_1}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t) \right\|_2 \\ &\leq \left( \frac{R^{k+n}}{(\alpha!)^{\frac{1}{2}}} \right) \sum_{r_1=0, r_2=0}^{n, \lfloor \frac{n}{2} \rfloor} \frac{t^{\frac{k+r_1+r_2}{2}}}{2^{r_1} 2^{\frac{r_2}{2}}} \sum_{\substack{\boldsymbol{s_{r_1}} \in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}} \in \bar{A}_{nr_2}}} \prod_{i=0}^{m} \frac{1}{(\bar{\beta}_i!)^{\frac{1}{2}} (\bar{\gamma}_i!)^{\frac{1}{4}} (\gamma_i!)^{\frac{1}{2}}} \\ &\leq \left( \frac{R^{k+n} t^{\frac{k}{2}}}{(\alpha!)^{\frac{1}{2}} (\beta!)^{\frac{1}{4}}} \right) \sum_{r_1=0, r_2=0}^{n, \lfloor \frac{n}{2} \rfloor} \frac{t^{\frac{r_1+r_2}{2}}}{2^{r_1} 2^{\frac{r_2}{2}}} \sum_{\substack{\boldsymbol{s_{r_1}} \in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}} \in \bar{A}_{nr_2}}} \prod_{i=0}^{m} \frac{\beta!}{\bar{\beta}_i! \bar{\gamma}_i! \gamma_i!} \\ &\leq \left( \frac{R^{k+n} t^{\frac{k}{2}}}{(\alpha!)^{\frac{1}{2}} (\beta!)^{\frac{1}{4}}} \right) 3^n \sum_{r_1=0}^{n} \frac{t^{\frac{r_1}{2}}}{2^{r_1}} \binom{n}{r_1} \sum_{r_2=0}^{n} \frac{t^{\frac{r_2}{2}}}{2^{\frac{r_2}{2}}} \binom{n}{r_2} \\ &\leq \frac{(R\sqrt{t})^k (3\sqrt{2}R(\sqrt{t}+2)(\sqrt{t}+\sqrt{2}))^n}{4^n (\alpha!)^{\frac{1}{2}} (\beta!)^{\frac{1}{4}}} \\ &< \frac{(R\sqrt{t})^k (\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\alpha!)^{\frac{1}{2}} (\beta!)^{\frac{1}{4}}}. \end{aligned}$$

■

## IV. Absolute global convergence of Fliess operators

The next theorem presents the main result of the paper.

*Theorem 5:* Let $c \in \mathbb{R}^\ell \langle\langle XY \rangle\rangle$ be a globally convergent series. Then for any finite $T > 0$ and $w \in \mathcal{UV}^m[0,T]$, the series (4) converges absolutely in the mean square sense to a well-defined random vector $y(t) = F_c[w](t)$, $t \in [0,T]$.

*Proof:* Without loss of generality it is assumed that $\ell = 1$. Pick a $t \in [0,T]$ and any $w \in \mathcal{UV}^m[0,T]$. Let $R = \max\{\|u\|_{L_2}, \|v\|_{L_2}, \|v_0\|_{L_2}, \|v\|_{L_4}\}$. Define

$$a_{k,n}(t) = \sum_{\eta \in X^k Y^n} (c,\eta) E_\eta[w](t).$$

Note that the language $L_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \{\eta \in X^k Y^n : |\eta|_{x_i} = \alpha_i, |\eta|_{y_i} = \beta_i, i = 0, \ldots, m\}$ consists of $(k+n)!/(\alpha!\beta!)$ words. Applying Theorem 4,

$$\begin{aligned} \|a_{k,n}(t)\|_2 &\leq \sum_{\eta \in X^k Y^n} |(c,\eta)| \, \|E_\eta[w](t)\|_2 \\ &\leq K M^{n+k} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{(R\sqrt{t})^k (\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\alpha!)^{\frac{1}{2}} (\beta!)^{\frac{1}{4}}} \frac{(k+n)!}{\alpha!\beta!}. \end{aligned}$$

Without loss of generality, it is assumed that $R \geq 1$. If $R' \triangleq 4R(R+4)$, then from the multinomial theorem,

$$\begin{aligned} \|a_{k,n}(t)\|_2 &\leq K(MR')^{k+n} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{(k+n)!}{(\alpha!)^{\frac{3}{2}} (\beta!)^{\frac{5}{4}}} \\ &\leq K(2MR')^{k+n} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{k!\, n!}{(\alpha!)^{\frac{3}{2}} (\beta!)^{\frac{5}{4}}} \\ &\leq \frac{K(2MR')^{k+n}}{(k!)^{\frac{1}{2}} (n!)^{\frac{1}{4}}} \sum_{\|\alpha\|=k} \frac{(k!)^{\frac{3}{2}}}{(\alpha!)^{\frac{3}{2}}} \sum_{\|\beta\|=n} \frac{(n!)^{\frac{5}{4}}}{(\beta!)^{\frac{5}{4}}} \\ &\leq \frac{K(2MR')^{k+n}}{(k!)^{\frac{1}{2}} (n!)^{\frac{1}{4}}} \left( \sum_{\|\alpha\|=k} \frac{k!}{\alpha!} \right)^2 \left( \sum_{\|\beta\|=n} \frac{n!}{\beta!} \right)^2 \\ &\leq \frac{K(2MR'(m+1)^2)^{k+n}}{(k!)^{\frac{1}{2}} (n!)^{\frac{1}{4}}}. \end{aligned} \quad (10)$$

To show that (4) is mean square convergent, it is sufficient to show that it is a Cauchy series. Since $|\eta| = |\eta|_X + |\eta|_Y =$

$k+n \triangleq j$, it follows immediately from the triangle inequality that

$$\left\| \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta \in X^k Y^{j-k}} (c,\eta) E_\eta[w](t) \right\|_2 \leq \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \|a_{k,j-k}(t)\|_2$$

for any $N_2 > N_1 \in \mathbb{N}$. Now for any $\epsilon > 0$ there exist an $N > 0$ such that by equation (10) it follows

$$\begin{aligned} &\sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \|a_{k,j-k}(t)\|_2 \\ &\quad \leq \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta \in X^k Y^{j-k}} |(c,\eta)| \, \|E_\eta[w](t)\|_2 \\ &\quad \leq K \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(2MR'(m+1)^2)^k (2MR'(m+1)^2)^{j-k}}{(k!)^{\frac{1}{2}}((j-k)!)^{\frac{1}{4}}} \\ &\quad = K \sum_{k=0}^{\infty} \frac{(M'')^k}{(k!)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(M'')^n}{(n!)^{\frac{1}{4}}} < \epsilon \end{aligned}$$

for all $N_2 > N_1 > N$, where $M'' \triangleq (2MR'(m+1)^2)$. Note that $(M'')^k/(k!)^{\frac{1}{2}}$ and $(M'')^n/(n!)^{\frac{1}{4}}$ are the $k$-th and $n$-th terms of an absolutely convergent series, respectively. Hence, by the ratio test, the series (4) is Cauchy. This implies that $\sum_{\eta \in XY^*} |(c,\eta)| \, \|E_\eta[w]\|_2 < \infty$, and thus, it is an absolutely convergent series in the mean-square sense. ■

*Example 1:* Consider the following integral system

$$\begin{aligned} z(t) &= z(0) - a \int_0^t z(s) ds \\ &\quad + b \left( \int_0^t z(t) u(s)\, ds + \int_0^t v(s)\, dW(s) \right) \\ y(t) &= z(t), \end{aligned} \tag{11}$$

where $a, b \in \mathbb{R}$, $z(0) = 1$ and $w \in \mathcal{UV}[0,T]$. Assume there exist a series $c \in \mathbb{R}\langle\langle XY \rangle\rangle$ such that $y(t) = F_c[w](t)$. Then (11) is equivalent to

$$c - 1 = -ax_0\, c + b\,(x_1 + y_1)c.$$

Solving for $c$ gives

$$c = (1 + ax_0 - b(x_1 + y_1))^{-1}.$$

Observe that $c$ is a rational series, and therefore, it is easy to prove that $c$ is globally convergent. From Theorem 5, it is known that $y(t) = F_c[w](t)$ converges to a well-defined random variable for all $t \in [0,T]$. The operator $F_c[w]$ can also be computed directly as

$$\begin{aligned} F_c[w](t) &= F_{(1+ax_0 - b(x_1+y_1))^{-1}}[w](t) \\ &= \sum_{k=0}^{\infty} F_{(-ax_0 + b(x_1+y_1))^k}[w](t). \end{aligned}$$

Riccomagno in [14] proved the identity

$$\begin{aligned} (ax_0 + by_0)^{\,\sqcup\!\sqcup\, k} &\triangleq \underbrace{(ax_0 + by_0) \sqcup\!\sqcup \cdots \sqcup\!\sqcup (ax_0 + by_0)}_{k \text{ times}} \\ &= k!\,(ax_0 + by_0)^k, \end{aligned}$$

for any $a, b \in \mathbb{R}$. Using the minor extension,

$$(ax_0 + bx_i + dy_i)^{\,\sqcup\!\sqcup\, k} = k!\,(ax_0 + bx_i + dy_i)^k$$

for $a, b, d \in \mathbb{R}$, observe that

$$\begin{aligned} F_c[w](t) &= \sum_{k=0}^{\infty} \frac{1}{k!} F_{(-ax_0 + b(x_1+y_1))^{\sqcup\!\sqcup\, k}}[w](t) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( F_{(-ax_0 + b(x_1+y_1))}[w](t) \right)^k \\ &= \sum_{k=0}^{\infty} \frac{(-at + bw(t))^k}{k!} \\ &= e^{-at + bw(t)}. \end{aligned}$$

Clearly, $F_c[w](t)$ is a well-defined random variable for all $t \in [0,T]$, where $T$ can be arbitrarily large but finite. This is consistent with Theorem 5. □

## V. Conclusions

A proposition was proved that a Fliess operator with inputs from $\mathcal{UV}^m[0,T]$, $T > 0$ is absolutely globally mean square convergent when $c$ is globally convergent. Future work includes obtaining an analogous condition for locally convergent series.

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