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# On the Rationality of the Composition Product: A Survey

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***Abstract*— When two bilinear state space systems are interconnected in a cascade fashion, the resulting input-output map may not have a bilinear realization. So in 1972, Brockett asked under what conditions is bilinearity preserved under composition. In 1979, Ferfera produced the least restrictive sufficient condition which is presently known using formal power series representations of the input-output maps. The primary intent of this paper is to supply a self-contained proof of Ferfera's sufficient condition for preserving rationality under composition, or equivalently, preserving bilinearity under composition. The proof provided here, which in the large follows the original, is somewhat less complicated as it employs different tools from the theory of rational transductions.**

## I. Introduction

Consider a state space system of the form

$$\begin{aligned}\dot{z}(t) &= Az(t) + \sum_{j=1}^{m} N_j z(t) u_j(t), \quad z(0) = z_0 \\ y(t) &= Cz(t),\end{aligned}$$

where $z(t) \in \mathbb{R}^n$; $u_j(t) \in \mathbb{R}$; $y(t) \in \mathbb{R}^\ell$; and $A$, $N_j$ and $C$ are matrices of appropriate dimensions. Systems of this type are called *bilinear systems*. They have been employed in a wide number of applications from engineering to ecology and medicine [17]. From a theoretical perspective, they can be viewed as a *bridge* between the class of linear systems and the class of nonlinear affine-input systems, i.e., systems of the form

$$\begin{aligned}\dot{z}(t) &= f(z(t)) + \sum_{j=1}^{m} g_j(z(t)) u_j(t), \quad z(0) = z_0 \\ y(t) &= h(z(t)),\end{aligned}$$

where $f$, $g_j$ and $h$ are vector fields defined in terms of local coordinates on a state space manifold [15]. Nonlinear affine-input have a considerable literature, for example, in the controls area [13].

It is easily verified that if two linear state space systems are interconnected in a cascade fashion, that is, if $m = \ell$ and one sets $u_2 = y_1$, then the resulting input-output system always has a linear realization. The same closure property also holds for the class of nonlinear affine-input systems. Unfortunately, this very convenient property does *not* hold in general for the bilinear case. For example, if $(A_i, N_{\cdot,i}, C_i)$, $i = 1, 2$ are two bilinear systems then one possible state space realization for the input-output mapping $u_1 \mapsto y_2$ is clearly

$$\begin{aligned}\dot{z}_1(t) &= A_1 z_1(t) + \sum_{j=1}^{m} N_{j,1} z_1(t) u_{j,1}(t) \\ \dot{z}_2(t) &= A_2 z_2(t) + \sum_{j=1}^{m} N_{j,2} z_2(t) (C_1 z_1(t))_j \\ y_2(t) &= C_2 z_2(t),\end{aligned}$$

which appears at first inspection to be only in the nonlinear affine-input class. (Here $(v)_j$ denotes the $j$-th component of $v \in \mathbb{R}^m$). In 1972, Brockett asked in [4] under what conditions is bilinearity preserved under composition. One trivial sufficient condition can be identified immediately from the state space system above: when a bilinear system is followed by a linear system, the resulting system is bilinear since in this case $N_{j,2} = 0$, $j = 1, 2, \ldots, m$. But this condition is very restrictive and not necessary. In 1979, Ferfera produced in [5], [6] a much less restrictive sufficient condition using formal power series representations of the input-output mappings, namely, $F_{c_i} : u_i \mapsto y_i$, where $c_i$ is a generating series written in terms of a noncommutative alphabet $X = \{x_0, x_1, \ldots, x_m\}$ [8], [9]. In this setting, system composition can be modeled by $F_{c_2} \circ F_{c_1} = F_{c_2 \circ c_1}$, where $c_2 \circ c_1$ denotes the composition product of two formal power series [5], [6], [12], [16]. Bilinearity, in this context, is equivalent to having a *rational* or *regular* generating series [2]. Ferfera introduced the notion of an *input-limited* rational series (linear series, i.e., the generating series for linear systems being a special case) and showed that rationality was preserved under composition when an arbitrary rational series is followed by an input-limited rational series. It is easily demonstrated, however, that this condition is not necessary. In fact, at present, no necessary condition is available in the literature concerning this property.

The primary intent of this paper is to supply a self-contained proof of Ferfera's sufficient condition for preserving rationality under composition, or equivalently, preserving bilinearity under composition. The proof provided here, which in the large follows the original, is somewhat less complicated as it employs different tools from the theory of rational transductions [1], [3], [7], [14], [18], [19]. The paper is organized as follows. In Section II, Ferfera's sufficient condition is described, as well as some basic background for the problem. In Section III, the necessary tools from the theory of rational transductions are provided. In the final section, the proof of Ferfera's sufficient condition is given.

## II. Ferfera's Sufficient Condition via the Composition Product

A finite nonempty set of symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an alphabet. Each element of $X$ is called a letter, and any finite string of letters from $X$, $\eta = x_{i_k} \cdots x_{i_1}$, is

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called a word over $X$. The length of $\eta$, $|\eta|$, is the number of letters in $\eta$. The set of all words with length $k$ will be denoted by $X^k$. The set of all words including the empty word $\phi$ will be denoted by $X^*$, which forms a monoid under catenation. A language is any subset of $X^*$. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a formal power series. The value of $c$ at $\eta \in X^*$ is denoted by $(c, \eta)$. Typically $c$ is written as a formal sum

$$c = \sum_{\eta \in X^*} (c, \eta)\eta.$$

The collection of all formal power series over $X$ is denoted $\mathbb{R}^\ell \langle\langle X \rangle\rangle$. It forms an $\mathbb{R}$-algebra under the catenation product. Given $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$, the subset of $X^*$ defined by

$$\mathrm{supp}(c) = \{\eta : (c, \eta) \neq 0\}$$

is called the support of $c$. The subset of $\mathbb{R}^\ell \langle\langle X \rangle\rangle$ consisting of all the series with finite support is denoted by $\mathbb{R}^\ell \langle X \rangle$, and its elements are called polynomials. $c$ is called proper if $\phi \notin \mathrm{supp}(c)$. The set $\mathbb{R}^\ell \langle\langle X \rangle\rangle$ is known to be a complete ultrametric space under the ultrametric

$$\begin{aligned} \mathrm{dist} &: \mathbb{R}^\ell \langle\langle X \rangle\rangle \times \mathbb{R}^\ell \langle\langle X \rangle\rangle \to \mathbb{R} \\ &: (c, d) \mapsto \sigma^{\mathrm{ord}(c-d)}, \end{aligned}$$

where $0 < \sigma < 1$ and $\mathrm{ord}(c) := \inf\{|\eta| : (c, \eta) \neq 0\}$. A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is called *invertible* if there exists a series $c^{-1} \in \mathbb{R}\langle\langle X \rangle\rangle$ such that $cc^{-1} = c^{-1}c = 1$. In the event that $c$ is not proper, it is always possible to write

$$c = (c, \emptyset)(1 - c'),$$

where $c' \in \mathbb{R}\langle\langle X \rangle\rangle$ is proper. It then follows that

$$c^{-1} = \frac{1}{(c, \emptyset)}(1 - c)^{-1} = \frac{1}{(c, \emptyset)}(c')^*,$$

where

$$(c')^* := \sum_{i=0}^{\infty} (c')^i \;.$$

In fact, it can be shown that $c$ is invertible if and only if $c$ is not proper. Now let $S$ be any subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}\langle\langle X \rangle\rangle$. $S$ is said to be *rationally closed* when every invertible $c \in S$ has $c^{-1} \in S$. The *rational closure* of any subset $E \in \mathbb{R}\langle\langle X \rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X \rangle\rangle$ containing $E$.

*Definition 1:* [2] A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is **rational** if it belongs to the rational closure of $\mathbb{R}\langle X \rangle$.

Thus, a given rational series can be obtained from a finite set of polynomials by performing a finite number of additions, scalar products, catenation products and inversions (or star operations), the so called *rational operations*. The following definition is important for characterizing rational series.

*Definition 2:* A **linear representation** of a series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is any triple $(\mu, \gamma, \lambda)$, where $\mu : X^* \to \mathbb{R}^{n \times n}$ is a monoid morphism, and, $\gamma, \lambda^T \in \mathbb{R}^{n \times 1}$ are such that

$$(c, \eta) = \lambda\mu(\eta)\gamma, \;\; \forall \eta \in X^*.$$

The integer n is the dimension of the representation.

*Definition 3:* [2] A series is called **recognizable** if it has a linear representation.

The following theorem of Schützenberger establishes a link between rationality and bilinear state space representations [20].

*Theorem 1:* (Schützenberger) A formal power series is rational if and only if it is recognizable.

For each $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$, one can formally associate a causal $m$-input, $\ell$-output operator, $F_c$, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, $u_i$, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Define recursively for each $\eta \in X^*$ the mapping $E_\eta : L^m_1[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by $E_\emptyset = 1$, and

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$ and $u_0(t) \equiv 1$. The input-output operator corresponding to $c$ is then

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator* [8]–[12], [16]. When there exist real numbers $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!$ for all $\eta \in X^*$, where $|z| := \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, then $F_c$ constitutes a well-defined operator from $B^m_\mathfrak{p}(R)[t_0,\, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e. $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [11]. Such a power series $c$ is said to be *locally convergent*. It can be easily shown via Schützenberger's theorem that every rational series is locally convergent. Given any linear representation $(\mu, \gamma, \lambda)$ of a rational $c$, it follows that

$$\begin{aligned} c &= \sum_{\eta \in X^*} (\lambda\mu(\eta)\gamma)\, \eta \\ &= \sum_{k=0}^{\infty} \sum_{i_1, \ldots, i_k = 0}^{m} (\lambda N_{i_k} \cdots N_{i_1} \gamma)\, x_{i_k} \cdots x_{i_1}, \end{aligned}$$

where $\mu(x_i) = N_i$. The corresponding Fliess operator, $F_c$, is realized by the bilinear realization

$$\begin{aligned} \dot{z}(t) &= N_0 z(t) + \sum_{i=1}^{m} N_i z(t)\, u_i(t), \;\; z(t_0) = z_0 \quad (1) \\ y(t) &= C z(t) \end{aligned}$$

in the sense that (1) has a well-defined solution $\Phi(t, t_0, z_0, u)$ on some interval $[t_0, t_1]$ for every $u \in B^m_p(R)[t_0, t_1]$ with $p \geq 1$ and $R > 0$ sufficiently small, and

$$F_c[u](t) = C\Phi(t, t_0, z_0, u), \;\; \forall t \in [t_0, t_1].$$

The composition of two input-output operators $F_c$ and $F_d$, where $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$, can be described in terms of the composition product given below.

*Definition 4:* [5], [6] For any $\eta \in X^*$ and series $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$, the composition of $\eta$ with $d$ is defined in a recursive manner by

$$\eta \circ d = \begin{cases} \eta & : \quad |\eta|_{x_i} = 0, \ \forall\ i \neq 0 \\ x_0^{k+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \quad \eta = x_0^k x_i \eta',\ k \in \mathbb{N}, \\ & \quad\ \ i \neq 0,\ \eta' \in X^*, \end{cases}$$

where $\sqcup\!\sqcup$ denotes the shuffle product on $\mathbb{R}\langle\langle X \rangle\rangle$, $|\eta|_{x_1}$ is the number of times the letter $x_1$ appears in $\eta$, and $d_i : \xi \mapsto (d,\xi)_i$ with $(d,\xi)_i$ being the $i$-th component of the coefficient $(d,\xi)$. The composition of any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ with $d$ is

$$c \circ d = \sum_{\eta \in X^*} (c,\eta)\eta \circ d.$$

*Theorem 2:* [12] Let $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$. The composition $F_c \circ F_d$ has generating series $c \circ d$, i.e., $F_c \circ F_d = F_{c \circ d}$. In addition, if $c$ and $d$ are locally convergent then $c \circ d$ is also locally convergent.

The following example (due to Ferfera [5]) shows that the composition product does not preserve rationality.

*Example 1:* Consider the bilinear realization

$$\dot{z}(t) = z(t)u(t),\ z(0) = 1$$
$$y(t) = z(t),$$

where $z(t) \in \mathbb{R}$. Then locally the input-output mapping is given by $F_c : u \mapsto y$, where

$$c = \sum_{j=0}^{\infty} x_1^j = (1 - x_1)^{-1},$$

which is clearly rational. The claim, however, is that $c$ composed with itself is *not* rational. Observe that

$$c \circ c = \sum_{j=0}^{\infty} x_1^j \circ c$$

with

$$x_1^j \circ c = x_0[c \sqcup\!\sqcup (x_1^{j-1} \circ c)],\ \ j > 0.$$

One can show by induction that

$$x_0^{-k}(x_1^j \circ c) = \begin{cases} c^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^{j-k} \circ c) & : \quad j \geq k \\ 0 & : \quad j < k, \end{cases}$$

where $x_0^{-1}$ denotes the *left-shift operator*, $x_0^{-k} := (x_0^k)^{-1}$, and $c^{\sqcup\!\sqcup\, k}$ is the *shuffle power* of $c$. To see that $c \circ c$ is not rational, consider the following coefficients:

$$\begin{aligned} (c \circ c, x_0^k x_1^k) &= (x_1^{-k} x_0^{-k}(c \circ c), \emptyset) \\ &= \sum_{j=0}^{\infty} (x_1^{-k} x_0^{-k}(x_1^j \circ c), \emptyset) \\ &= \sum_{j=k}^{\infty} (x_1^{-k}(c^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^{j-k} \circ c)), \emptyset) \\ &= \sum_{j=k}^{\infty} (x_1^{-k}(c^{\sqcup\!\sqcup\, k}), \emptyset)\ (x_1^{j-k} \circ c, \emptyset) \\ &= x_1^{-k}(c^{\sqcup\!\sqcup\, k}, \emptyset) \\ &= (c^{\sqcup\!\sqcup\, k}, x_1^k). \end{aligned}$$

The identity

$$\left(\sum_{j=0}^{\infty} x_1^j\right)^{\sqcup\!\sqcup\, k} = \sum_{j=0}^{\infty} k^j x_1^j,\ \ k \geq 1$$

yields the final expression

$$(c \circ c, x_0^k x_1^k) = k^k,\ \ k \geq 1.$$

The key observation is that these coefficients are growing faster than any sequence of coefficients from a rational series can possibly grow, namely, $KM^{|\eta|}$ for real numbers $K, M > 0$ (e.g., see [11]). Hence, the series $c \circ c$ can not be rational.

□

While the composition product is not a rational operation in general, it will preserve rationality under certain conditions

*Definition 5:* [5], [6] A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is **limited relative to $\boldsymbol{x_i}$** if there exists an integer $\mathcal{N}_i \geq 0$ such that

$$\sup_{\eta \in \text{supp}(c)} |\eta|_{x_i} = \mathcal{N}_i < \infty.$$

If $c$ is limited relative to $x_i$ for every $i = 1, \ldots, m$ then c is said to be **input-limited**. In such cases, let $\mathcal{N}_c := \max_i \mathcal{N}_i$. A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is input-limited if each component series, $c_j$, is input-limited for $j = 1, \ldots, \ell$. In this case, $\mathcal{N}_c := \max_j \mathcal{N}_{c_j}$.

Now, the most important result in this paper is the following.

*Proposition 1:* [5], [6] Let $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ be two rational series. If $c$ is input-limited then the series $c \circ d$ is rational.

*Example 2:* Let $c = \sum_{j=0}^{\infty} x_1^j = (1 - x_1)^{-1}$, which is obviously rational but *not* input-limited, and $d = 1$. Trivially $c \circ d = c$. Thus, having $c$ input-limited is not a necessary condition for the composition product to preserve rationality.

□

## III. Rational Transductions

To prove Proposition 1, some results are needed from the theory of rational transductions. For brevity, the focus is on scalar output case, i.e., when $\ell = 1$. Let $X$ and $Y$ be two arbitrary alphabets. Any linear mapping $\tau : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle Y \rangle\rangle$ is called a *transduction*. It is completely specified by

$$\tau(\eta) = \sum_{\xi \in Y^*} (\tau(\eta), \xi)\xi,\ \ \forall \eta \in X^*.$$

With any $\tau$ one can canonically associate a series in $\mathbb{R}\langle\langle X \otimes Y \rangle\rangle$, namely

$$\begin{aligned} \hat{\tau} &= \sum_{\eta \in X^*} \eta \otimes \tau(\eta) \\ &= \sum_{\eta \in X^*,\ \xi \in Y^*} (\tau(\eta), \xi)\eta \otimes \xi. \end{aligned}$$

From $\hat{\tau}$ one can define a second transduction $\tau' : \mathbb{R}\langle\langle Y \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle$ via

$$\tau'(\xi) = \sum_{\eta \in X^*} (\tau(\eta), \xi)\eta,\ \ \forall \xi \in Y^*.$$

$\tau'$ is called the *inverse* of $\tau$. A transduction $\tau$ is called *rational* if the series $\hat{\tau}$ is a rational series in $\mathbb{R}\langle\langle X \otimes Y \rangle\rangle$, in which case every rational series in $\mathbb{R}\langle\langle X \rangle\rangle$ is mapped to a rational series in $\mathbb{R}\langle\langle Y \rangle\rangle$. Clearly, if $\tau$ is a rational transduction, then $\tau'$ is also a rational transduction. A transduction is *recognizable* if there exists an integer $N \geq 1$, a linear representation $\mu : X^* \rightarrow (\mathbb{R}<X>)^{N\times N}$, $\gamma$, $\lambda^T \in (\mathbb{R}<X>)^{N\times 1}$ such that $\tau(\eta) = \lambda\mu(\eta)\gamma, \ \ \forall\eta \in X^*$.

*Theorem 3:* [7], [19] A transduction $\tau : \mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$ is rational if and only if there exists an alphabet $W$, two monoid morphisms $\varrho_1 : W^* \rightarrow X^*$ and $\varrho_2 : W^* \rightarrow Y^*$, and a rational series $c_W \in \mathbb{R}\langle\langle W\rangle\rangle$ such that for all series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ the following identity is satisfied

$$\tau(c) = \varrho_2(\varrho_1^{-1}(c) \odot c_W),$$

where

$$\varrho_1^{-1}(\xi) := \sum_{\substack{\eta\in W^* \\ \varrho_1(\eta)=\xi}} \eta, \ \ \forall\xi \in X^*$$

is linearly extended to $\mathbb{R}\langle\langle X\rangle\rangle$ (likewise for $\varrho_2$), and $e \odot f$ is the Hadamard product on $\mathbb{R}\langle\langle W\rangle\rangle$, i.e.,

$$e \odot f = \sum_{\eta\in W^*} (e,\eta)(f,\eta)\eta$$

(see [2], [19]).

*Definition 6:* [7], [14], [19] A transduction $\tau : \mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$ has **finite image** if $\mathrm{supp}(\tau(\eta))$ is a finite subset of $Y^*$ for all $\eta \in X^*$.

*Theorem 4:* [14] If transduction $\tau : \mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$ has finite image and is recognizable then it is a rational transduction.

For the next pair of lemmas, it is necessary to introduce the following framework. Let $Y$ and $Y'$ be two finite disjoint alphabets and define a third alphabet $Z = Y \cup Y'$. It is easy to see that $Z^*$ can be partitioned as $Z^* = Y^* \cup Z^*Y'Y^*$, i.e., a word $\eta \in Z^*$ is either from $Y^*$ or can be constructed by a catenation of a word from $Z^*$, a letter from $Y'$ and a word in $Y^*$. In what follows, $\bar{Z}$ and $\bar{\bar{Z}}$ will be two distinct copies of $Z$. Let $\bar{\bar{\mathbf{Z}}}^*$ be the characteristic series of the language $\bar{\bar{Z}}^*$, i.e., $\bar{\bar{\mathbf{Z}}}^* = \sum_{\theta\in\bar{\bar{Z}}^*} \theta$, and let $\bar{\varrho} : Z^* \mapsto \bar{Z}^*$ be the natural monoid isomorphism defined by $\bar{\varrho}(z_i) = \bar{z}_i$ for all $z_i \in Z$. Define the transduction $\tau_1$ as the linear extension of the monoid mapping

$$\tau_1 : Z^* \rightarrow \mathbb{R}\langle\langle Z \cup \bar{Z} \cup \bar{\bar{Z}}\rangle\rangle$$
$$: \eta \mapsto \begin{cases} \bar{\bar{\mathbf{Z}}}^*(y_{i_n}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) : \eta = y_{i_n}\cdots y_{i_1} \in Y^* \\ \bar{\varrho}(\nu)\bar{\bar{\mathbf{Z}}}^*(y_{i_n}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) : \eta = \nu\xi, \ \nu \in Z^*Y', \\ \qquad \xi = y_{i_n}\cdots y_{i_1} \in Y^*. \end{cases}$$

Similarly, define the transduction $\tau_2$ from the monoid morphism

$$\tau_2 \ : \ Z^* \rightarrow \mathbb{R}\langle\langle Z \cup \bar{\bar{Z}}\rangle\rangle$$
$$: \ \eta = z_{i_n}\cdots z_{i_1} \mapsto (\bar{\bar{z}}_{i_n}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*).$$

By definition $\tau_1(\phi) = \bar{\bar{\mathbf{Z}}}^*$ and $\tau_2(\phi) = \phi$. The following result is essential.

*Lemma 1:* [5] The transductions $\tau_1$ and $\tau_2$ are rational.

*Proof:* Consider the corresponding transductions $\tau_1'$ and $\tau_2'$. One can construct the following linear representations by creating two corresponding Mealy finite-state machines. For $\tau_1'$:

$$\mu_1(y_i) = \begin{bmatrix} 0 & y_i & 0 & 0 \\ 0 & 0 & 0 & y_i \\ 0 & 0 & 0 & y_i \\ 0 & y_i & 0 & 0 \end{bmatrix}, \quad \mu_1(y_i') = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mu_1(\bar{z}_i) = \begin{bmatrix} 0 & 0 & z_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mu_1(\bar{\bar{z}}_i) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T, \quad \lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

For $\tau_2'$:

$$\mu_2(y_i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mu_2(y_i') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mu_2(\bar{\bar{z}}_i) = \begin{bmatrix} 0 & z_i & 0 \\ 0 & 0 & z_i \\ 0 & z_i & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$\lambda_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Trivially, one can see that $\tau_1'$ and $\tau_2'$ have finite images. Thus, by Theorem 4, $\tau_1$ and $\tau_2$ are rational transductions.

■

The next concept needed is a variation of the shuffle product as described below.

*Definition 7:* [5] Let $Z = Y \cup Y'$, where $Y$ and $Y'$ are two disjoint alphabets. The ***restricted shuffle product*** of two series $c, d \in \mathbb{R}\langle\langle Z\rangle\rangle$ is defined as

$$c \sqcup\!\sqcup_Y d = \sum_{\eta,\xi\in Z^*} (c,\eta)(d,\xi)\eta \sqcup\!\sqcup_Y \xi,$$

where

$$\eta \sqcup\!\sqcup_Y \xi = \begin{cases} \eta \sqcup\!\sqcup \xi & : \ \eta \in Y^*, \ \xi \in Z^* \\ \nu(\eta' \sqcup\!\sqcup \xi) & : \ \eta = \nu\eta', \ \nu \in Z^*Y', \\ & \quad \eta' \in Y^*, \ \xi \in Z^*. \end{cases}$$

The following lemma provides a particularly simple sufficient condition for a certain class of transductions to be rational. It will be used subsequently to show that the restricted shuffle product is a rational operation.

*Lemma 2:* [2] Suppose $X$ and $Y$ are two arbitrary alphabets. Let $\varrho : X \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$ be a map such that the series $\varrho(x_i)$ is a proper rational series for all $x_i \in X$. Then $\varrho$ can be extended uniquely to a monoid morphism and then extended again uniquely to a morphism of semirings $\mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$, which is the identity on $\mathbb{R}$ (i.e, $\varrho(\alpha) = \alpha, \ \forall\alpha \in \mathbb{R}$) and a rational transduction.

*Lemma 3:* [5] If $c, d \in \mathbb{R}\langle\langle Z\rangle\rangle$ are rational series then $c \sqcup\!\sqcup_Y d$ is also a rational series.

*Proof:* In light of Lemma 2, let $\varphi : \mathbb{R}\langle\langle Z \cup \bar{Z} \cup \bar{\bar{Z}}\rangle\rangle \rightarrow \mathbb{R}\langle\langle Z\rangle\rangle$ be the rational semiring epimorphism defined by setting

$$\varphi(\bar{\bar{z}}_i) = \varphi(\bar{z}_i) = \varphi(z_i) = z_i$$

for all $z_i \in Z$, $\bar{z}_i \in \bar{Z}$ and $\bar{\bar{z}}_i \in \bar{\bar{Z}}$. It will first be shown that

$$\eta \sqcup\!\sqcup_Y \xi = \varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)), \ \forall\eta,\xi \in Z^*. \qquad (2)$$

Since $Z^*$ is partitioned by $Y^*$ and $Z^*Y'Y^*$, one can divide the proof of identity (2) into two cases for a given $\eta \in Z^*$, namely when $\eta \in Y^*$ and when $\eta \in Z^*Y'$.

<u>Case 1</u>: If $\eta \in Y^*$ then for all $\nu \in \mathrm{supp}(\tau_1(\eta))$ it is clear that $|\nu|_{\bar{Z}} = 0$. In addition, all the words in $\mathrm{supp}(\bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi))$ have prefixes belonging to $\bar{Z}^*$. So the only word from

$\bar{Z}^*$ that can serve as a prefix in the set $\mathrm{supp}(\tau_1(\eta)) \cap \mathrm{supp}(\bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi))$ is the empty word. Therefore,

$$\varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) = \varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi)).$$

The next step is to show that for all $\eta \in Y^*$ and $\xi \in Z^*$

$$\varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi)) = \eta \sqcup\!\sqcup \xi. \tag{3}$$

This will be proven by induction on the sum of the lengths of $\eta$ and $\xi$. Let $\ell := |\eta| + |\xi|$ and suppose that $\ell = 0$, i.e., $\eta = \phi$ and $\xi = \phi$. Applying the definition of the Hadamard product and $\varphi$, it follows directly that

$$\begin{aligned} \varphi(\tau_1(\phi) \odot \mathbf{Y}^*\tau_2(\phi)) &= \varphi(\bar{\bar{\mathbf{Z}}}^* \odot \mathbf{Y}^*) \\ &= \varphi(\phi) = \phi \\ &= \phi \sqcup\!\sqcup \phi. \end{aligned}$$

Now assume identity (3) is true up to some $\ell = n+p-1 \geq 0$, and consider the words $\eta' = y_{i_{n-1}} y_{i_{n-2}} \cdots y_{i_1} \in Y^*$ and $\xi' = z_{i_{p-1}} z_{i_{p-2}} \cdots z_{i_1} \in Z^*$. Note that if $Y = \{y_1, \ldots, y_k\}$ and $\bar{\bar{Z}} = \{\bar{\bar{z}}_1, \bar{\bar{z}}_2, \ldots, \bar{\bar{z}}_l\}$ then

$$\begin{aligned} \mathbf{Y}^* &= \phi + y_1\mathbf{Y}^* + \cdots + y_k\mathbf{Y}^* \\ \bar{\bar{\mathbf{Z}}}^* &= \phi + \bar{\bar{z}}_1\bar{\bar{\mathbf{Z}}}^* + \cdots + \bar{\bar{z}}_l\bar{\bar{\mathbf{Z}}}^*. \end{aligned}$$

In which case, if $\eta = y_{i_n}\eta'$ and $\xi = z_{i_p}\xi'$ then

$$\begin{aligned} &\varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi)) \\ &\quad = \varphi(\bar{\bar{\mathbf{Z}}}^*(y_{i_n}\bar{\bar{\mathbf{Z}}}^*)(y_{i_{n-1}}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) \odot \\ &\qquad\quad \mathbf{Y}^*(\bar{\bar{z}}_{i_p}\mathbf{Y}^*)(\bar{\bar{z}}_{i_{p-1}}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*)) \\ &\quad = \varphi(y_{i_n}(\bar{\bar{\mathbf{Z}}}^*(y_{i_{n-1}}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) \odot \\ &\qquad\quad \mathbf{Y}^*(\bar{\bar{z}}_{i_p}\mathbf{Y}^*)(\bar{\bar{z}}_{i_{p-1}}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*)) + \\ &\qquad\quad \bar{\bar{z}}_{i_p}(\bar{\bar{\mathbf{Z}}}^*(y_{i_n}\bar{\bar{\mathbf{Z}}}^*)(y_{i_{n-1}}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) \odot \\ &\qquad\quad \mathbf{Y}^*(\bar{\bar{z}}_{i_{p-1}}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*))) \\ &\quad = y_{i_n}\varphi(\tau_1(\eta') \odot \mathbf{Y}^*\tau_2(\xi)) + z_{i_p}\varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi')) \\ &\quad = y_{i_n}(\eta' \sqcup\!\sqcup \xi) + z_{i_p}(\eta \sqcup\!\sqcup \xi') \\ &\quad = \eta \sqcup\!\sqcup \xi. \end{aligned}$$

Hence, identity (3) is proved for words of arbitrary length. Observe that since $\eta \in Y^*$ then $\eta \sqcup\!\sqcup_Y \xi = \eta \sqcup\!\sqcup \xi$, which implies that identity (2) is also proved.

Case 2: If $\eta = \nu\eta'$, where $\nu \in Z^*Y'$ and $\eta' \in Y^*$, then from the definition of $\tau_1$ it follows that

$$\varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) = \varphi(\bar{\nu}\tau_1(\eta') \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)).$$

Applying $\varphi$ and using the definition of the Hadamard product, it is clear that

$$\varphi(\bar{\nu}\tau_1(\eta') \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) = \nu\varphi(\tau_1(\eta') \odot \mathbf{Y}^*\tau_2(\xi)).$$

The remainder of the proof for identity (2) is now identical to that of Case 1.

To extend identity (2) to rational $c, d \in \mathbb{R}\langle\langle Z \rangle\rangle$, observe from the fact that $\varphi$ is a semiring epimorphism which induces the identity over $\mathbb{R}$, one can write

$$\begin{aligned} &c \sqcup\!\sqcup_Y d \\ &\quad = \sum_{\eta,\xi \in Z^*} (c,\eta)(d,\xi)\eta \sqcup\!\sqcup_Y \xi \\ &\quad = \sum_{\eta,\xi \in Z^*} (c,\eta)(d,\xi)\varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) \\ &\quad = \varphi\left(\sum_{\eta,\xi \in Z^*} (c,\eta)\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*(d,\xi)\tau_2(\xi)\right) \\ &\quad = \varphi\left(\left(\sum_{\eta \in Z^*} (c,\eta)\tau_1(\eta)\right) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\left(\sum_{\xi \in Z^*} (d,\xi)\tau_2(\xi)\right)\right) \\ &\quad = \varphi(\tau_1(c) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(d)). \end{aligned}$$

Finally, since the Hadamard product preserves rationality, and the transductions $\varphi$, $\tau_1$ and $\tau_2$ are rational (cf. Lemma 1), the restricted shuffle product also preserves rationality. ■

## IV. Proof of Proposition 1

Only the single-input, single-output case is considered, so the underlying alphabet is $X = \{x_0, x_1\}$. If $c$ is input-limited relative to $x_1$, then there exists an $\mathcal{N}_c \in \mathbb{N}$ such that

$$\mathrm{supp}(c) \subset \cup_{j=0}^{\mathcal{N}_c} x_0^j (x_1 x_0^*)^j.$$

Let $c_j$ be the restriction of $c$ to $x_0^*(x_1x_0^*)^j$, $j = 0, \ldots, \mathcal{N}_c$. The idea is to prove the rationality of $c \circ d = \sum_{j=0}^{\mathcal{N}_c} c_j \circ d$ by showing that each $c_j \circ d$ is rational when $c$ and $d$ are rational. For a specific $j$, it is possible to write

$$c_j = \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0},$$

where for brevity $\alpha_{k_j,\ldots,k_0} := (c, x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0})$. Introducing the alphabets $Z_0 = \{x_0\}$, $Z_1 = \{x_0, x_1, y_1, y_1'\}$, $Z_2 = \{x_0, x_1, y_1, y_2, y_1', y_2'\}$, $\ldots$, $Z_j = \{x_0, x_1, y_1, \ldots, y_j, y_1', \ldots, y_j'\}$, define iteratively the series $\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_j$ as :

$$\begin{aligned} \bar{c}_0 &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' x_0^{k_0} \qquad (4) \\ \bar{c}_i &= \bar{c}_{i-1} \sqcup\!\sqcup_{Z_{i-1}} d, \quad i = 1, 2, \ldots, j. \end{aligned}$$

Also let $\psi : \mathbb{R}\langle\langle Z_j \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle$ be the rational semiring epimorphism defined by: $\psi(x_0) = x_0$, $\psi(x_1) = x_1$, and $\psi(y_i) = \psi(y_i') = x_0$, for $i = 1, 2, \ldots, j$ (cf. Lemma 2). The first objective is to show that $c_j \circ d = \psi(\bar{c}_j)$. This can be done by an inductive procedure. From the commutativity of the (normal) shuffle product, observe

$$\begin{aligned} \bar{c}_1 &= \bar{c}_0 \sqcup\!\sqcup_{Z_0} d \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0} \left[y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' x_0^{k_0} \sqcup\!\sqcup_{Z_0} d\right] \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' \left[x_0^{k_0} \sqcup\!\sqcup_{Z_0} d\right] \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' \left[d \sqcup\!\sqcup x_0^{k_0}\right]. \end{aligned}$$

Now, assume that the procedure has been applied up to $\bar{c}_{j-1}$. Then

$$\begin{aligned}\bar{c}_j &= \bar{c}_{j-1} \sqcup\!\sqcup_{z_{j-1}} d \\ &= \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0} \left[y_j^{k_j} y_j' y_{j-1}^{k_{j-1}} y_{j-1}' \left[d \sqcup\!\sqcup y_{j-2}^{k_{j-2}} y_{j-2}' \cdot \right.\right. \\ &\quad \left.\left.\left[\cdots\left[d \sqcup\!\sqcup y_1^{k_1} y_1' \left[d \sqcup\!\sqcup x_0^{k_0}\right]\cdots\right] \sqcup\!\sqcup_{z_{j-1}} d\right]\right]\right] \\ &= \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0}\, y_j^{k_j} y_j' \left[y_{j-1}^{k_{j-1}} y_{j-1}' \left[d \sqcup\!\sqcup y_{j-2}^{k_{j-2}} y_{j-2}' \cdot \right.\right. \\ &\quad \left.\left.\left[\cdots\left[d \sqcup\!\sqcup y_1^{k_1} y_1' \left[d \sqcup\!\sqcup x_0^{k_0}\right]\cdots\right] \sqcup\!\sqcup_{z_{j-1}} d\right]\right]\right] \\ &= \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0}\, y_j^{k_j} y_j' \left[d \sqcup\!\sqcup y_{j-1}^{k_{j-1}} y_{j-1}' \cdot \right. \\ &\quad \left.\left[d \sqcup\!\sqcup y_{j-2}^{k_{j-2}} y_{j-2}' \left[\cdots\left[d \sqcup\!\sqcup y_1^{k_1} y_1' \left[d \sqcup\!\sqcup x_0^{k_0}\right]\cdots\right]\right]\right]\right].\end{aligned}$$

With the form of $\bar{c}_j$ established, apply the epimorphism $\psi$ to $\bar{c}_j$ :

$$\begin{aligned}\psi(\bar{c}_j) &= \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0}\, x_0^{k_j+1} \cdot \\ &\quad \left[d \sqcup\!\sqcup x_0^{k_{j-1}+1}\left[\cdots\left[d \sqcup\!\sqcup x_0^{k_1+1}\left[d \sqcup\!\sqcup x_0^{k_0}\right]\cdots\right]\right]\right].\end{aligned}$$

But recall from the definition of the composition product, if

$$\xi = x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0}$$

then

$$\xi \circ d = x_0^{k_j+1}\left[d \sqcup\!\sqcup x_0^{k_{j-1}+1}\left[\cdots\left[d \sqcup\!\sqcup x_0^{k_1+1}\left[d \sqcup\!\sqcup x_0^{k_0}\right]\cdots\right]\right]\right].$$

Therefore, $c_j \circ d = \psi(\bar{c}_j)$.

The final step is to show that $\bar{c}_j$ is rational. This can be done by showing that $\bar{c}_0$ is rational, since $\bar{c}_j$ is then computed using only rational operations applied to rational series. Let $\tilde{Z} = Z - \{x_1\}$ and consider the transduction

$$\delta_j : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle \tilde{Z} \rangle\rangle : c \mapsto \bar{c}_0.$$

Theorem 3 is employed to show that $\delta_j$ is rational. Consider the alphabet $W = A \cup A'$, where $A = \{a_0, \dots, a_j\}$ and $A' = \{a_1', \dots, a_j'\}$. Define two monoid morphisms $\varrho_1$ and $\varrho_2$ in the following way: $\varrho_1(a_0) = x_0$, $\varrho_1(a_i) = x_0$, $\varrho_1(a_i') = x_1$, $\varrho_2(a_0) = x_0$, $\varrho_2(a_i) = y_i$, and $\varrho_2(a_i') = y_i'$, where $i = 1, \dots, j$. Clearly

$$\varrho_1^{-1}(x_0) = \sum_{k=0}^{j} a_k, \quad \varrho_1^{-1}(x_1) = \sum_{k=1}^{j} a_k'.$$

Let $\mathbf{L} = \sum_{\theta\in\mathcal{L}} \theta$ be the characteristic series of the language $L = a_j^* a_j' \cdots a_1^* a_1' a_0^* \subset W^*$. Note that

$$\varrho_2(L) = y_j^* y_j' \cdots y_1^* y_1' x_0^*.$$

If $\tilde{X}^j := \{\xi = x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0} \in X^* : k_{i_\ell} \geq 0,\ \ell = 0, 1, \dots, j\}$ then

$$\varrho_1^{-1}(\xi) = \sum_{\substack{\eta_i \in A^{k_i} \\ \ell_i \in \{1,\dots,j\} \\ i=0,\dots,j}} \eta_j a_{\ell_j}' \cdots \eta_1 a_{\ell_1}' \eta_0, \quad \forall \xi \in \tilde{X}^j$$

with $a_{\ell_0}' := \emptyset$ (suppressed). By linearity

$$\varrho_1^{-1}(c) = \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0}\, \varrho_1^{-1}(x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0}),$$

and thus, using equation (4):

$$\begin{aligned}\varrho_1^{-1}(c) \odot \mathbf{L} &= \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0}\, a_j^{k_j} a_j' \cdots a_1^{k_1} a_1' a_0^{k_0} \\ \varrho_2(\varrho_1^{-1}(c) \odot \mathbf{L}) &= \sum_{k_j,\dots,k_0\geq 0} \alpha_{k_j,\dots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' x_0^{k_0} \\ &= \bar{c}_0 = \delta_j(c).\end{aligned}$$

Therefore, $\delta_j$ is rational, and the sufficient condition of Ferfera is proved.

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