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PHYSICAL REALIZABILITY AND PRESERVATION OF COMMUTATION AND ANTICOMMUTATION RELATIONS FOR n-LEVEL QUANTUM SYSTEMS*

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Abstract. The purpose of this paper is to address the problem of physical realizability for n-level quantum systems. We provide necessary and sufficient conditions for quantum stochastic differential equations that ensure the existence of physical parameters characterizing the unitary evolution required by the laws of quantum mechanics. Also, these conditions guarantee the preservation of the commutation and anticommutation relations of the underlying Lie algebra $su(n)$.

Key words. quantum stochastic differential equations, open quantum systems, physical realizability, finite level quantum systems, $su(n)$

AMS subject classifications. 81R25, 81R50, 93E03, 93E99

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1. Introduction. In quantum technology, when a coherent controller is synthesized using H^∞ or LQG approaches, one has to ensure that the synthesized controller can be implemented as a *physical quantum system* [16, 23]. For example, quantum finite level systems having Hamiltonian dynamics perturbed by environmental inputs are qualified as quantum systems with multiplicative noise inputs and their realization structure, in simplified form, is

$$dX = A_0 + AX\,dt + BX\,dW, \quad dY = CX\,dt + dW, \tag{1.1}$$

where X is the system variable, Y is the output, W accounts for the input environment, and A_0, A, B, C are matrices of suitable dimension. The question of whether a synthesized system in the form (1.1) corresponds to a physical quantum system leads to the question about the existence of quantum operators as well as input fields in

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a quantum mechanical setting. In this context a physical quantum system is a system whose evolution can be parameterized by the set of quantum operators (S, L, H), where H is the usual *Hamiltonian* operator, L is a vector of *coupling* operators expressing the interaction of the system variables with the environments fields in which the quantum system is immersed, and S is an operator matrix, known as the *scattering* operator, describing the interaction of the environment fields among themselves [1, 2, 12, 15]. These quantum operators S, L, and H lead to a set of equations in the Heisenberg picture that describe the evolution of physical quantities of the system represented by operators. More concretely, this is the realm of open quantum systems. That is, quantum systems that interact with their surrounding environment as provided by the operator L. For example, the environment can comprise several quantum fields (such as lasers) arranged in an experimental setting. Mathematically one may also consider operator equations that a priori may not have any association with any physical system. A dynamical system is said to have the property of *physical realizability* if such an (S, L, H) parameterization exists. In other words, physical realizability concerns the problem of determining conditions under which such equations do in fact correspond to a physical system. This property has attracted considerable interest in recent years with its main motivation being its role in *quantum coherent feedback control* [3, 29, 5, 19]. Thus, the physical realizability problem can be stated as follows: given a system's model, does there exist an (S, L, H) parameterization so that the system's model corresponds to a physical system with this parametrization?

There exist two *equivalent* points of view under which quantum systems are studied: the Schrödinger picture and the Heisenberg picture [4]. In the case of open quantum systems, the latter description is naturally mapped into the context of quantum stochastic differential equations (QSDEs) in which the system variables are now operators, and the QSDE model resembles the state space models in classical systems and control theory [26]. This allows the translation of standard control techniques into a quantum mechanical setting [16, 8, 1, 5, 19, 20, 29, 30]. Not every QSDE describes a quantum system. For instance, when a system has an (S, L, H) parameterization, these operators determine the unitary evolution of the system, and as a consequence some commutation relations are preserved. The noncommutativity of operators is a fundamental difference between quantum systems and classical systems. These commutation relations amount to the *Heisenberg uncertainty principle* [18]. The commutation relations constrain the coefficients of the QSDE corresponding to such a system. On the other hand, an arbitrary QSDE whose coefficients are not bound by these constraints need not correspond to a physical system. Therefore, satisfying these commutation relations is necessary for a QSDE to have a quantum behavior. Physical realizability for QSDEs addresses the question, given a QSDE, does there exist a corresponding (S, L, H) parameterization? The question of physical realizability has proven to be important in recent synthesis problems such as H^∞ control and model reduction of open quantum systems [16, 24, 25]. Early versions of physical realizability can be found in the work of Belavkin [2].

Solutions for the physical realizability problem are known as *physical realizability conditions*. These conditions provide simple testable matrix equations for a QSDE to correspond to a physical quantum system and thus have an (S, L, H) parameterization. The case for linear QSDEs is well understood [16, 24] and serves as an illustration of the physical realizability problem and its solution in this introduction. The operators constituting the system variables for linear QSDEs are operators on the space of square integrable complex sequences [8]. These are closely related to the standard quantum harmonic oscillator, and its system variables in terms of the annihilator operator a

and creation operator $a^\dagger$, which have commutation relations $[a,a]=[a^\dagger,a^\dagger]=0$ and $[a,a^\dagger]=1$. It is common to provide the system variables in self-adjoint form by applying the transformation $X_1=a+a^\dagger$ and $X_2=\boldsymbol{i}a^\dagger-\boldsymbol{i}a$, which provide the commutation relations $[X_1,X_1]=[X_2,X_2]=0$ and $[X_1,X_2]=\boldsymbol{i}$. Here X_1 and X_2 are proportional to the position and momentum operators, respectively. For an n-dimensional vector of self-adjoint operators X the commutation relations can be written in the following form (e.g., see [16]):

$$[X,X^T]=2\boldsymbol{i}\Theta \quad \text{with} \quad \Theta:=I_n\otimes J \quad \text{and} \quad J:=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The general form for a linear QSDE satisfying these commutation relations is

$$dX=AX\,dt+B\,dW,\quad dY=CX\,dt+dW, \tag{1.2}$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times n_w}$, and $C\in\mathbb{R}^{n_w\times n}$, and W represents an n_w-dimensional vector of interacting fields corresponding to the environment in which the system is immersed. n_w is an even number since W is assumed in quadrature form; therefore $C=(C_1^T\;C_2^T)^T$ with $C_i\in\mathbb{R}^{\frac{n_w}{2}\times n}$ for $i=1,2$. It was shown in [16, 8] that the QSDE (1.2) corresponds to a Hamiltonian $H=X^TRX$ with $R\in\mathbb{R}^{n\times n}$ symmetric, a coupling operator $L=\Gamma X$ for $\Gamma\in\mathbb{R}^{n_w\times n}$, and a scattering matrix operator $S=\hat{I}$ if and only if the following conditions are satisfied:

$$A\Theta+\Theta A+BJB=0, \tag{1.3a}$$

$$B=\Theta C^T(J\otimes I_n). \tag{1.3b}$$

Moreover, $R=\frac{1}{4}\left(-\Theta A+A^T\Theta\right)$ and $\Gamma=\frac{1}{2}\left(C_1+\boldsymbol{i}C_2\right)$, which uniquely identify H and L. In particular, condition (1.3a) is required to preserve the commutation relations, and it can be obtained solely from algebraic considerations.

Systems such as n-level quantum systems are widely used in quantum technology. For instance, the unit in quantum information is a two-level system known as a *qubit*. However, n-level systems are beyond the realm in which the physical realizability of linear QSDEs has been studied. Linear QSDEs do not have enough structure for modeling n-level systems in that the interacting fields act in a multiplicative manner on the system variables. This is in principle due to the commutation relations of n-level systems, which correspond to the commutation relations of the Lie algebra $su(n)$ since n-level systems evolve in the special unitary group $SU(n)$. Specifically, the n-level quantum systems treated in this paper correspond to bilinear QSDEs of the form

$$dX=A_0\,dt+AX\,dt+\sum_{k=1}^{n_w}B_{1k}X\,dW_{1k}+\sum_{k=1}^{n_w}B_{2k}X\,dW_{2k}, \tag{1.4}$$

$$\begin{pmatrix} dY_{1k} \\ dY_{2k} \end{pmatrix}=\begin{pmatrix} (C_1)_k \\ (C_2)_k \end{pmatrix}X\,dt+\begin{pmatrix} dW_{1k} \\ dW_{2k} \end{pmatrix},\quad k=1,\dots,n_w, \tag{1.5}$$

where A_0,A,B_{1k},B_{2k},C_1, and C_2 are suitable matrices, $(\cdot)_k$ denotes the kth row of a matrix, and W_{1k} and W_{2k} represent quantum interacting fields to be introduced in section 2. The special case of physical realizability conditions for QSDEs evolving in $SU(2)$, which corresponds to the case of two-level open quantum systems, was provided in [10]. Preliminary results for bilinear QSDEs evolving in $SU(n)$ were provided in [9, 11]. In these papers, it was shown that the extension of the formalism in [9] to general n-level quantum systems is nontrivial since now the QSDEs of interest must preserve anticommutation relations in addition to the commutation relations of $su(n)$. One of the main complications in this problem arises due to the so-called anomaly coefficients of $su(n)$, which are represented by the completely symmetric tensor d_{ijk} and appear exclusively due to the anticommutation relations of the generators of $su(n)$. In the case of QSDEs evolving in $SU(2)$, it was sufficient to consider only the preservation of commutation relations of $su(2)$, since in this case the tensor d_{ijk} is zero. Therefore, the first goal of this paper is to establish explicitly the correspondence between bilinear QSDEs and the preservation in time of the commutation and anticommutation relations of $su(n)$. The second goal consists of developing conditions under which a bilinear QSDE possesses an underlying (S, L, H) parameterization and thus corresponds to a physical quantum system. As expected, the Lie algebra $su(n)$ and its generators play a central role in the development presented in this paper [22, 21, 17]. More precisely, we will show that there exists an underlying (S, L, H) parameterization for the system (1.4)–(1.5) if and only if the matrices $A_0, A, B_{1k}, B_{2k}, C_1$, and C_2 satisfy

$$A_0 = \frac{1}{n}\sum_{k=1}^{n_w}(B_{1k} + \boldsymbol{i}B_{2k})\left((C_1)_k + \boldsymbol{i}(C_2)_k\right)^\dagger , \tag{1.6a}$$

$$B_{1k} = \Theta^-((C_2)_k), \tag{1.6b}$$

$$B_{2k} = -\Theta^-((C_1)_k), \tag{1.6c}$$

$$A + A^T + \sum_{i,k=1}^{2,n_w} B_{ik}{B_{ik}}^T = \frac{n}{2}\Theta^+(A_0), \tag{1.6d}$$

where the definition of the mappings $\Theta^-(\cdot)$ and $\Theta^+(\cdot)$ is provided in section 3. In the manuscript, both results are proved independently of each other. Moreover, these two results are then related by showing that a physically realizable quantum system does preserve the commutation and anticommutation relations of $su(n)$, as expected from the physics of n-level quantum systems.

The manuscript is organized as follows. Section 2 provides a summary on open quantum systems. The necessary tools regarding $SU(n)$ and its Lie algebra $su(n)$ are given in section 3. In section 4, a characterization of open n-level quantum systems, as well as the definition of physical realizability for such systems, is provided. This is followed by section 5, in which conditions for the preservation of the commutation and anticommutation relations of $su(n)$ are developed. In section 6, necessary and sufficient conditions for QSDEs to be physically realizable as n-level open quantum systems are given. Here, it is also shown that the conditions for preserving the commutation and anticommutation relations are implied by the physical realizability conditions. Section 7 gives the conclusions. Finally, the proofs to all lemmas in the paper are given in the appendix.

1.1. Notation. Let $\mathbb{R}$ denote the real numbers and $\mathbb{C}$ the complex numbers with imaginary unit $\boldsymbol{i}$. The set of real and complex n-dimensional vectors is denoted $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. The set of real and complex n by m matrices is denoted $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$, respectively. The n-dimensional identity matrix is denoted by I_n. A separable Hilbert space is denoted by $\mathfrak{H}$. The set of linear operators on $\mathfrak{H}$ is denoted by $\mathfrak{T}(\mathfrak{H})$, the set of n-dimensional column vectors of operators in $\mathfrak{T}(\mathfrak{H})$ is denoted by $\mathfrak{T}(\mathfrak{H})^n$. That is, an element X of $\mathfrak{T}(\mathfrak{H})^n$ is $X = (X_1 \cdots X_n)^T$, $X_i \in \mathfrak{T}(\mathfrak{H})$. The set $\mathfrak{T}(\mathfrak{H})^{n\times m}$ comprises $n \times m$ dimensional arrays of operators of the form $\{X_{ij}\}_{i,j=1}^n$ with $X_{ij} \in \mathfrak{T}(\mathfrak{H})$. The operator $\hat{I}$ denotes the identity in $\mathfrak{T}(\mathfrak{H})$. The adjoint of $X \in \mathfrak{T}(\mathfrak{H})^n$ is denoted by $X^\# := (X_1^* \, X_2^* \, \cdots \, X_n^*)^T$ and $X^\dagger := (X^\#)^T$, where $(\cdot)^T$ denotes the transpose operation and $(\cdot)^*$ denotes the adjoint of operators in $\mathfrak{T}(\mathfrak{H})$ (or the complex conjugate in the case of complex vectors or matrices). The operation $[\cdot,\cdot] : \mathfrak{T}(\mathfrak{H}) \times \mathfrak{T}(\mathfrak{H}) \to \mathfrak{T}(\mathfrak{H})$ is known as the *commutator*, and it is defined as $[X, Y] = XY - XY$. For vectors $X \in \mathfrak{T}(\mathfrak{H})^n$ and $Y \in \mathfrak{T}(\mathfrak{H})^m$, the commutator is given as $[X, Y^T] := XY^T - (YX^T)^T \in \mathfrak{T}(\mathfrak{H})^{n\times m}$ and satisfies $[X, Y^T]^T = -YX^T + \left(XY^T\right)^T = -[Y, X^T]$. In a quantum mechanical framework, it is common to multiply either vectors or matrices by arrays of operators. For example, letting $A \in \mathbb{C}^{m\times n}$ and $X \in \mathfrak{T}(\mathfrak{H})^{n\times r}$, the (i, j) element of the multiplication of a matrix by an operator matrix is $(AX)_{ij} := \sum_{k=1}^n a_{ik}X_{kj} \in \mathfrak{T}(\mathfrak{H})$. This product obeys the usual matrix multiplication rules. These considerations allow us to treat operators as system variables since in quantum mechanics they will play the role of states and therefore allow us to use state space systems notation.

Remark. The operations between complex matrices and operators follow the guidelines of the standard *canonical quantization* [6], which in simple words is a recipe that promotes the system variables from a classical mechanical framework into an operator framework in order to obtain a quantum mechanical description of the system.

2. Open quantum systems. *Open quantum systems* are systems governed by the laws of quantum mechanics that interact with an external environment. A quantum mechanical system is described in terms of *observables* and *states*. Observables represent physical quantities that can be measured, as self-adjoint operators on a separable Hilbert space $\mathfrak{H}$, while states give the current status of the system, allowing the computation of expected values of observables. In [3, 26], the evolution of open quantum systems is given in terms of QSDEs. For this purpose, observables may be thought of as quantum random variables that do not commute in general. The noncommutativity of observables is a fundamental difference between quantum systems and classical systems in which the former must satisfy certain commutation relations, which lead to the *Heisenberg uncertainty principle* [18].

The environment consists of a set of quantum fields. Each quantum field is a collection of oscillator systems, each oscillator having an annihilation field operator $w(t)$ and a creation field operator $w^*(t)$ representing the annihilation and creation of quanta at point t, and commonly known as the *boson quantum field* (with parameter t). Here it is assumed that t is a real time parameter. The field operators $w(t)$ and $w^*(t)$ satisfy commutation relations as well. That is, $[w(t), w^*(t')] = \delta(t - t')$ for all $t, t' \in \mathbb{R}$, where $\delta(t)$ denotes the Dirac delta. The mathematical description of the environment is given in terms of a Hilbert space called a *Fock space* [26, section 19]. When a boson quantum field is in the vacuum state, i.e., no physical particles are present, it then represents a natural quantum extension of white noise and may be described using the quantum Itô calculus [3, 26]. This amounts to having three interacting signals (inputs) in the evolution of the system: the annihilation processes $W(t)$, the creation process $W^*(t)$, and the counting process $\Lambda_w(t)$.

An open quantum system is described by putting together the evolutions of the system and the environment in a unitary fashion. That is, if ψ is an initial state, then $\psi(t) = U(t)\psi$, where $U(t)$ is unitary for all t, and is the solution of

$$dU(t) = \left((S - \hat{I})\, d\Lambda_w(t) + L\, dW^*(t) - L^*S\, dW(t) - \frac{1}{2}(L^*L + \boldsymbol{i}H)\, dt\right) U(t)$$

with initial condition $U(0) = \hat{I}$. Here, H is a fixed self-adjoint operator representing the *Hamiltonian* of the system, L is an operator determining the *coupling* of the system to the field, and S is unitary. The evolution of ψ is equivalent to the evolution of the observable X given by $X(t) = U^*(t)(X \otimes \hat{I})\, U(t)$, whose evolution is referred to as the *Heisenberg picture*, while the one for ψ is known as the *Schrödinger picture*. This paper exclusively takes the point of view of the Heisenberg picture.

The quantum stochastic calculus in [26] allows us to express the Heisenberg picture evolution of an operator X interacting with a boson field as

$$dX = (S^*XS - X)\, d\Lambda_w + \mathcal{L}(X)\, dt + S^*[X, L]\, dW^* + [L^*, X]S\, dW, \tag{2.1}$$

where $\mathcal{L}(X)$ is the Lindblad operator defined as

$$\mathcal{L}(X) = -\boldsymbol{i}[X, H] + \frac{1}{2}\left(L^*[X, L] + [L^*, X]L\right).$$

The output field is given by $Y(t) = U(t)^*W(t)U(t)$, which amounts to $dY = Ldt + SdW$. These equations can be extended in a straightforward way to the case of multiple environment fields. In this case, L becomes a vector of operators and S becomes a matrix of operators.

In summary, the dynamics of an open quantum system are uniquely determined by the parameterization (S, L, H). As mentioned before, the matrix of operators S indicates how several fields interact among each other. For example, a common experimental setting is that fields are lasers in a controlled environment where these do not interact with each other. The matrix of operators S in this case is the identity [13, 28]. However, there are situations where one can force the fields to interact in a particular manner, and in this case S is not the identity as in the model of a beam splitter. For instance, when a vacuum light field is reflected inside an interferometer a specific interconnection of systems such as *optical feedback* can be achieved [12]. The latter is not the focus of this manuscript, and hereafter the matrix of operators S is assumed to be the identity. This choice is also equivalent to the assumption of not considering jumps in standard Itô calculus. Note this assumption eliminates the first term on the right-hand side of (2.1), which cancels the action of the noise Λ_w representing the quantum version of Poisson jumps. Then when one considers n_w interacting boson fields, the evolution equation (2.1) is rewritten as

$$dX = \mathcal{L}(X)\, dt + dW^\dagger\, [X, L] + [L^\dagger, X]\, dW,$$

where $\mathcal{L}(X) := -\boldsymbol{i}[X, H] + \frac{1}{2}\left(L^\dagger\, [X, L] + [L^\dagger, X]\, L\right)$, $[X, dW] = [X, dW^\#] = 0$, $L = (L_1 \cdots L_s)^T$, $dW = (dW_1 \cdots dW_{n_w})^T$, and $dW^\dagger = \left(dW_1^* \cdots dW_{n_w}^*\right)$.

Consider the vector of operators $X = (X_1 \cdots X_s)^T \in \mathfrak{T}(\mathfrak{H})^s$. By stacking (columnwise) the scalar evolutions for each X_i, it follows that

$$dX = \mathcal{L}(X)\, dt + [X, L^T]\, dW^\# - [X, L^\dagger]\, dW,$$

where $\mathcal{L}(X) - \boldsymbol{i}[X, H] + \frac{1}{2}\left(\left(L^\dagger\, [X, L^T]^T\right)^T + [L^\#, X^T]^T\, L\right)$. This amounts to the following Heisenberg evolution equation:

$$dX = \mathcal{L}(X)\, dt + [X, L^T]\, dW^\# - [X, L^\dagger]\, dW, \tag{2.2}$$

where

$$\mathcal{L}(X) := -\boldsymbol{i}[X,H] + \frac{1}{2}\left(\left(L^\dagger\ \left[X, L^T\right]^T\right)^T + \left[L^\#, X^T\right]^T\ L\right). \tag{2.3}$$

It is customary to express QSDEs in terms of their interaction with quadrature fields. The quadrature fields are defined by the transformation

$$\begin{pmatrix}\bar{W}_1\\ \bar{W}_2\end{pmatrix} = \begin{pmatrix} I_{n_w} & I_{n_w}\\ -\boldsymbol{i}I_{n_w} & \boldsymbol{i}I_{n_w}\end{pmatrix}\begin{pmatrix}W\\ W^\#\end{pmatrix}, \tag{2.4}$$

where the operators $\bar{W}_1$ and $\bar{W}_2$ are now self-adjoint. The Itô table of W and $W^\dagger$ is nonzero only for the product $dW dW^\dagger = I_{n_w} dt$ [14]. In terms of the quadrature fields the Itô table amounts to

$$\begin{pmatrix}d\bar{W}_1\\ d\bar{W}_2\end{pmatrix}\begin{pmatrix}d\bar{W}_1^T & d\bar{W}_2^T\end{pmatrix} = \begin{pmatrix} I_{n_w} & \boldsymbol{i}I_{n_w}\\ -\boldsymbol{i}I_{n_w} & I_{n_w}\end{pmatrix} dt. \tag{2.5}$$

Thus, $dX = \mathcal{L}(X)\,dt + \frac{1}{2}\left([X,L^T] - [X,L^\dagger]\right) d\bar{W}_1 - \frac{\boldsymbol{i}}{2}\left([X,L^T] + [X,L^\dagger]\right) d\bar{W}_2$. The quadrature form of the output fields is obtained from the quadrature transformations $\bar{Y}_1 = Y + Y^\dagger$ and $\bar{Y}_2 = -\boldsymbol{i}Y + \boldsymbol{i}Y^\dagger$. This gives

$$\begin{pmatrix}d\bar{Y}_1\\ d\bar{Y}_2\end{pmatrix} = \begin{pmatrix}L + L^\#\\ \boldsymbol{i}(L^\# - L)\end{pmatrix} dt + \begin{pmatrix}d\bar{W}_1\\ d\bar{W}_2\end{pmatrix}. \tag{2.6}$$

Before specializing the theory of open quantum systems to n-level systems, one requires first to consider the algebraic setting in which n-level systems evolve. The next section is dedicated to introducing this algebraic framework.

3. The algebra of $SU(n)$.

In what follows, tools arising from the algebra of the special unitary group $SU(n)$ will be provided; see [22, 21, 17, 27] for more details. This algebra is the Lie algebra $su(n)$. A basis for this algebra is formed by all $n\times n$ complex matrices that are Hermitian and have zero trace. Consider the set of elementary vectors spanning $\mathbb{C}^n$, namely, $\{e_1,\ldots,e_n\}$. Define $P_{k,l}\in\mathbb{C}^{n\times n}$ as $P_{k,l} = e_k e_l^T$, where $k,l = 1,\ldots,n$. A standard way of constructing a complete basis for $su(n)$ is

$$\begin{aligned} u_{jk} &= P_{j,k} + P_{k,j}, \quad v_{jk} = \boldsymbol{i}\left(P_{j,k} - P_{k,j}\right),\\ w_l &= -\sqrt{\frac{2}{l(l+1)}}\left(\sum_{s=1}^{l} P_{s,s} - lP_{l+1,l+1}\right)\end{aligned} \tag{3.1}$$

for $1\le j<k\le n$, $1\le l\le n-1$. Note that the identity matrix I must be included in the basis in order to form a complete set. The identity, the $(n^2-n)/2$ symmetric matrices u_{jk}, the $(n^2-n)/2$ antisymmetric matrices v_{jk}, and the $n-1$ mutually commutative matrices w_l together form the *generators of* $su(n)$. These generators are known as the *generalized Gell–Mann matrices*. Without any particular order, the generators are relabeled $\{I,\lambda_1,\ldots,\lambda_s\}$, where $s = n^2-1$. Here these matrices satisfy $\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Their commutation and anticommutation relations are

$$[\lambda_i,\lambda_j] = 2\boldsymbol{i}\sum_{k=1}^{s} f_{ijk}\lambda_k, \quad \{\lambda_i,\lambda_j\} = \frac{4}{n}\delta_{ij} + 2\sum_{k=1}^{s} d_{ijk}\lambda_k,$$

where the real completely antisymmetric tensor f_{ijk} and the real completely symmetric tensor d_{ijk} are called the *structure constants* of $su(n)$. Then, the product $\lambda_i\lambda_j$ can be easily computed as $\lambda_i\lambda_j = \frac{1}{2}\left([\lambda_i, \lambda_j] + \{\lambda_i, \lambda_j\}\right) = \frac{2}{n}\delta_{ij} + \sum_{k=1}^{s}\left(\boldsymbol{i}f_{ijk} + d_{ijk}\right)\lambda_k$. The tensors f_{ijk} and d_{ijk} satisfy

$$f_{ilm}f_{mjk} + f_{jlm}f_{imk} + f_{klm}f_{ijm} = 0, \tag{3.2a}$$

$$f_{ilm}d_{mjk} + f_{jlm}d_{imk} + f_{klm}d_{ijm} = 0, \tag{3.2b}$$

$$\sum_{k=1}^{s} f_{ilk}f_{mjk} = \frac{2}{n}\left(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\right) + \sum_{k=1}^{s}\left(d_{imk}d_{ljk} - d_{ijk}d_{lmk}\right), \tag{3.2c}$$

$$\sum_{m,k=1}^{s} f_{imk}f_{jmk} = n\delta_{ij}. \tag{3.2d}$$

Define $F_i, D_i \in \mathbb{R}^{s\times s}$, $i \in \{1, \ldots, s\}$, such that their (j,k) component is $(F_i)_{jk} = f_{ijk}$ and $(D_i)_{jk} = d_{ijk}$, respectively. In particular, the set $\{-\boldsymbol{i}F_1, \ldots, -\boldsymbol{i}F_s\}$ is the adjoint representation of $su(n)$. In [21, 17], identities (3.2a)–(3.2c) were employed to obtain the following useful relationships:

$$[F_i, F_j] = -\sum_{k}^{s} f_{ijk}F_k, \tag{3.3a}$$

$$[F_i, D_j] = -\sum_{k}^{s} f_{ijk}D_k, \tag{3.3b}$$

$$F_iD_j + F_jD_i = \sum_{k}^{s} d_{ijk}F_k, \tag{3.3c}$$

$$D_iF_j + D_jF_i = \sum_{k}^{s} d_{ijk}F_k, \tag{3.3d}$$

$$\left(D_iD_j - F_jF_i\right)_{ml} = \sum_{k}^{s} d_{ijk}(D_k)_{ml} + \frac{2}{n}\left(\delta_{ij}\delta_{ml} - \delta_{im}\delta_{jl}\right). \tag{3.3e}$$

DEFINITION 3.1. *Let $\beta \in \mathbb{C}^s$. The linear mappings $\Theta^-, \Theta^+ : \mathbb{C}^s \to \mathbb{C}^{s\times s}$ are defined as*

$$\Theta^-(\beta) = \left(F_1^T\beta \cdots F_s^T\beta\right), \tag{3.4a}$$

$$\Theta^+(\beta) = \left(D_1^T\beta \cdots D_s^T\beta\right). \tag{3.4b}$$

Observe that the nature of the f and d tensors make $\Theta^-(\beta)$ and $\Theta^+(\beta)$ antisymmetric and symmetric, respectively. When β is an s-dimensional row vector, then it will be understood hereafter that $\Theta^-(\beta) = \Theta^-(\beta^T)$ and $\Theta^+(\beta) = \Theta^+(\beta^T)$. Consider now the *stacking operator* $\mathrm{vec} : \mathbb{C}^{n\times m} \to \mathbb{C}^{nm}$ whose action on a matrix creates a column vector by stacking its columns on top of one another. That is, for a matrix $B = (B_1\ B_2\ \cdots\ B_m) \in \mathbb{C}^{n\times m}$ with $B_i \in \mathbb{C}^{n\times 1}$, one has that $\mathrm{vec}(B) = ({B_1}^T \cdots {B_n}^T)^T \in \mathbb{C}^{nm\times 1}$. A key property of the stacking operator vec to be used in this paper is

$$\mathrm{vec}(ABC) = (C^T \otimes A)\,\mathrm{vec}(B) \tag{3.5}$$

for $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times l}$, and $C \in \mathbb{C}^{l\times r}$ with $n, m, l, r \in \mathbb{N}$, and $\otimes$ denoting the Kronecker product. With the help of the operator vec, the matrices $\Theta^-(\beta)$ and $\Theta^+(\beta)$ can be reorganized so that

$$\text{(3.6)}\quad \operatorname{vec}(\Theta^-(\beta)) = \begin{pmatrix} \Theta_1^-(\beta) \\ \vdots \\ \Theta_s^-(\beta) \end{pmatrix} = F\beta \quad \text{and} \quad \operatorname{vec}(\Theta^+(\beta)) = \begin{pmatrix} \Theta_1^+(\beta) \\ \vdots \\ \Theta_s^+(\beta) \end{pmatrix} = D\beta,$$

where $\Theta_i^-(\beta) = F_i^T\beta$, $F = (F_1 \cdots F_s)^T$, $\Theta_i^+(\beta) = D_i\beta$, $D = (D_1 \cdots D_s)^T$, and $\beta \in \mathbb{C}^s$. From (3.2d), F satisfies

$$(F^TF)_{ij} = -\sum_{k,m=1}^{s} (F_k)_{im}(F_k)_{mj} = -\sum_{k,m=1}^{s} f_{kim}f_{kmj} = \sum_{k,m=1}^{s} f_{imk}f_{jmk} = n\delta_{ij},$$

which implies

$$F^TF = nI. \tag{3.7}$$

The properties of $\Theta^-(\cdot)$ and $\Theta^+(\cdot)$ are summarized in the next lemma.

Lemma 3.2. *Let $\beta, \gamma \in \mathbb{C}^s$ be given. Then, the mappings $\Theta^-(\cdot)$ and $\Theta^+(\cdot)$ satisfy*

$$\Theta^-(\beta)\gamma = -\Theta^-(\gamma)\beta, \tag{3.8a}$$
$$\Theta^+(\beta)\gamma = \Theta^+(\gamma)\beta, \tag{3.8b}$$
$$\Theta^-(\beta)\beta = 0, \tag{3.8c}$$
$$\Theta^-\left(\Theta^-(\beta)\gamma\right) = [\Theta^-(\beta), \Theta^-(\gamma)], \tag{3.8d}$$
$$\Theta^-\left(\Theta^+(\beta)\gamma\right) = \Theta^-(\beta)\Theta^+(\gamma) + \Theta^-(\gamma)\Theta^+(\beta), \tag{3.8e}$$
$$\Theta^+\left(\Theta^-(\beta)\gamma\right) = [\Theta^+(\beta), \Theta^-(\gamma)] = [\Theta^-(\beta), \Theta^+(\gamma)], \tag{3.8f}$$
$$\Theta^+\left(\Theta^+(\beta)\gamma\right) = \Theta^+(\beta)\Theta^+(\gamma) - \Theta^-(\gamma)\Theta^-(\beta) - \frac{2}{n}\left(\beta^T\gamma I - \beta\gamma^T\right). \tag{3.8g}$$

Some additional identities regarding the matrices F and D with respect to the Kronecker product are given next. They will become useful when proving the main results in section 4. Define the tensor permutation matrix $\mathbb{1}_\otimes \in \mathbb{R}^{s^2\times s^2}$. This is a symmetric block matrix $\mathbb{1}_\otimes = \{\mathbb{1}_{ji}\}_{i,j=1}^s$, where $\mathbb{1}_{ij}$ is an elementary matrix having 1 at position (i,j) and 0 everywhere else. Note that $\mathbb{1}_\otimes$ satisfies $\mathbb{1}_\otimes(A\otimes B)\mathbb{1}_\otimes = (B\otimes A)$ for any $A, B \in \mathbb{C}^{s\times s}$.

Lemma 3.3. *Let $F = (F_1 \cdots F_s)^T$ and $D = (D_1 \cdots D_s)^T$, and $A, B \in \mathbb{R}^{s\times s}$. Then*

$$F = -\mathbb{1}_\otimes F, \tag{3.9a}$$
$$D = \mathbb{1}_\otimes D, \tag{3.9b}$$
$$F^T(A\otimes B)F = F^T(B\otimes A)F, \tag{3.9c}$$
$$D^T(A\otimes B)D = D^T(B\otimes A)D. \tag{3.9d}$$

4. Open n-level quantum systems and physical realizability. The interest in this paper is in systems evolving with respect to the special unitary group $SU(n)$. Consider the Hilbert space for these systems to be $\mathfrak{H} = \mathbb{C}^n$ and the corresponding space $\mathfrak{T}(\mathfrak{H}) = \mathbb{C}^{n\times n}$. It is standard to associate a vector $\beta = (\beta_1 \cdots \beta_s) \in \mathbb{C}^s$ with the vector of operators $\hat{\beta} \in \mathfrak{T}(\mathfrak{H})^s$ by simply inserting the identity operator in each component of the vector, i.e., $\hat{\beta} = (\beta_1\hat{I} \cdots \beta_s\hat{I})$. In a similar manner, any complex matrix is associated to a matrix of operators by inserting the identity operator $\hat{I}$ in each component. The identity operator $\hat{I}$ is usually suppressed. Therefore, abusing the notation slightly, the mappings $\Theta^-(\cdot)$ and $\Theta^+(\cdot)$ are allowed to act on a vector of operators to produce a matrix of operators. In this manner, it is valid to multiply complex matrices and operator matrices; again, this will produce a matrix of operators. Taking into account these considerations and the remarks in the notation section (section 1.1), this convention allows us to express the commutation and anticommutation relations of the system variables X of system (2.2) as

$$[X, X^T] = 2\boldsymbol{i}\Theta^-(X), \tag{4.1a}$$

$$\{X, X^T\} = \frac{4}{n}I + 2\Theta^+(X). \tag{4.1b}$$

Therefore,

$$XX^T = \frac{1}{2}\left([X, X^T] + \{X, X^T\}\right) = \frac{2}{n}I + \boldsymbol{i}\Theta^-(X) + \Theta^+(X). \tag{4.2}$$

The vector of system variables for (2.2) evolving in $SU(n)$ is then $X = (X_1, \ldots, X_s)^T$, where $X_1, \ldots, X_s$ are self-adjoint and a linear combination of the generalized Gell–Mann matrices. Here spanned means that these operators are linear combinations of the generalized Gell–Mann matrices. They are usually called *spin operators*. The initial values of the system variables are set to

$$X(0) = (\lambda_1^T \cdots \lambda_s^T)^T \tag{4.3}$$

with $\lambda_1, \ldots, \lambda_s$ being the generators of $su(n)$ introduced in section 2. This choice is made since one is looking at the evolution of the basis generators of a system evolving in $SU(n)$. Moreover, the natural assumption is that at time zero the initial conditions satisfy the commutation relations of $su(n)$.

Due to the product relation (4.2), any polynomial function of spin operators can be written as a linear combination of generalized Gell–Mann matrices. For example, a common Hamiltonian has the quadratic form $H = X^T RX$ with R being a symmetric matrix of a suitable dimension. Using (4.2), one can write $H = \sum_{i=1}^s \sum_{j=1}^s r_{ij} X_i X_j = \sum_{k=1}^s \alpha_k X_k = \alpha X$ with $\alpha^T \in \mathbb{R}^s$. In this example, α is a function of r_{ij} and the structure constants forming Θ^- and Θ^+ in (4.2). Moreover, any polynomial Hamiltonian can be transformed into a linear combination of the components of X. The same argument can be made for the coupling operator L. This allows us to restrict the attention in this paper, without significant loss of generality, to the class of linear Hamiltonians $H = \alpha X$ with $\alpha^T \in \mathbb{R}^s$ and also to consider the class of vector coupling operators of the form $L = \Gamma X$ with $\Gamma \in \mathbb{C}^{n_w \times s}$. Using the characterization of $su(n)$ introduced in section 2, the following expressions in terms of X are used for the characterization of n-level systems in terms of QSDEs.

LEMMA 4.1. *Let $A, B \in \mathbb{C}^{n_w \times s}$, and denote by A_i, B_i their respective rows, $i = 1, \ldots, n_w$. Then*

$$[X, (AX)^T] = -2\boldsymbol{i}\left(\Theta^-(A_1)X \cdots \Theta^-(A_{n_w})X\right), \tag{4.4a}$$

$$[X, (AX)^T]\, BX = -2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\frac{2}{n}\Theta^-(A_k)B_k^T \right.$$

$$\left. + \Theta^-(A_k)\Theta^+(B_k)X - \boldsymbol{i}\Theta^-(A_k)\Theta^-(B_k)X\right), \tag{4.4b}$$

$$(BX)^T[AX, X^T] = 2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\frac{2}{n}\Theta^-(A_k)B_k^T + \Theta^-(A_k)\Theta^+(B_k)X + \boldsymbol{i}\Theta^-(A_k)\Theta^-(B_k)X\right)^T. \tag{4.4c}$$

The explicit computation of the vector fields in (2.2) for n-level systems with linear Hamiltonian and coupling operators is given in the next lemma.

Lemma 4.2. *Let $H = \alpha X$ and $L = \Gamma X$. The component coefficients of* (2.2) *and* (2.3) *are*

$$[X, H] = -2\boldsymbol{i}\Theta^-(\alpha)X, \tag{4.5a}$$

$$[X, L^T] = -2\boldsymbol{i}\left(\Theta^-(\Gamma_1)X \cdots \Theta^-(\Gamma_{n_w})X\right), \tag{4.5b}$$

$$[X, L^\dagger] = -2\boldsymbol{i}\left(\Theta^-(\Gamma_1^\#)X \cdots \Theta^-(\Gamma_{n_w}^\#)X\right), \tag{4.5c}$$

$$\left[L^\#, X^T\right]^T L = \sum_{k=1}^{n_w}\left(\frac{4\boldsymbol{i}}{n}\Theta^-(\Gamma_k^\#)\Gamma_k^T + 2\boldsymbol{i}\Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k)X + 2\Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k)X\right), \tag{4.5d}$$

$$\left(L^\dagger \left[X, L^T\right]^T\right)^T = \sum_{k=1}^{n_w}\left(\frac{4\boldsymbol{i}}{n}\Theta^-(\Gamma_k^\#)\Gamma_k^T - 2\boldsymbol{i}\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#)X + 2\Theta^-(\Gamma_k)\Theta^-(\Gamma_k^\#)X\right). \tag{4.5e}$$

In summary, it follows from Lemma 4.2 that the evolution of X in quadrature form and initial condition (4.3) is described by a set of bilinear QSDEs of the form

$$dX = A_0\,dt + AX\,dt + (B_{11}X \cdots B_{1n_w}X\ B_{21}X \cdots B_{2n_w}X)\begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \tag{4.6}$$

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} X\,dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \tag{4.7}$$

where $A_0 \in \mathbb{R}^s$, $A, B_{1k}, B_{2k} \in \mathbb{R}^{s\times s}$, and $C_1, C_2 \in \mathbb{R}^{n_w\times s}$ for $k = 1, \ldots, n_w$. All matrices in (4.6) and (4.7) are real due to the fact that the class of quantum systems considered in this paper are in quadrature form and hence all operators being considered are self-adjoint. The central question of this paper is to find conditions under which a QSDE of the form (4.6)–(4.7) corresponds to an open quantum system evolving in $SU(n)$. In this context, we present the following explicit definition of *physical realizability* for bilinear QSDEs.

Definition 4.3. *A system described by* (4.6) *and* (4.7) *is said to be* physically realizable *if there exist operators $H = \alpha X$ and $L = \Gamma X$ with $\alpha^T \in \mathbb{R}^s$ and $\Gamma \in \mathbb{C}^{n_w\times s}$ such that* (4.6) *and* (4.7) *can be written as in* (2.2) *and* (2.6).

5. Preservation of commutation and anticommutation relations. In this section, we first consider under what conditions the QSDE (4.6) preserves the commutation and anticommutation relations of $su(n)$ at each instant of time during its evolution. That is, we establish conditions for the solution to (4.6) to satisfy

$$[X(t), X(t)] = 2\boldsymbol{i}\Theta^-(X(t)), \tag{5.1a}$$

$$\{X(t), X(t)\} = \frac{4}{n}I + 2\Theta^+(X(t)) \tag{5.1b}$$

for all $t \geq 0$. This preservation is necessary in order for the evolution of (4.6) to satisfy (4.2). Every n-level quantum system must satisfy (4.2) [22, section 2.2.2.4]. In order to tackle this problem, a preliminary lemma is needed first. Recall that any $G \in \mathbb{C}^{s\times s}$ can be regarded as $G \in \mathfrak{T}(\mathfrak{H})^{s\times s}$ since the identity operator in $\mathfrak{T}(\mathfrak{H})$ can be considered to be multiplying each component of G.

LEMMA 5.1. *Let $X = (X_1 \cdots X_s)^T$ be such that $\{I\ X_1 \cdots X_s\}$ is a linearly independent set. If the equation*

$$G\Theta^-(X) + \Theta^-(X)G^T - \Theta^-(GX) = 0 \tag{5.2}$$

has an antisymmetric solution $G \in \mathbb{C}^{s\times s}$, then there exists a $g \in \mathbb{C}^s$ such that

$$G = \Theta^-(g). \tag{5.3}$$

Such a g is unique and is given by

$$g := -\frac{1}{n}(\mathrm{Tr}(F_1 G)\cdots\mathrm{Tr}(F_s G))^T. \tag{5.4}$$

Conversely, for any $g \in \mathbb{C}^s$, the matrix G defined in (5.3) *satisfies* (5.2) *for any $X \in \mathfrak{T}(\mathfrak{H})^s$.*

The conditions that the system variables in (4.6) must satisfy in order to preserve the commutation and anticommutation relations of $su(n)$ over time amount to

$$d[X, X^T] - 2\boldsymbol{i}\Theta^-(dX) = 0, \tag{5.5a}$$

$$d\{X, X^T\} - 2\Theta^+(dX) = 0. \tag{5.5b}$$

Note by the linearity of $\Theta^-(\cdot)$ and $\Theta^+(\cdot)$ that

$$\begin{aligned}\Theta^-(dX) &= \Theta^-(A_0)dt + \Theta^-(AX)dt + \Theta^-(B_1X)d\bar{W}_1 + \Theta^-(B_2X)d\bar{W}_2,\\ \Theta^+(dX) &= \Theta^+(A_0)dt + \Theta^+(AX)dt + \Theta^+(B_1X)d\bar{W}_1 + \Theta^+(B_2X)d\bar{W}_2.\end{aligned}$$

Thus, a condition for the system (4.6) to satisfy (5.5a) and (5.5b) is given in the next theorem, which is the first main result of this paper.

THEOREM 5.2. *The system* (4.6) *satisfies relations* (5.5a) *and* (5.5b) *for all $t \geq 0$ if and only if*

$$B_i = \Theta^-(b_i), \tag{5.6a}$$

$$\sum_{k=1}^{n_w} B_{1k}B_{2k}^T - B_{2k}B_{1k}^T = \frac{n}{2}\Theta^-(A_0), \tag{5.6b}$$

$$A = \Theta^-(a) + \frac{1}{2}\sum_{k=1}^{n_w}\left(-B_{1k}B_{1k}^T - B_{2k}B_{2k}^T + B_{2k}\Theta^+(b_{1k}) - B_{1k}\Theta^+(b_{2k})\right), \tag{5.6c}$$

where b_{ik} and a are s-dimensional vectors as in (5.4) *for $i = 1, \dots, s$ and $k = 1, \dots, n_w$.*

Hereafter, the notation $X = X(t)$ is used where it does not lead to confusion.

Proof. First it is shown that (5.1a) is equivalent to

$$B_1\Theta^-(X) + \Theta^-(X)B_1^T - \Theta^-(B_1X) = 0, \tag{5.7a}$$

$$B_2\Theta^-(X) + \Theta^-(X)B_2^T - \Theta^-(B_2X) = 0, \tag{5.7b}$$

$$\frac{2}{n}B_1B_2^T - \frac{2}{n}B_2B_1^T - \Theta^-(A_0) = 0, \tag{5.7c}$$

$$\begin{aligned} &A\Theta^-(X) + \Theta^-(X)A^T + B_1\Theta^-(X)B_1^T + B_2\Theta^-(X)B_2^T \\ &+ B_1\Theta^+(X)B_2^T - B_2\Theta^+(X)B_1^T - \Theta^-(AX) = 0. \end{aligned} \tag{5.7d}$$

Without loss of generality, one can consider (4.6) interacting with only one quadrature field. Using the Itô table (2.5), it follows that

$$\begin{aligned} d(XX^T) &= (dX)X^T + X(dX)^T + (dX)(dX)^T \\ &= (A_0X^T + XA_0^T)\,dt + (AXX^T + XX^TA^T)\,dt \\ &\quad + (B_1XX^T + XX^TB_1^T)\,d\bar{W}_1 + (B_2XX^T + XX^TB_2^T)\,d\bar{W}_2 \\ &\quad + B_1XX^TB_1^T dt + \boldsymbol{i}B_1XX^TB_2^T dt - \boldsymbol{i}B_2XX^TB_1^T dt + B_2XX^TB_2^T dt. \end{aligned}$$

Similarly,

$$\begin{aligned} \big(d(XX^T)\big)^T &= (A_0X^T + XA_0^T)\,dt + (A(XX^T)^T + (XX^T)^TA^T)\,dt \\ &\quad + (B_1(XX^T)^T + (XX^T)^TB_1^T)\,d\bar{W}_1 + (B_2(XX^T)^T + (XX^T)^TB_2^T)\,d\bar{W}_2 \\ &\quad + B_1(XX^T)^TB_1^T dt + \boldsymbol{i}B_1(XX^T)^TB_2^T dt \\ &\quad - \boldsymbol{i}B_2(XX^T)^TB_1^T dt + B_2(XX^T)^TB_2^T dt. \end{aligned}$$

Computing $d\left[X, X^T\right]$ then gives

$$\begin{aligned} d([X, X^T]) &= d(XX^T) - \big(d(XX^T)\big)^T \\ &= 2\boldsymbol{i}\left(\frac{2}{n}B_1B_2^T - \frac{2}{n}B_2B_1^T\right) dt + 2\boldsymbol{i}\bigg(A\Theta^-(X) + \Theta^-(X)A^T\bigg)\, dt \\ &\quad + 2\boldsymbol{i}\bigg(B_1\Theta^-(X)B_1^T + B_2\Theta^-(X)B_2^T\bigg)\, dt \\ &\quad + 2\boldsymbol{i}\bigg(B_1\Theta^+(X)B_2^T - B_2\Theta^+(X)B_1^T\bigg)\, dt \\ &\quad + 2\boldsymbol{i}\left(B_1\Theta^-(X) + \Theta^-(X)B_1^T\right)d\bar{W}_1 + 2\boldsymbol{i}\left(B_2\Theta^-(X) + \Theta^-(X)B_2^T\right)d\bar{W}_2. \end{aligned} \tag{5.8}$$

Substituting (5.8) into (5.5a), which is the differential form of (5.1a), leads to

$$\begin{aligned} &2\boldsymbol{i}\left(\frac{2}{n}B_1B_2^T - \frac{2}{n}B_2B_1^T - \Theta^-(A_0) + A\Theta^-(X) + \Theta^-(X)A^T\right. \\ &\left. + B_1\Theta^-(X)B_1^T + B_2\Theta^-(X)B_2^T - \Theta^-(AX)\right) dt \\ &+ 2\boldsymbol{i}\left(B_1\Theta^-(X) + \Theta^-(X)B_1^T - \Theta^-(B_1X)\right)d\bar{W}_1 \\ &+ 2\boldsymbol{i}\left(B_2\Theta^-(X) + \Theta^-(X)B_2^T - \Theta^-(B_2X)\right)d\bar{W}_2 = 0. \end{aligned} \tag{5.9}$$

From [26, Proposition 23.7], one can also equate the integrands in (5.9) to zero. That is, (5.7a), (5.7b), and

$$\begin{aligned} \frac{2}{n}B_1B_2^T - \frac{2}{n}B_2B_1^T - \Theta^-(A_0) + A\Theta^-(X) + \Theta^-(X)A^T \\ + B_1\Theta^-(X)B_1^T + B_2\Theta^-(X)B_2^T - \Theta^-(AX) = 0 \end{aligned} \tag{5.10}$$

hold for all $t \geq 0$. Applying the vec operator and using property (3.5), (5.10) can be rewritten as $\mathbf{A}X = \mathbf{b}$, where

$$\begin{aligned} \mathbf{b} &= -\operatorname{vec}\left(\frac{2}{n}B_1B_2^T - \frac{2}{n}B_2B_1^T - \Theta^-(A_0)\right), \\ \mathbf{A}X &= \operatorname{vec}\left(A\Theta^-(X) + \Theta^-(X)A^T + B_1\Theta^-(X)B_1^T + B_2\Theta^-(X)B_2^T - \Theta^-(AX)\right) \\ &= \left(\left((I\otimes A) + (A\otimes I) + (B_1\otimes B_1) + (B_2\otimes B_2)\right)F + FA\right)X. \end{aligned}$$

Now consider (5.10) at $t = 0$. Recall that the components of $X(0)$ belong to the set of generalized Gell–Mann matrices, i.e., the set of generators of $su(n)$ excluding the identity. This implies that any linear combination $a_0I + \sum_{k=0}^{s} a_iX_i(0) \neq 0$ unless $a_i = 0$ for all i. In addition, no linear combination of generalized Gell–Mann matrices generates the identity. So, given that $X(0) \neq 0$, (5.10) implies $\mathbf{b} = 0$ and $\mathbf{A} = 0$. The first condition yields (5.7c), while the second condition implies (5.7d). Conversely, substituting (5.7a)–(5.7d) into (5.8) one obtains (5.1a), which proves the equivalence between (5.7a)–(5.7d) and (5.1a).

Also, using the same argument we show that (5.1b) is equivalent to

$$B_1\Theta^+(X) + \Theta^+(X)B_1^T - \Theta^+(B_1X) = 0, \tag{5.11a}$$

$$B_2\Theta^+(X) + \Theta^+(X)B_2^T - \Theta^+(B_2X) = 0, \tag{5.11b}$$

$$A + A^T + B_1B_1^T + B_2B_2^T - \frac{n}{2}\Theta^+(A_0) = 0, \tag{5.11c}$$

$$B_1 + B_1^T = B_2 + B_2^T = 0, \tag{5.11d}$$

$$\begin{aligned} A_0\,X^T + X\,A_0^T + A\Theta^+(X) + \Theta^+(X)A^T - \Theta^+(AX) \\ + B_1\Theta^+(X)B_1^T + B_2\Theta^+(X)B_2^T - B_1\Theta^-(X)B_2^T + B_2\Theta^-(X)B_1^T = 0. \end{aligned} \tag{5.11e}$$

We now prove the necessity part of the theorem. Assume that (5.7a)–(5.7d) and (5.11a)–(5.11e) hold. Condition (5.6b) appears explicitly in (5.7b). From (5.7a), (5.7b), and (5.11d), condition (5.6a) is obtained by direct application of Lemma 5.1, i.e., $B_i = \Theta^-(b_i)$ holds, where b_i is given in (5.4) for $i \in \{1, 2\}$. In order to show (5.6c), A_0 needs to be expressed in a more convenient form. This is achieved by using (3.8d) and (5.7b). That is,

$$\Theta^-(A_0) = \frac{2}{n}\left(\Theta^-(b_2)\Theta^-(b_1) - \Theta^-(b_2)\Theta^-(b_1)\right) = \frac{2}{n}\Theta^-\left(\Theta^-(b_2)b_1\right).$$

Due to the linearity of $\Theta^-(\cdot)$, A_0 is then uniquely determined by $A_0 = \frac{2}{n}\Theta^-(b_2)b_1$. Computing $\Theta^+(A_0)$ in terms of b_1 and b_2 using (3.8f) gives

$$\begin{aligned} \Theta^+(A_0) = \frac{2}{n}\Theta^+(\Theta^-(b_2)b_1) = \frac{1}{n}\big(&\Theta^+(b_2)\Theta^-(b_1) - \Theta^-(b_1)\Theta^+(b_2) \\ &+\Theta^-(b_2)\Theta^+(b_1) - \Theta^+(b_1)\Theta^-(b_2)\big). \end{aligned} \tag{5.12}$$

Using (5.11c) and (5.12), one obtains the following expression:

$$
\begin{aligned}
(\Theta^-(b_1))^2 + (\Theta^-(b_2))^2 = A &- \frac{1}{2}\big(\Theta^-(b_2)\Theta^+(b_1) - \Theta^-(b_1)\Theta^+(b_2)\big) \\
&+ \left(A - \frac{1}{2}\big(\Theta^-(b_2)\Theta^+(b_1) - \Theta^-(b_1)\Theta^+(b_2)\big)\right)^T.
\end{aligned} \tag{5.13}
$$

Now let $\bar{P} := A - \frac{1}{2}(\Theta^-(b_2)\Theta^+(b_1) - \Theta^-(b_1)\Theta^+(b_2))$. One can trivially decompose $\bar{P}$ into its symmetric and antisymmetric parts. The antisymmetric part is $P = \frac{1}{2}\big(\bar{P} - \bar{P}^T\big)$. Note from (5.13) that $\bar{P} + \bar{P}^T = \Theta^-(b_1)\Theta^-(b_1) + \Theta^-(b_2)\Theta^-(b_2)$. Hence, P can be written as

$$
P = A - \frac{1}{2}\big(\Theta^-(b_1)\Theta^-(b_1) + \Theta^-(b_2)\Theta^-(b_2)\big) - \frac{1}{2}\big(\Theta^-(b_2)\Theta^+(b_1) - \Theta^-(b_1)\Theta^+(b_2)\big).
$$

It is only left to show that P can be written in terms of $\Theta^-(\cdot)$. The formula for P allows us to calculate $\Theta^-(PX)$ as

$$
\begin{aligned}
\Theta^-(PX) = \Theta^-(AX) &- \frac{1}{2}\Theta^-\big(\Theta^-(b_1)\Theta^-(b_1)X + \Theta^-(b_2)\Theta^-(b_2)X \\
&+ \Theta^-(b_2)\Theta^+(b_1)X - \Theta^-(b_1)\Theta^+(b_2)X\big) \\
= \Theta^-(AX) &+ \Theta^-(b_1)\Theta^-(X)\Theta^-(b_1) + \Theta^-(b_2)\Theta^-(X)\Theta^-(b_2) \\
&- \frac{1}{2}\big(\Theta^-(b_1)\Theta^-(b_1)\Theta^-(X) + \Theta^-(b_2)\Theta^-(b_2)\Theta^-(X) \\
&+ \Theta^-(X)\Theta^-(b_1)\Theta^-(b_1) + \Theta^-(X)\Theta^-(b_2)\Theta^-(b_2) \\
&+ \Theta^-(b_2)\Theta^+(b_1)\Theta^-(X) + \Theta^-(b_2)\Theta^+(X)\Theta^-(b_1) \\
&- \Theta^-(b_1)\Theta^+(X)\Theta^-(b_2) - \Theta^-(X)\Theta^+(b_1)\Theta^-(b_2) \\
&- \Theta^-(b_1)\Theta^+(b_2)\Theta^-(X) - \Theta^-(b_1)\Theta^+(X)\Theta^-(b_2) \\
&+ \Theta^-(b_2)\Theta^+(X)\Theta^-(b_1) + \Theta^-(X)\Theta^+(b_2)\Theta^-(b_1)\big).
\end{aligned}
$$

In the previous calculation, the fact that $\Theta^-(\cdot) = -(\Theta^-(\cdot))^T$ was used to obtain

$$
\Theta^-\big(\Theta^+(b_2)X\big) = \Theta^+(X)\Theta^-(b_2) + \Theta^+(b_2)\Theta^-(X).
$$

Also,

$$
\begin{aligned}
P\Theta^-(X) + \Theta^-(X)P^T = A\Theta^-(X) + \Theta^-(X)A^T \\
&- \frac{1}{2}\big(\Theta^-(b_1)\Theta^-(b_1)\Theta^-(X) + \Theta^-(b_2)\Theta^-(b_2)\Theta^-(X) \\
&+ \Theta^-(X)\Theta^-(b_1)\Theta^-(b_1) + \Theta^-(X)\Theta^-(b_2)\Theta^-(b_2) \\
&+ \Theta^-(b_2)\Theta^+(b_1)\Theta^-(X) - \Theta^-(X)\Theta^+(b_1)\Theta^-(b_2) \\
&- \Theta^-(b_1)\Theta^+(b_2)\Theta^-(X) + \Theta^-(X)\Theta^+(b_2)\Theta^-(b_1)\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P\Theta^-(X) + \Theta^-(X)P^T - \Theta^-(PX) = A\Theta^-(X) &+ \Theta^-(X)A^T - \Theta^-(AX) \\
&- \Theta^-(b_1)\Theta^-(X)\Theta^-(b_1) - \Theta^-(b_2)\Theta^-(X)\Theta^-(b_2) \\
&- \Theta^-(b_1)\Theta^+(X)\Theta^-(b_2) + \Theta^-(b_2)\Theta^+(X)\Theta^-(b_1).
\end{aligned} \tag{5.14}
$$

From (5.7d), it is thus evident that P satisfies $P\Theta^-(X)+\Theta^-(X)P^T-\Theta^-(PX)=0$. Given that this equation is satisfied for all t, a direct application of Lemma 5.1 when $t=0$ shows that $P=\Theta^-(a)$ for a vector a as in (5.4), which implies (5.6c).

For the sufficiency part of the theorem, assume that (5.6a)–(5.6c) are true. By Lemma 5.1, (5.7a)–(5.7c) and (5.11c) are automatically satisfied. Equation (5.11d) it is easily obtained by computing $A+A^T$ using (5.6c). Furthermore, (5.11a) and (5.11b) are a direct result of (3.8f). The validity of (5.7d) is shown by computing $A\Theta^-(X)+\Theta^-(X)A^T-\Theta^-(AX)$. This calculation is analogous to the computation of (5.14), but in this case it is already known that $P=\Theta^-(a)$. Thus, (5.7d) is true given that (5.6a)–(5.6c) hold. To show that (5.11e) holds, one first computes $\Theta^+(AX)$ using (5.6c) as follows:

$$\begin{aligned}\Theta^+(AX) = {}& \Theta^+\left(\Theta^-(a)X\right)+\frac{1}{2}\left(\Theta^-(b_1)\Theta^+\left(\Theta^-(b_1)X\right)-\Theta^+\left(\Theta^-(b_1)X\right)\Theta^-(b_1)\right.\\ &\left.+\Theta^-(b_2)\Theta^+\left(\Theta^-(b_2)X\right)-\Theta^+\left(\Theta^-(b_2)X\right)\Theta^-(b_2)\right)\\ &+\frac{1}{2}\left(\Theta^-(b_2)\Theta^+\left(\Theta^+(b_1)X\right)-\Theta^+\left(\Theta^+(b_1)X\right)\Theta^-(b_2)\right.\\ &\left.-\Theta^-(b_1)\Theta^+\left(\Theta^+(b_2)X\right)+\Theta^+\left(\Theta^+(b_2)X\right)\Theta^-(b_1)\right).\end{aligned}$$

Now, from the fact that $\Theta^+(\cdot)=(\Theta^+(\cdot))^T$ and (3.8g), one has that

$$\Theta^+\left(\Theta^+(b_i)X\right)=\Theta^+(X)\Theta^+(b_2)-\Theta^-(b_2)\Theta^-(X)-\frac{2}{n}b_i^TXI+\frac{2}{n}Xb_i^T.$$

Therefore,

$$\begin{aligned}\Theta^+(AX) = {}& \Theta^+\left(\Theta^-(a)X\right)+\frac{1}{2}\left(\Theta^-(b_1)\Theta^+(X)\Theta^-(b_1)-\Theta^-(b_1)\Theta^-(b_1)\Theta^+(X)\right.\\ &-\Theta^+(X)\Theta^-(b_1)\Theta^-(b_1)+\Theta^-(b_1)\Theta^+(X)\Theta^-(b_1)\\ &+\Theta^-(b_2)\Theta^+(X)\Theta^-(b_2)-\Theta^-(b_2)\Theta^-(b_2)\Theta^+(X)\\ &\left.-\Theta^+(X)\Theta^-(b_2)\Theta^-(b_2)+\Theta^-(b_2)\Theta^+(X)\Theta^-(b_2)\right)\\ &+\frac{1}{2}\Bigg(\Theta^-(b_2)\Theta^+(b_1)\Theta^+(X)-\Theta^-(b_2)\Theta^-(X)\Theta^-(b_1)\\ &-\frac{2}{n}\Theta^-(b_2)b_1^TX+\frac{2}{n}\Theta^-(b_2)b_1X^T\\ &-\Theta^+(X)\Theta^+(b_1)\Theta^-(b_2)+\Theta^-(b_1)\Theta^-(X)\Theta^-(b_2)\\ &+\frac{2}{n}b_1^TX\,\Theta^-(b_2)-\frac{2}{n}Xb_1^T\Theta^-(b_2)\\ &-\Theta^-(b_1)\Theta^+(b_2)\Theta^+(X)+\Theta^-(b_1)\Theta^-(X)\Theta^-(b_2)\\ &+\frac{2}{n}\Theta^-(b_1)b_2^TX-\frac{2}{n}\Theta^-(b_1)b_2X^T\\ &+\Theta^+(X)\Theta^+(b_2)\Theta^-(b_1)-\Theta^-(b_2)\Theta^-(X)\Theta^-(b_1)\\ &-\frac{2}{n}b_2^TX\,\Theta^-(b_1)+\frac{2}{n}Xb_2^T\Theta^-(b_1)\Bigg).\end{aligned}$$

Also,

$$\begin{aligned}A\Theta^+(X)+\Theta^+(X)A^T = {}& \Theta^-(a)\Theta^+(X)+\Theta^+(X)\Theta^-(a)^T\\ &+\frac{1}{2}\left(\Theta^-(b_1)\Theta^-(b_1)\Theta^+(X)+\Theta^-(b_2)\Theta^-(b_2)\Theta^+(X)\right.\end{aligned}$$

$$+ \Theta^+(X)\Theta^-(b_1)\Theta^-(b_1) + \Theta^+(X)\Theta^-(b_2)\Theta^-(b_2)$$
$$+ \Theta^-(b_2)\Theta^+(b_1)\Theta^+(X) - \Theta^+(X)\Theta^+(b_1)\Theta^-(b_2)$$
$$- \Theta^-(b_1)\Theta^+(b_2)\Theta^+(X) + \Theta^+(X)\Theta^+(b_2)\Theta^-(b_1)\big).$$

Since $\frac{2}{n}\Theta^-(b_2)b_1X^T = A_0X^T$ and $\frac{2}{n}\Theta^-(b_2)b_1X^T = XA_0^T$, it then follows that

$$\begin{aligned} A\Theta^-(X) + \Theta^-(X)A^T - \Theta^+(AX) = {} & \Theta^-(a)\Theta^+(X) + \Theta^+(X)\Theta^-(a)^T - \Theta^+(\Theta^-(a)X) \\ & + \Theta^-(b_1)\Theta^+(X)\Theta^-(b_1) + \Theta^-(b_2)\Theta^+(X)\Theta^-(b_2) \\ & + \Theta^-(b_1)\Theta^-(X)\Theta^-(b_2) + \Theta^-(b_2)\Theta^-(X)\Theta^-(b_1) \\ & - \frac{1}{n}\left(A_0X^T + XA_0^T\right). \end{aligned} \tag{5.15}$$

Identity (3.8f) shows directly that $\Theta^-(a)\Theta^+(X) + \Theta^+(X)\Theta^-(a)^T - \Theta^+(\Theta^-(a)X) = 0$. Hence, (5.15) implies (5.11e), which completes the proof. ∎

6. Necessary and sufficient conditions for physical realizability of *n*-level quantum systems. In this section the main results of the paper concerned with physical realizability of the QSDE (4.6) having output equation (4.7) are developed. First we show that the matrices comprising the system formed by (4.6) and (4.7) can be written as in the next theorem. Next, we present the main result of the paper, which shows that the QSDE (4.6) equipped with the output equation (4.7) admits physical parameters (S, L, H) if and only if conditions (1.6a)–(1.6d) given in the introduction hold. This result is presented in Theorem 6.2. Furthermore, we show that these conditions guarantee the preservation of the commutation and anti-commutation relations of the underlying algebra $su(n)$ by verifying that conditions (1.6a)–(1.6d) imply the conditions of Theorem 5.2.

The explicit form of matrices $A_0, A, B_{1k}, B_{2k}, C_1$, and C_2 of a physically realizable QSDE in terms of the Hamiltonian and coupling operator is given next.

THEOREM 6.1. *Let $H = \alpha X$, with $\alpha^T \in \mathbb{R}^s$, and $L = \Gamma X$, with $\Gamma \in \mathbb{C}^{n_w \times s}$. Then the corresponding QSDEs* (4.6) *and* (4.7) *are such that*

$$A_0 = \frac{4\boldsymbol{i}}{n}\sum_{k=1}^{n_w}\Theta^-(\Gamma_k^\#)\Gamma_k^T, \tag{6.1a}$$

$$A = -2\Theta^-(\alpha) + \sum_{k=1}^{n_w}(R_k - \boldsymbol{i}Q_k), \tag{6.1b}$$

$$B_{1k} = \Theta^-\left(\boldsymbol{i}(\Gamma_k^\# - \Gamma_k)\right), \tag{6.1c}$$

$$B_{2k} = -\Theta^-(\Gamma_k + \Gamma_k^\#), \tag{6.1d}$$

$$C_1 = \Gamma + \Gamma^\#, \tag{6.1e}$$

$$C_2 = \boldsymbol{i}\left(\Gamma^\# - \Gamma\right), \tag{6.1f}$$

where

$$R_k := \Theta^-(\Gamma_k)\Theta^-(\Gamma_k^\#) + \Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k),$$
$$Q_k := \Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k).$$

Proof. The proof follows by direct application of Lemmas 3.2, 4.1, and 4.2. That is, (2.2) is rewritten using (4.5a)–(4.5e) as the following bilinear QSDE:

$$
\begin{aligned}
dX = & -2\Theta^-(\alpha)X\,dt + \frac{4\boldsymbol{i}}{n}\sum_{k=1}^{n_w}\Theta^-(\Gamma_k^\#)\Gamma_k^T\,dt \\
& + \sum_{k=1}^{n_w}\Big(\Theta^-(\Gamma_k)\Theta^-(\Gamma_k^\#) + \Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k)\Big)X\,dt \\
& - \boldsymbol{i}\sum_{k=1}^{n_w}\Big(\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k)\Big)X\,dt \\
& + \boldsymbol{i}\Big(\Theta^-(\Gamma_1^\# - \Gamma_1)X \cdots \Theta^-(\Gamma_{n_w}^\# - \Gamma_{n_w})X\Big)\,d\bar{W}_1 \\
& - \Big(\Theta^-(\Gamma_1 + \Gamma_1^\#)X \cdots \Theta^-(\Gamma_{n_w} + \Gamma_{n_w}^\#)X\Big)\,d\bar{W}_2.
\end{aligned} \tag{6.2}
$$

Also, as mentioned in section 2, the output fields $\bar{Y}_1$ and $\bar{Y}_2$ depend linearly on L, $L^\dagger$ and the input fields $\bar{W}_1$ and $\bar{W}_2$; i.e.,

$$
\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} \Gamma + \Gamma^\# \\ \boldsymbol{i}(\Gamma^\# - \Gamma) \end{pmatrix} X\,dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}. \tag{6.3}
$$

It is now easy to identify the matrices $A_0, A, B_{1k}, B_{2k}, C_1$, and C_2, which completes the proof. ◻

Remark. Note that all matrices involved in (6.3) are real. To confirm that, observe that $\Gamma^\# - \Gamma$ is purely imaginary and $\Gamma + \Gamma^\#$ is purely real. Now fix k and compute the real part of $\Theta^-(\Gamma_k)\Gamma_k^\dagger$ and $\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k)$. Given that $(\Theta^-(\Gamma_k)\Gamma_k^\dagger)^\# = -\Theta^-(\Gamma_k)\Gamma_k^\dagger$, one has that

$$
Re\{\Theta^-(\Gamma)\Gamma^\dagger\} = \frac{1}{2}\left(\Theta^-(\Gamma)\Gamma^\dagger + (\Theta^-(\Gamma)\Gamma^\dagger)^\#\right) = 0.
$$

Also,

$$
\begin{aligned}
& Re\{\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k)\} \\
& = \frac{1}{2}\Big(\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k) \\
& \quad + \Big(\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k)\Big)^\#\Big) = 0.
\end{aligned}
$$

Moreover, by direct inspection ${B_{ik}}^T = -B_{ik}$ for $i = 1, 2$ and $k = 1, \ldots, n_w$.

The next theorem establishes necessary and sufficient conditions for the physical realizability of the bilinear QSDE (4.6)–(4.7) in terms of the matrices (A_0, A, B_i, C_k).

THEOREM 6.2. *The system* (4.6) *with output equation* (4.7) *is physically realizable if and only if conditions* (1.6a)–(1.6d) *hold. In this case, the coupling matrix can be identified to be*

$$
\Gamma = \frac{1}{2}(C_1 + \boldsymbol{i}C_2), \tag{6.4}
$$

and α*, determining the system Hamiltonian, is given by*

$$
\alpha = \frac{1}{4n}\,\mathrm{vec}\left(A^T - A + \frac{1}{2}\sum_{k=1}^{n_w}\left(\{B_{1k}, \Theta^+((C_1)_k)\} + \{B_{2k}, \Theta^+((C_2)_k)\}\right)\right)^T F. \tag{6.5}
$$

Proof. We first prove necessity. Assuming that (4.6) and (4.7) are physically realizable implies that (6.1a)–(6.1f) are satisfied. Hence, conditions (1.6b)–(1.6c) hold. This is easy to verify by direct calculation.

Condition (1.6a) is obtained by rewriting (6.1a) using (6.1e) and (6.1f). That is,

$$
\begin{aligned}
A_0 &= -\frac{4\boldsymbol{i}}{n}\sum_{k=1}^{n_w}\Theta^-(\Gamma_k)\Gamma_k^\dagger = \frac{1}{n}\sum_{k=1}^{n_w}\Theta^-((C_2)_k - \boldsymbol{i}(C_1)_k)((C_1)_k + \boldsymbol{i}(C_2)_k)^\dagger \\
&= \frac{1}{n}\sum_{k=1}^{n_w}(B_{1k} + \boldsymbol{i}B_{2k})((C_1)_k + \boldsymbol{i}(C_2)_k)^\dagger .
\end{aligned}
$$

Now, one has that

$$
B_{1k}B_{1k}^T = \Theta^-(\Gamma^\#)\Theta^-(\Gamma^\#) - \Theta^-(\Gamma^\#)\Theta^-(\Gamma) - \Theta^-(\Gamma)\Theta^-(\Gamma^\#) + \Theta^-(\Gamma)\Theta^-(\Gamma).
$$

Similarly,

$$
B_{2k}B_{2k}^T = -\Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k) - \Theta^-(\Gamma_k)\Theta^-(\Gamma_k^\#) - \Theta^-(\Gamma_k)\Theta^-(\Gamma_k).
$$

Thus, $B_{1k}B_{1k}^T + B_{2k}B_{2k}^T = -2R_k$. One can now rewrite A in terms of α, B_{1k}, and B_{2k} as

$$
A = -\,2\Theta^-(\alpha) - \frac{1}{2}\sum_{k=1}^{n_w}\left(B_{1k}B_{1k}^T + B_{2k}B_{2k}^T\right) - \boldsymbol{i}Q_k. \tag{6.6}
$$

Similarly,

$$
A^T = 2\Theta^-(\alpha) - \frac{1}{2}\sum_{k=1}^{n_w}\left(B_{1k}B_{1k}^T + B_{2k}B_{2k}^T\right) - \boldsymbol{i}Q_k^T. \tag{6.7}
$$

Adding (6.6) and (6.7) gives $A + A^T = -\sum_{k=1}^{n_w}\left(B_{1k}B_{1k}^T + B_{2k}B_{2k}^T\right) - \boldsymbol{i}\left(Q_k + Q_k^T\right)$. The (i,j) component of $Q_k + Q_k^T$ is computed as

$$
\begin{aligned}
\left(Q_k + Q_k^T\right)_{ij} &= \Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#) - \Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k) \\
&\quad - \Theta^+(\Gamma_k^\#)\Theta^-(\Gamma_k) + \Theta^+(\Gamma_k)\Theta^-(\Gamma_k^\#) \\
&= -\,\Gamma_k^T F_i D_j \Gamma_k^\# + \Gamma_k^\dagger F_i D_j \Gamma_k - \Gamma_k^\dagger D_i F_j \Gamma_k + \Gamma_k^T D_i F_j \Gamma_k^\# .
\end{aligned}
$$

Note that every summand is a scalar, which is equal to its transpose. By (3.3c) and (3.3d), it follows that

$$
\begin{aligned}
\left(Q_k + Q_k^T\right)_{ij} &= \Gamma_k^T(D_iF_j + D_jF_i)\Gamma_k^\# + \Gamma_k^T(F_iD_j + F_jD_i)\Gamma_k^\# \\
&= 2\sum_{k=1}^{s} d_{ijk}\Gamma_k^T F_k \Gamma_k^\# = 2\left(\Theta^+(\Theta^-(\Gamma_k)\Gamma_k^\#)\right)_{ij} = \frac{n\boldsymbol{i}}{2}\left(\Theta^+(A_0)\right)_{ij} .
\end{aligned}
$$

Therefore, $A + A^T + \sum_{i,k=1}^{2,n_w} B_{ik}B_{ik}^T = \frac{n}{2}\Theta^+(A_0)$, which is condition (1.6d).

For the sufficiency part of the theorem, one needs to show that if conditions (1.6a)–(1.6d) of the theorem are satisfied, then there exists a vector α and a matrix Γ such that the system (4.6)–(4.7) is physically realizable. Let

$$
\Theta^-(\alpha) := \frac{1}{4}\left(A^T - A + \frac{1}{2}\sum_{k=1}^{n_w}\left(\{B_{1k}, \Theta^+((C_1)_k)\} + \{B_{2k}, \Theta^+((C_2)_k)\}\right)\right). \tag{6.8}
$$

It is trivial to check that the right-hand side of (6.8) is antisymmetric and hence this equation uniquely defines α via (3.4a). Also, let

$$\Gamma = \frac{1}{2}(C_1 + \boldsymbol{i}C_2). \tag{6.9}$$

Then, Q_k can be written in terms of B_1, B_2, C_1, and C_2 as follows:

$$Q_k = -\frac{\boldsymbol{i}}{2}\left(\Theta^-((C_1)_k)\Theta^+((C_2)_k) - \Theta^-((C_2)_k)\Theta^+((C_1)_k)\right).$$

From (1.6b), (1.6c) and recalling that $B_{1k} = \Theta^-((C_2)_k)$ and $B_{2k} = -\Theta^-((C_1)_k)$, it follows that

$$Q_k = \frac{\boldsymbol{i}}{2}\left(B_{2k}\Theta^+((C_2)_k) + B_{1k}\Theta^+((C_1)_k)\right). \tag{6.10}$$

Then,

$$\begin{aligned} Q_k - Q_k^T &= +\frac{\boldsymbol{i}}{2}\left(B_{2k}\Theta^+((C_2)_k) + B_{1k}\Theta^+((C_1)_k)\right) \\ &\quad + \frac{\boldsymbol{i}}{2}\left(\Theta^+((C_2)_k)B_{2k} + \Theta^+((C_1)_k)B_{1k}\right) \\ &= \frac{\boldsymbol{i}}{2}\left(\{B_{1k}, \Theta^+((C_1)_k)\} + \{B_{2k}, \Theta^+((C_2)_k)\}\right). \end{aligned}$$

Similarly, it is simple to write R_k in terms of C_1 and C_2. That is,

$$R_k = \frac{1}{2}\left(\Theta^-((C_1)_k)\Theta^-((C_1)_k) + \Theta^-((C_2)_k)\Theta^-((C_2)_k)\right).$$

It is clear that R_k is symmetric. From (1.6b) and (1.6c), one obtains that

$$R_k = \Theta^-(\Gamma_k)\Theta^-(\Gamma_k^\#) + \Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k) = -\frac{1}{2}\left(B_{1k}B_{1k}^T + B_{2k}B_{2k}^T\right).$$

From (1.6a) and (6.9), $\Theta^+(A_0) = -\frac{2\boldsymbol{i}}{n}\left(Q_k + Q_k^T\right)$. Since (1.6d) implies $A^T = -A - \sum_{i,k=1}^{2,n_w} B_{ik}B_{ik}^T + \frac{n}{2}\Theta^+(A_0)$, one can use (6.8) to obtain (6.1b). Moreover, using (3.7), (6.8) and applying the stacking operator to $\Theta^-(\alpha)$, α is explicitly obtained as $\left(F^T \operatorname{vec}(\Theta^-(\alpha))\right)^T = \left(F^T F \alpha^T\right)^T = n\alpha$. Hence,

$$\alpha = \frac{1}{4n}\operatorname{vec}\left(A^T - A + \frac{1}{2}\sum_{k=1}^{n_w}\{B_{1k}, \Theta^+((C_1)_k)\} + \{B_{2k}, \Theta^+((C_2)_k)\}\right)^T F,$$

which completes the proof. ∎

Remark. Note that the above physical realizability conditions do not require the computation of the Hamiltonian (6.5), which depends on the structure constants d and f, in order to know whether the system given by (4.6) and (4.7) is physically realizable.

Finally, the next theorem shows that a physically realizable system does indeed preserve the commutation and anticommutation relations of $su(n)$ as expected from the unitary evolution of the system described by (I, L, H).

THEOREM 6.3. *A physically realizable system* (4.6)–(4.7) *satisfies the conditions of Theorem* 5.2.

Proof. Consider, without loss of generality, that $n_w = 1$. Define $C = \frac{1}{2}(C_1 + \boldsymbol{i}C_2)$. This means that C is playing the role of Γ in Theorem 6.1. Using conditions (1.6b) and (1.6c), the fact that $B_1 = \Theta^-(b_1)$ and $B_2 = \Theta^-(b_2)$ follows directly by considering $C_2 = b_1^T$ and $C_1 = -b_2^T$. Next, one has from (1.6a) that

$$A_0 = \frac{1}{n}\left(\frac{1}{2}(B_1 + \boldsymbol{i}B_2)(C_1 + \boldsymbol{i}C_2)^\dagger\right) = -\frac{2}{n}\boldsymbol{i}\Theta^-(C)C^\dagger.$$

Now, from (3.8d), $\Theta^-(A_0)$ is given by

$$\frac{n}{2}\Theta^-(A_0) = -2i\left(\Theta^-(C)\Theta^-(C^\dagger) - \Theta^-(C^\dagger)\Theta^-(C)\right).$$

Since B_i is antisymmetric, it follows that

$$B_1B_2^T - B_2B_1^T = -2\boldsymbol{i}(\Theta^-(C)\Theta^-(C^\dagger) - \Theta^-(C^\dagger)\Theta^-(C)).$$

Therefore, (5.6b) holds. Since $\Theta^-(\cdot)$ is linear, and (6.10) holds due to physical realizability, one can rewrite (6.1b) as

$$\begin{aligned} A =& -2\Theta^-(\alpha) + R - \boldsymbol{i}Q = \Theta^-(-2\alpha) + R - \boldsymbol{i}Q \\ =& \Theta^-(a) - \frac{1}{2}\left(B_1B_1^T + B_2B_2^T\right) - \boldsymbol{i}\left(\frac{\boldsymbol{i}}{2}\left(B_2\Theta^+((C_2)) + B_1\Theta^+((C_1))\right)\right) \\ =& \Theta^-(a) - \frac{1}{2}\left(B_1B_1^T + B_2B_2^T\right) + \frac{1}{2}\left(B_2\Theta^+((b_1)) - B_1\Theta^+((b_2))\right), \end{aligned}$$

where $a := -2\alpha$. Hence, (5.6c) holds, which concludes the proof. ∎

Note that a QSDE, as given by (4.6) and (4.7), describes an open n-level quantum system when it satisfies the physical realizability conditions. This theorem shows in particular that the physical realizability conditions ensure preservation of the commutation and anticommutation relations of $su(n)$ over time.

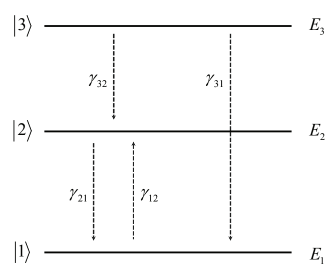

FIG. 1. *Three-level system diagram.*

Example 1. In what follows, an example is provided as a proof of concept in order to illustrate the physical realizability conditions. Consider the three-level quantum system in Figure 1, where relaxation and excitation of populations among energy levels are due to an interaction with a Boson field of quantum oscillators. Let $\mathfrak{H} = \mathbb{C}^3$, and consider the following vector of self-adjoint operators in $\mathfrak{T}(\mathbb{C}^3)(= \mathbb{C}^{3\times3})$:

(6.11) $$X = (u_{12}, u_{13}, u_{23}, v_{12}, v_{13}, v_{23}, w_1, w_2)^T,$$

where u_{ij}, v_{ij}, and w_k were given in (3.1). To apply the results of this paper, recall the only requirements for the elements of x is that they are self-adjoint and expanded by a complete set of generators (Gell–Mann matrices in this case). A system evolving with these system variables is in general a three-level system, and its (S, L, H) description is commonly given in terms of raising and lowering operators σ_{ij}^+ and σ_{ij}^- whose actions on quantum states are $\sigma_{ij}^+ |i\rangle = |j\rangle$ and $\sigma^- |j\rangle = |i\rangle$ for $i < j$, and their relationship with respect to (6.11) is

$$u_{ij} = \sigma_{ij}^+ + \sigma_{ij}^-, \quad v_{ij} = \boldsymbol{i}(\sigma_{ij}^- - \sigma_{ij}^+), \quad \text{and} \quad \sigma_{z_i} = w_i. \tag{6.12}$$

Physically, σ_{ij}^+ corresponds to the creation of a quanta of energy in the system, while σ_{ij}^- corresponds to destruction of quanta. Note also that σ_{ij}^+ is the adjoint of σ_{ij}^- ($\sigma_{ij}^+ = (\sigma_{ij}^-)^*$). Based on the previous argument, the three-level system in Figure 1 has the following (S, L, H) parameterization:

$$S = I, \quad L = \sqrt{\gamma_{12}}\sigma_{12}^+ + \sum_{\substack{i,j=1\\ i<j}}^{3} \sqrt{\gamma_{ji}}\sigma_{ij}^-, \quad H = \frac{(E_2 - E_1)}{2}\sigma_{z1} - \frac{(E_1 + E_2 - 2E_3)}{2\sqrt{3}}\sigma_{z2},$$

where $\gamma_{ij} > 0$ is the rate of population change from energy level $|i\rangle$ to energy level $|j\rangle$. Observe that S and H are already in self-adjoint form and that, using (6.12), the self-adjoint form of L is $L = \Gamma x$, where

$$\Gamma = \left(\frac{(\sqrt{\gamma_{12}} + \sqrt{\gamma_{21}})}{2}, \frac{\sqrt{\gamma_{31}}}{2}, \frac{\sqrt{\gamma_{32}}}{2}, \frac{\boldsymbol{i}(\sqrt{\gamma_{12}} - \sqrt{\gamma_{21}})}{2}, -\frac{\boldsymbol{i}\sqrt{\gamma_{31}}}{2}, -\frac{\boldsymbol{i}\sqrt{\gamma_{32}}}{2}, 0, 0 \right).$$

The state space representation of the system is then given by the following matrices:

$$A = \begin{pmatrix}
\frac{2\gamma_{12}'\gamma_{21}' - \gamma - \gamma_{12}}{2} & 0 & 0 & 2\Delta E_{12} & 0 & 0 & 0 & -\frac{\sqrt{3}\gamma_{31}'\gamma_{32}'}{3} \\
\gamma_{12}'\gamma_{31}' & \frac{\gamma_{32} - \gamma}{2} & -\frac{\gamma_{31}'\gamma_{32}'}{2} & 0 & 2\Delta E_{13} & 0 & -\gamma_{12}'\gamma_{32}' & \frac{\sqrt{3}\gamma_{12}'\gamma_{32}'}{3} \\
\gamma_{21}'\gamma_{32}' & -\frac{\gamma_{31}'\gamma_{32}'}{2} & -\frac{\gamma_{12} + \gamma_{32}}{2} & 0 & 0 & 2\Delta E_{23} & \gamma_{21}'\gamma_{31}' & \frac{\sqrt{3}\gamma_{21}'\gamma_{31}'}{3} \\
2\Delta E_{21} & 0 & 0 & \frac{\gamma_{21} - \gamma - (\gamma_{12}' + \gamma_{21}')^2}{2} & 0 & 0 & 0 & 0 \\
0 & 2\Delta E_{31} & 0 & -\gamma_{12}'\gamma_{31}' & \frac{\gamma_{32} - \gamma}{2} & -\frac{\gamma_{31}'\gamma_{32}'}{2} & 0 & 0 \\
0 & 0 & 2\Delta E_{32} & \gamma_{21}'\gamma_{32}' & -\frac{\gamma_{31}'\gamma_{32}'}{2} & -\frac{\gamma_{12} + \gamma_{32}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma + \gamma_{21} + 2\gamma_{12}}{2} & \frac{\sqrt{3}(2\gamma_{12} - 2\gamma_{21} - \gamma_{31} + \gamma_{32})}{6} \\
-\sqrt{3}\gamma_{31}'\gamma_{32}' & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}(\gamma_{32} - \gamma_{31})}{2} & \frac{\gamma_{21} - \gamma}{2}
\end{pmatrix},$$

$$A_0 = \left(-\frac{2\gamma_{31}'\gamma_{32}'}{3}, \frac{2\gamma_{12}'\gamma_{32}'}{3}, \frac{2\gamma_{21}'\gamma_{31}'}{3}, 0, 0, 0, \frac{2\gamma_{12} - 2\gamma_{21} - \gamma_{31} + \gamma_{32}}{3}, \frac{\sqrt{3}(\gamma_{31} + \gamma_{32})}{3}, \right)^T,$$

$$C_1 = \begin{pmatrix} (\gamma_{12}' + \gamma_{21}') & \gamma_{31}' & \gamma_{32}' & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 0 & 0 & 0 & (\gamma_{12}' - \gamma_{21}') & -\gamma_{31}' & -\gamma_{32}' & 0 & 0 \end{pmatrix},$$
$$B_1 = \boldsymbol{i}(\Theta^-(\Gamma^\# - \Gamma)), \quad \text{and} \quad B_2 = -\Theta^-(\Gamma + \Gamma^\#),$$

where $\gamma = \gamma_{21} + \gamma_{31} + \gamma_{32}$ and $\gamma_{ij}' = \sqrt{\gamma_{ij}}$. Note that these matrices satisfy

$$A + A^T + (B_1 B_1^T + B_2 B_2^T) = \frac{3}{2}\Theta^+(A_0)$$

and also the other conditions for physical realizability in Theorem 6.2.

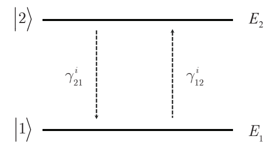

FIG. 2. *Two-level system diagram.*

Example 2. In the following example, physical realizability is illustrated by showing that the concatenation of two noninteracting two-level systems (qubits) is physically realizable by the tensor product of the corresponding QSDEs. Then, the composite system is used to recover the concatenation product of quantum systems in [12]. The diagram of a two-level open quantum system is shown in Figure 2.

Each qubit is characterized by

$$(S_i, L_i, H_i) = \left(\hat{I}_2, \frac{\sqrt{\gamma_{12}^i + \gamma_{21}^i}}{2}\sigma_x + \boldsymbol{i}\frac{\sqrt{\gamma_{12}^i - \gamma_{21}^i}}{2}\sigma_y, \Delta E_{21}\sigma_z \right)$$

with $\hat{I}_2$ the identity unitary operator on $SU(2)$, $\Delta E_{21} := E_2 - E_1$, and γ_{12}^i and γ_{21}^i the rates of population change from energy levels $1 \to 2$ and $2 \to 1$ for system i, respectively. As mentioned in Example 1, n-level system diagrams are better understood in terms of lowering and rising operators. The operator L in terms of the lowering and raising operators is $L = \sqrt{\gamma_{21}}\sigma_{12}^- + \sqrt{\gamma_{12}}\sigma_{12}^+$, where one can see explicitly how the change rates relate to the energy levels. Similarly as in Example 1, the QSDEs for the qubit systems have the physically realizable state space models

$$dX_i = A_{0,i}\,dt + A_iX_i\,dt + B_{1,i}X_i\,dW_{1,i} + B_{1,i}X_i\,dW_{1,i}, \quad i = 1, 2, \tag{6.13}$$

and output equations $dY_i = \begin{pmatrix} C_{1,i} \\ C_{2,i} \end{pmatrix} X_i\,dt + \begin{pmatrix} dW_{1,i} \\ dW_{2,i} \end{pmatrix}$ for $i = 1, 2$, where

$$A_{0,i} = \begin{pmatrix} 0 \\ 0 \\ \gamma_{12}^i - \gamma_{21}^i \end{pmatrix}, \qquad A_i = \begin{pmatrix} -\frac{(\sqrt{\gamma_{12}^i} - \sqrt{\gamma_{21}^i})^2}{2} & 2\Delta E_{12} & 0 \\ 2\Delta E_{21} & -\frac{(\sqrt{\gamma_{12}^i} + \sqrt{\gamma_{21}^i})^2}{2} & 0 \\ 0 & 0 & -(\gamma_{12}^i + \gamma_{21}^i) \end{pmatrix},$$

$$B_{1,i} = \begin{pmatrix} 0 & 0 & \sqrt{\gamma_{21}^i} - \sqrt{\gamma_{12}^i} \\ 0 & 0 & 0 \\ \sqrt{\gamma_{12}^i} - \sqrt{\gamma_{21}^i} & 0 & 0 \end{pmatrix}, B_{2,i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -(\sqrt{\gamma_{12}^i} + \sqrt{\gamma_{21}^i}) \\ 0 & \sqrt{\gamma_{12}^i} + \sqrt{\gamma_{21}^i} & 0 \end{pmatrix},$$

$$C_{1,i} = \left(\sqrt{\gamma_{12}^i} + \sqrt{\gamma_{21}^i}, 0, 0 \right), \qquad C_{2,i} = \left(0, \sqrt{\gamma_{12}^i} - \sqrt{\gamma_{21}^i}, 0 \right).$$

The state space concatenation of the qubits is realized by the tensor product of dX_1 and dX_2 in (6.13). This gives a QSDE of the form

$$dX = A_0\,dt + AX\,dt + \sum_{i=1}^{2} B_1^{\{i\}}X\,dW_{1,i} + B_2^{\{i\}}X\,dW_{2,i} \;\text{ with }\; X := \begin{pmatrix} X_1 \otimes I \\ I \otimes X_2 \\ X_1 \otimes X_2 \end{pmatrix}.$$

It is observed that all coefficients of the terms involving $X_1 \otimes X_2$ are zero; therefore the vector system variables of interest can be reduced to

$$X = \begin{pmatrix} X_1 \otimes I \\ I \otimes X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Note that since X_1 and X_2 are in different sides of the tensor product the components of X_1 and X_2 commute between each other. Having established the commutativity of X_1 and X_2 allows us to omit the tensor product without further discussion. This fact provides a great deal of simplification when writing the commutation relations of the composite system. In this regard,

$$[X, X^T] = \begin{pmatrix} \Theta^-(X_1) & 0_{3\times 3} \\ 0_{3\times 3} & \Theta^-(X_2), \end{pmatrix} =: \Theta(X). \tag{6.14}$$

It then follows that the specific structure of a two-qubit system makes the anomaly tensor d_{ijk} zero. As a consequence, the physical realizability condition (1.6a), for instance, reduces to $A + A^T + \sum_{i=1}^{2} \left(B_{1,i}B_{1,i}^T + B_{2,i}B_{2,i}^T\right) = 0$. Thus, the commutation relations of the composite system reduces to the commutation relations of the individual systems. Moreover, the composite system in $SU(2) \times SU(2)$ is easily embedded into $SU(4)$ by a similarity transformation that takes X to a description in terms of u_{ij}, v_{ij}, and w_k for $1 \le i < j \le 4$ and $1 \le k \le 3$, which are the generators of $SU(4)$ following (3.1). Also, the transformation of the raising and lowering operators into $SU(4)$ follows (6.12). Therefore, it is a simple exercise to compute matrices A_0, A, $B_1^{\{1\}}, B_2^{\{1\}}, B_1^{\{2\}}, B_2^{\{2\}}, C_1$, and C_2 in $SU(4)$ coordinates, and it can be verified that the physical realizability conditions (1.6a)–(1.6d) are satisfied by the composite QSDE. Recall that these conditions are simplified due to the commutation relations of the algebra of $SU(2) \otimes SU(2)$ and that $d_{ijk} = 0$ for any indices. One can now use (6.4) and (6.5) to obtain the (S, L, H) parametrization of the composite system. That is, one can compute α and Γ corresponding to $H = \alpha X$ and $L = \Gamma X$.

Given the commutation relations for a noninteracting two-qubit system in (6.14), (3.6) needs to be adapted to the case of $\Theta(\cdot)$ in (6.14). First define

$$F'^T := \begin{pmatrix} F_1 & 0_{3\times3} & F_2 & 0_{3\times3} & F_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & F_1 & 0_{3\times3} & F_2 & 0_{3\times3} & F_3 \end{pmatrix}.$$

It is now a simple exercise to check that $\mathrm{vec}\,(\Theta(\alpha)) = F'\alpha$ holds. Using (3.7), it is also easy to check that $F'^T F' = 2I_{6\times 6}$. From (6.5), (6.14), and the fact that d_{ijk} for the composite system is zero, one has that

$$\Theta(\alpha) = \frac{1}{4}(A + A^T) = \begin{pmatrix} \mathcal{E}_1 & 0_{3\times3} \\ 0_{3\times3} & \mathcal{E}_2 \end{pmatrix}, \quad \text{where} \quad \mathcal{E}_i := \begin{pmatrix} 0 & \Delta E_{12}^{\{i\}} & 0 \\ \Delta E_{21}^{\{i\}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for $i = 1, 2$, and A has been transformed back to $SU(2) \otimes SU(2)$ coordinates. Thus,

$$\alpha = \frac{1}{8}\,\mathrm{vec}(A + A^T)F' = \begin{pmatrix} 0 & 0 & \Delta E_{21}^{\{1\}} & 0 & 0 & \Delta E_{21}^{\{2\}} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}.$$

The composite Hamiltonian is then

$$H = \alpha X = \alpha_1 X_1 + \alpha_2 X_2 = H_1 + H_2. \tag{6.15}$$

Similarly, the coupling operator matrix Γ is

$$\Gamma = \frac{1}{2}(C_1 + \boldsymbol{i} C_2) = \begin{pmatrix} \frac{\sqrt{\gamma_{12}^1} + \sqrt{\gamma_{21}^1}}{2} & \boldsymbol{i}\frac{\sqrt{\gamma_{12}^1} - \sqrt{\gamma_{21}^1}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\gamma_{12}^2} + \sqrt{\gamma_{21}^2}}{2} & \boldsymbol{i}\frac{\sqrt{\gamma_{12}^2} - \sqrt{\gamma_{21}^2}}{2} & 0 \end{pmatrix}.$$

Note that if Γ is transformed into its ladder operator form (raising and lowering operators), then the coupling operator matrix is

$$\Gamma_{ladder} = \begin{pmatrix} \sqrt{\gamma_{12}^1} & \sqrt{\gamma_{21}^1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma_{12}^2} & \sqrt{\gamma_{21}^2} & 0 \end{pmatrix}.$$

Here one can see clearly the decoupled channels of the noninteracting qubits. The composite coupling operator can then be written as

$$L = \Gamma X = \begin{pmatrix} \Gamma_1 & 0_{1\times 3} \\ 0_{1\times 3} & \Gamma_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}. \tag{6.16}$$

In [12], it was shown that the concatenation of noninteracting systems can be achieved by means of the concatenation product of (S, L, H) parametrizations of the component systems. That is, the concatenation of $(\hat{I}_2, L_1, H_1)$ and $(\hat{I}_2, L_2, H_2)$ is given by

$$(S, L, H) = (\hat{I}_2, L_1, H_1) \boxplus (\hat{I}_2, L_2, H_2) = \left(\hat{I}_4, \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, H_1 + H_2\right)$$

with $\hat{I}_4$ the identity unitary operator on $SU(4)$. It is now obvious that H and L in the previous formula coincide with (6.15) and (6.16), respectively. This agreement confirms the hypothesis that direct tensoring of noninteracting qubits is equivalent to the concatenation of their corresponding (S, L, H) parametrization and, therefore, indicates the physical realizability of direct tensoring of two noninteracting qubits. This example can be easily extended to arbitrarily many qubits.

7. Conclusions. A full description of QSDEs of the form (4.6)–(4.7) as n-level open quantum systems was provided via necessary and sufficient conditions for physical realizability of such QSDEs. These results explicitly use $su(n)$, the Lie algebra of $SU(n)$, and in particular algebraic expressions involving the anomaly tensor d are employed. Also, it was shown that the physical realizability conditions imply the preservation of the commutation and anticommutation relations of $su(n)$. Under the physical realizability conditions, the system given by (4.6) and (4.7) describes an open n-level quantum system. Moreover, it was observed that the relationship between these two conditions is that physically realizable QSDEs require a specific coupling between the evolution equation and the output equation of the system.

Appendix. The proofs of all lemmas in the paper are provided in this section.

Proof of Lemma 3.2. Using (3.4a), one can decompose the left-hand side of (3.8a) in terms of the matrices F_i. The ith component of the vector $\Theta^-(\beta)\gamma$ is written as

$$\beta^T F_i^T \gamma = -\sum_{k,l=1}^{s} \beta_l f_{ilk} \gamma_k = \sum_{k,l=1}^{s} \gamma_k f_{ikl} \beta_l,$$

which implies $\beta^T F_i^T \gamma = -\gamma^T F_i^T \beta$. Therefore $\Theta^-(\beta)\gamma = -\Theta^-(\gamma)\beta$.

Using (3.4b), a similar procedure is applied to the identity (3.8b) in terms of the matrices D_i. The ith component is given by

$$\beta^T D_i \gamma = \sum_{k,l=1}^{s} \beta_l d_{ilk} \gamma_k = \sum_{k,l=1}^{s} \gamma_k d_{ikl} \beta_l,$$

which gives $\beta^T D_i \gamma = \gamma^T D_i \beta$. Thus, $\Theta^+(\beta)\gamma = \Theta^+(\gamma)\beta$. The identity (3.8c) is true since $f_{ijj} = 0$ for all i and j, and

$$\begin{aligned}\sum_{k,l=1}^{s} \beta_l f_{ilk} \beta_k &= \sum_{\substack{k,l=1\\ k\neq l}}^{s} \beta_l f_{ilk} \beta_k = \sum_{k<l} \beta_l f_{ilk} \beta_k + \sum_{k>l} \beta_l f_{ilk} \beta_k \\ &= \sum_{k<l} \beta_l f_{ilk} \beta_k - \sum_{k<l} \beta_l f_{ilk} \beta_k = 0,\end{aligned}$$

where the negative sign in the last summand was obtained because of the antisymmetry of f_{ilk}. The (i,j) component of the left-hand side of the identity (3.8d) is $(\Theta^-(\Theta^-(\beta)\gamma))_{ij} = \sum_{k=1}^{s} f_{jik} \beta^T F_k \gamma$, which by the antisymmetry of f_{jik} and (3.3a) gives

$$\begin{aligned}\left(\Theta^-\left(\Theta^-(\beta)\gamma\right)\right)_{ij} &= -\sum_{k=1}^{s} f_{ijk} \beta^T F_k \gamma \\ &= \beta^T \left(-\sum_{k=1}^{s} f_{ijk} F_k\right) \gamma \\ &= \beta^T F_i^T F_j^T \gamma - \gamma^T F_i^T F_j^T \beta \\ &= \left(\Theta^-(\beta)\Theta^-(\gamma) - \Theta^-(\gamma)\Theta^-(\beta)\right)_{ij} \\ &= \left([\Theta^-(\beta), \Theta^-(\gamma)]\right)_{ij}.\end{aligned}$$

Similarly, decomposing the (i,j) component of the left-hand side of (3.8e) and using (3.3b) gives

$$\begin{aligned}\left(\Theta^-\left(\Theta^+(\beta)\gamma\right)\right)_{ij} &= -\beta^T \left(-\sum_{k=1}^{s} f_{ijk} D_k\right) \gamma \\ &= -\beta^T [F_i, D_j] \gamma \\ &= \beta^T F_i^T D_j \gamma + \gamma^T F_i^T D_j \beta \\ &= \left(\Theta^-(\beta)\Theta^+(\gamma) + \Theta^-(\gamma)\Theta^+(\beta)\right)_{ij}.\end{aligned}$$

Again, decomposing the (i,j) component of the left-hand side of (3.8f) and using (3.3c) gives

$$\begin{aligned}\left(\Theta^+\left(\Theta^-(\beta)\gamma\right)\right)_{ij} &= -\beta^T \left(\sum_{k=1}^{s} d_{ijk} F_k\right) \gamma \\ &= \beta^T F_i^T D_j \gamma - \gamma^T D_i F_j^T \beta \\ &= \left(\Theta^-(\beta)\Theta^+(\gamma) - \Theta^+(\gamma)\Theta^-(\beta)\right)_{ij} \\ &= \left([\Theta^-(\beta), \Theta^+(\gamma)]\right)_{ij}.\end{aligned}$$

Applying the same procedure but using (3.3d) instead gives

$$\left(\Theta^+\left(\Theta^-(\beta)\gamma\right)\right)_{ij} = -\beta^T \left(\sum_{k=1}^{s} d_{ijk} F_k\right) \gamma$$

$$
\begin{aligned}
&= \beta^T D_i F_j^T \gamma - \gamma^T F_i^T D_j \beta \\
&= \left(\Theta^+(\beta)\Theta^-(\gamma) - \Theta^-(\gamma)\Theta^+(\beta)\right)_{ij} \\
&= \left([\Theta^+(\beta), \Theta^-(\gamma)]\right)_{ij}.
\end{aligned}
$$

Finally, the (i,j) component of (3.8g) is $(\Theta^+(\Theta^+(\beta)\gamma))_{ij} = \sum_{k=1}^{s} d_{jik}\beta^T D_k \gamma$, which by the symmetry of d_{jik} and (3.3e) gives

$$
\begin{aligned}
\left(\Theta^+\left(\Theta^+(\beta)\gamma\right)\right)_{ij} &= \beta^T \left(\sum_{k=1}^{s} d_{ijk} D_k\right)\gamma \\
&= \beta^T \left(D_i D_j - F_j F_i\right)\gamma - \frac{2}{n}\left(\delta_{ij}\sum_{m,l=1}^{s}\beta_m \delta_{ml}\gamma_l - \sum_{m,l=1}^{s}\beta_m\delta_{im}\delta_{jl}\gamma_l\right) \\
&= \beta^T D_i D_j \gamma - \gamma^T F_i^T F_j^T \beta - \frac{2}{n}\left(\beta^T\gamma\delta_{ij} - \beta_i\gamma_j\right) \\
&= \left(\Theta^+(\beta)\Theta^+(\gamma) - \Theta^-(\gamma)\Theta^-(\beta)\right)_{ij} - \frac{2}{n}\left(\beta^T\gamma I - \beta\gamma^T\right)_{ij},
\end{aligned}
$$

which completes the proof. ∎

Proof of Lemma 3.3. The identity (3.9a) is proved by multiplying directly $\mathbb{1}_\otimes$ and F. Therefore,

$$
\mathbb{1}_\otimes F = -\begin{pmatrix} \mathbb{1}_{11} & \cdots & \mathbb{1}_{s1} \\ \vdots & \ddots & \vdots \\ \mathbb{1}_{1s} & \cdots & \mathbb{1}_{ss} \end{pmatrix}\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} = -\begin{pmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_s \end{pmatrix} \quad \text{with} \quad \tilde{F}_i := \begin{pmatrix} (F_1)_i \\ \vdots \\ (F_s)_i \end{pmatrix}
$$

and $(F_i)_j = (f_{ij1}, \ldots, f_{ijs})$. Observe that $(F_i)_j = -(F_j)_i$ since f_{ijk} is an antisymmetric tensor. Therefore, $\tilde{F}_i = F_i^T$, which proves the identity. An analogous procedure is used to show identity (3.9b), but in this case the permutation of indices in d_{ijk} does not produce a negative sign due to the fact that d_{ijk} is a completely symmetric tensor. For (3.9c), one employs (3.9a) as follows:

$$
F^T(A\otimes B)F = (-\mathbb{1}_\otimes F)^T(A\otimes B)(-\mathbb{1}_\otimes F) = F^T(\mathbb{1}_\otimes(A\otimes B)\mathbb{1}_\otimes)F = F^T(B\otimes A)F.
$$

Finally, the identity (3.9d) is proved similarly but using (3.9b) instead. ∎

Proof of Lemma 4.1. The goal is to rewrite (4.4a) in terms of $[X, X^T]$ and $\{X, X^T\}$ in order to apply (4.1a) and (4.1b). Indeed,

$$
\begin{aligned}
[X, (AX)^T] &= \left(XX^TA_1^T - ((A_1X)X^T)^T \cdots XX^TA_{n_w}^T - ((A_1X)X^T)^T\right) \\
&= \left(XX^TA_1^T - (XX^T)^TA_1^T \cdots XX^TA_{n_w}^T - (XX^T)^TA_{n_w}^T\right) \\
&= \left([X,X^T]A_1^T \cdots [X,X^T]A_{n_w}^T\right) \\
&= 2\boldsymbol{i}\left(\Theta^-(X)A_1^T \cdots \Theta^-(X)A_{n_w}^T\right).
\end{aligned}
$$

Thus, Lemma 3.2 gives $[X, (AX)^T] = -2\boldsymbol{i}\,(\Theta^-(A_1)X \cdots \Theta^-(A_{n_w})X)$.

For (4.4b), note that the operator B_iX commutes with $\Theta^-(A_j)$ for any i and j. Recall that $XX^T = \frac{1}{2}([X,X^T] + \{X,X^T\})$. It then follows that

$$
[X, (AX)^T]\,BX = -2\boldsymbol{i}\left(\Theta^-(A_1)X \ldots \Theta^-(A_{n_w})X\right)\begin{pmatrix} B_1X \\ \vdots \\ B_{n_w}X \end{pmatrix}
$$

$$= -2\boldsymbol{i}\sum_{k=1}^{n_w}\begin{pmatrix} A_k F_1^T X X^T B_k^T \\ \vdots \\ A_k F_s^T X X^T B_k^T \end{pmatrix}$$

$$= -2\boldsymbol{i}\sum_{k=1}^{n_w}\begin{pmatrix} A_k F_1^T \left(\frac{2}{n} I_s + \Theta^+(X) + \boldsymbol{i}\Theta^-(X)\right) B_k^T \\ \vdots \\ A_k F_s^T \left(\frac{2}{n} I_s + \Theta^+(X) + \boldsymbol{i}\Theta^-(X)\right) B_k^T \end{pmatrix}$$

$$= -2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\frac{2}{n}\Theta^-(A_k)B_k^T + \Theta^-(A_k)\Theta^+(B_k)X - \boldsymbol{i}\Theta^-(A_k)\Theta^-(B_k)X\right).$$

Using $[X,(AX)^T]^T = -[AX, X^T]$, one obtains (4.4c) by transposing (4.4b) and changing the sign. That is,

$$\begin{aligned}(BX)^T[AX, X^T] &= 2\boldsymbol{i}\sum_{k=1}^{n_w}\left(B_k X A_k F_1^T X \cdots B_k X A_k F_s^T X\right)\\ &= 2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\Theta_1^-(A_k)(B_k X)X \cdots \Theta_s^-(A_k)(B_k X)X\right)\\ &= 2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\Theta_1^-(A_k)(XX^T)^T B_k^T \cdots \Theta_s^-(A_k)(XX^T)^T B_k^T\right)\\ &= 2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\Theta^-(A_k)\left(\frac{2}{n}I_s + \Theta^+(X) - \boldsymbol{i}\Theta^-(X)\right)B_k^T\right)^T\\ &= 2\boldsymbol{i}\sum_{k=1}^{n_w}\left(\frac{2}{n}\Theta^-(A_k)B_k^T + \Theta^-(A_k)\Theta^+(B_k)X + \boldsymbol{i}\Theta^-(A_k)\Theta^-(B_k)X\right)^T. \quad \square\end{aligned}$$

Proof of Lemma 4.2. The commutators (4.5a)–(4.5c) follow directly from (4.4a). The commutator (4.5d) is computed using (4.4b) as

$$\begin{aligned}\left[L^\#, X^T\right]^T L &= -[X, L^\dagger]\,L\\ &= \sum_{k=1}^{n_w}\left(\frac{4\boldsymbol{i}}{n}\Theta^-(\Gamma_k^\#)\Gamma_k^T + 2\boldsymbol{i}\Theta^-(\Gamma_k^\#)\Theta^+(\Gamma_k)X + 2\Theta^-(\Gamma_k^\#)\Theta^-(\Gamma_k)X\right).\end{aligned}$$

Finally, the commutator (4.5e) is obtained using (4.4c) as

$$\begin{aligned}\left(L^\dagger\ \left[X, L^T\right]^T\right)^T &= -\left(L^\dagger\ \left[L^\#, X^T\right]\right)^T\\ &= \sum_{k=1}^{n_w}\left(\frac{4\boldsymbol{i}}{n}\Theta^-(\Gamma_k^\#)\Gamma_k^T - 2\boldsymbol{i}\Theta^-(\Gamma_k)\Theta^+(\Gamma_k^\#)X + 2\Theta^-(\Gamma_k)\Theta^-(\Gamma_k^\#)X\right). \quad \square\end{aligned}$$

Proof of Lemma 5.1. Assume (5.2) holds. By recalling that $F = (F_1, \ldots, F_s)^T$ and applying property (3.5) of the stacking operator vec to (5.2), it follows that

$$\operatorname{vec}\left(G\Theta^-(X) + \Theta^-(X)G^T - \Theta^-(GX)\right) = (I\otimes G)FX + (G\otimes I)FX - FGX.$$

Since by assumption, the components of X are linearly independent, it follows that

$$(I\otimes G)F + (G\otimes I)F - FG = 0. \tag{A.1}$$

The application of identity (3.9c) and equation (3.7) and multiplying on the left by F^T gives $2\sum_{k=1}^s F_k G F_k + nG = 0$. Then, the equation for the (i,j) component of G is

$$2\sum_{k,r,l=1}^{s} f_{kjr} f_{kil} G_{lr} - nG_{ij} = 0. \tag{A.2}$$

From (3.2a) and noting that $\mathrm{Tr}(AB) = \sum_{r,l=0}^s A_{rl}B_{lr}$ for any $A, B \in \mathbb{R}^{s\times s}$, (A.2) can be rewritten as

$$\begin{aligned} 0 =& \ 2\sum_{k,r,l=1}^{s} f_{kjr} f_{kil} G_{lr} - nG_{ij} \\ =& -2\sum_{r,l=1}^{s} G_{lr} \sum_{k=1}^{s} (f_{jlk} f_{ikr} + f_{rlk} f_{ijk}) - nG_{ij} \\ =& -2\sum_{k,r,l=1}^{s} f_{jlk} f_{ikr} (G^T)_{rl} - 2\sum_{k=1}^{s} f_{ijk} \mathrm{Tr}(F_k G) - nG_{ij}. \end{aligned} \tag{A.3}$$

Also (A.2) implies that

$$2\sum_{k,r,l=1}^{s} f_{jlk} f_{ikr} (G^T)_{rl} = 2\sum_{k,r,l=1}^{s} f_{kjl}(-f_{kir})(-G_{rl}) = n\,G_{ij}.$$

Substituting this into (A.3) gives $0 = -nG_{ij} - 2\sum_{k=1}^s f_{ijk}\mathrm{Tr}(F_k G) - nG_{ij}$, which implies that $G_{ij} = -\frac{1}{n}\sum_{k=1}^s f_{ijk}\mathrm{Tr}(F_k G)$. Then,

$$G_{ij} = -\frac{1}{n}\sum_{k=1}^{s} f_{ijk}\mathrm{Tr}(F_k G) = -\frac{1}{n}((F_j)_{i1}\cdots(F_j)_{is}) \begin{pmatrix} \mathrm{Tr}(F_1 G) \\ \vdots \\ \mathrm{Tr}(F_s G) \end{pmatrix}.$$

Therefore, the j-th column of G is $G_j = -\frac{1}{n}F_j^T\left(\mathrm{Tr}(F_1 G)\cdots\mathrm{Tr}(F_s G)\right)^T$. By defining g as in (5.4), one obtains that $G = (F_1^T g \cdots F_s^T g) = \Theta^-(g)$. Now, assuming that there exists a g such that $G = \Theta^-(g)$, it then follows directly from (3.8d) that the left-hand side of (5.2) satisfies

$$\Theta^-(g)\Theta^-(X) - \Theta^-(X)\Theta^-(g)^T - \Theta^-(\Theta^-(g)X) = 0$$

for an arbitrary $X \in \mathfrak{T}(\mathfrak{H})^s$. Finally, by the linearity of $\Theta^-(\cdot)$, if there exists another g' such that $G = \Theta^-(g')$ one concludes that $\Theta^-(g) - \Theta^-(g') = 0$ implies $g = g'$, which completes the proof. ∎

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