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Faà di Bruno Hopf algebra of the output feedback group for multivariable Fliess operators

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ABSTRACT

Given two nonlinear input–output systems written in terms of Chen–Fliess functional expansions, i.e., Fliess operators, it is known that the feedback interconnected system is always well defined and in the same class. An explicit formula for the generating series of a single-input, single-output closed-loop system was provided by the first two authors in earlier work via Hopf algebra methods. This paper is a sequel. It has four main innovations. First, the full multivariable extension of the theory is presented. Next, a major simplification of the basic setup is introduced using a new type of grading that has recently appeared in the literature. This grading also facilitates a fully recursive algorithm to compute the antipode of the Hopf algebra of the output feedback group, and thus, the corresponding feedback product can be computed much more efficiently. The final innovation is an improved convergence analysis of the antipode operation, namely, the radius of convergence of the antipode is computed.

1. Introduction

Given two nonlinear input–output systems written in terms of Chen–Fliess functional expansions [1], i.e., Fliess operators, it was shown in [2,3] that the feedback interconnected system is always well defined and in the same class. An explicit formula for the generating series of a single-input, single-output (SISO) closed-loop system was later provided in [4] using Hopf algebra methods. In particular, the so called *feedback product* of the two generating series for the component systems can be computed in terms of the antipode of a Faà di Bruno type Hopf algebra. This antipode was described in terms of a sequence of polynomials of increasing degree. While explicit, this somewhat brute force formula is not ideal for software implementation [5]. Nevertheless, this antipode can be used to provide a tractable formula for nonlinear system inversion from a purely input–output point of view, i.e., no state space model is required [6].

This paper is a sequel to [4]. It has four main innovations. First, the full multivariable extension of the theory in [4] is presented, which makes it more relevant to practical control problems. The second innovation is more technical, but it greatly simplifies the basic setup. Specifically, it was shown recently in [7] that the Hopf algebra for the SISO output feedback group is *connected* under a grading that is distinct from the one described in [4]. This important observation implies that the bialgebra presented in the original paper is *automatically* a Hopf algebra, and therefore, much of the technical analysis concerning the existence of the antipode can now be omitted. So here the method in [7] is extended to the multivariable case and applied throughout. The third innovation is related to the existence of this new grading. Namely, the *partially* recursive formula for the antipode of any connected graded Hopf algebra in [8] is exploited here to produce a *fully* recursive antipode algorithm for the Hopf algebra of the output feedback group. This in turn allows one to compute the feedback product much more efficiently. The approach involves carefully combining results from [7–9]. The SISO version of this algorithm was presented in [5] and compared against other existing methods. In a Mathematica implementation, this new algorithm provided an order of magnitude reduction in execution times. For the multivariable case, such gains are likely to be even larger, but this analysis is beyond the scope of this paper. The final innovation is an improved convergence analysis of the antipode operation, specifically, the radius of convergence of the antipode is computed using techniques presented

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in [10]. In [4] it was only shown that this radius of convergence is positive.

The paper is organized as follows. In the next section, some mathematical preliminaries and background are briefly summarized concerning Hopf algebras and the interconnection of Fliess operators. In Section 3, the Hopf algebra of the multivariable output feedback group is presented, including the recursive algorithm for the antipode and the radius of convergence for this operation. In the subsequent section, these results are used to define the multivariable feedback product, and the corresponding convergence analysis is presented. The theory is demonstrated on a simple steering example. The conclusions are given in the final section.

2. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product defined in terms of the shuffle product of two words

$$(x_i\eta) ⧢ (x_j\xi) = x_i(\eta ⧢ (x_j\xi)) + x_j((x_i\eta) ⧢ \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta ⧢ \emptyset = \emptyset ⧢ \eta = \eta$ [1,9]. Its restriction to polynomials over X is

$$\mathrm{sh} : \mathbb{R}\langle X\rangle \otimes \mathbb{R}\langle X\rangle \to \mathbb{R}\langle X\rangle : p \otimes q \mapsto p ⧢ q.$$

The corresponding adjoint map sh^* is the unique $\mathbb{R}$-linear map of the form $\mathbb{R}\langle X\rangle \to \mathbb{R}\langle X\rangle \otimes \mathbb{R}\langle X\rangle$ which satisfies the identity

$$(\mathrm{sh}(p \otimes q), r) = (p \otimes q, \mathrm{sh}^*(r))$$

for all $p, q, r \in \mathbb{R}\langle X\rangle$. The following theorem states an important duality.

Theorem 1 ([9]). *The adjoint map* sh^* *is an* $\mathbb{R}$*-algebra morphism for the catenation product* $\mathrm{cat} : p \otimes q \mapsto pq$. *That is,*

$$\mathrm{sh}^*(pq) = \mathrm{sh}^*(p)\,\mathrm{sh}^*(q)$$

for all $p, q \in \mathbb{R}\langle X\rangle$ *with* $\mathrm{sh}^*(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$.[2] *In particular, for* $x_i \in X$ *and* $\eta \in X^*$

$$\mathrm{sh}^*(x_i\eta) = (x_i \otimes \mathbf{1} + \mathbf{1} \otimes x_i)\mathrm{sh}^*(\eta).$$

2.1. Hopf algebras

In this section, a few basic facts and tools concerning Hopf algebras are summarized. The reader is referred to [8,11,12] for more complete treatments.

A *coalgebra* over $\mathbb{R}$ consists of a triple (C, Δ, ε). The coproduct $\Delta : C \to C \otimes C$ is coassociative, that is, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$, and $\varepsilon : C \to \mathbb{R}$ denotes the counit map. A *bialgebra* B is both a unital algebra and a coalgebra together with compatibility relations, such as both the algebra product, $m(x, y) = xy$, and unit map, $\mathrm{e} : \mathbb{R} \to B$, are coalgebra morphisms. This provides, for example, that $\Delta(xy) = \Delta(x)\Delta(y)$. The unit of B is denoted by $\mathbf{1} = \mathrm{e}(1)$. A bialgebra is called *graded* if there are $\mathbb{R}$-vector subspaces B_n, $n \geq 0$ such that $B = \bigoplus_{n\geq 0} B_n$ with $m(B_k \otimes B_l) \subseteq B_{k+l}$ and $\Delta B_n \subseteq \bigoplus_{k+l=n} B_k \otimes B_l$. Elements $x \in B_n$ are given a degree $\deg(x) = n$. Moreover, B is called *connected* if $B_0 = \mathbb{R}\mathbf{1}$. Define $B_+ = \bigoplus_{n>0} B_n$. For any $x \in B_n$ the coproduct is of the form

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \Delta'(x) \in \bigoplus_{k+l=n} B_k \otimes B_l,$$

where $\Delta'(x) := \Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x \in B_+ \otimes B_+$ is the *reduced* coproduct.

Suppose A is an $\mathbb{R}$-algebra with product m_A and unit e_A, e.g., $A = \mathbb{R}$ or $A = B$. The vector space $L(B, A)$ of linear maps from the bialgebra B to A together with the convolution product $\Phi \star \Psi := m_A \circ (\Phi \otimes \Psi) \circ \Delta : B \to A$, where $\Phi, \Psi \in L(B, A)$, is an associative algebra with unit $\iota := \mathrm{e}_A \circ \varepsilon$. A *Hopf algebra* H is a bialgebra together with a particular $\mathbb{R}$-linear map called an *antipode* $S : H \to H$ which satisfies the Hopf algebra axioms and has the property that $S(xy) = S(y)S(x)$. When $A = H$, the antipode $S \in L(H, H)$ is the inverse of the identity map with respect to the convolution product, that is,

$$S \star \mathrm{id} = \mathrm{id} \star S := m \circ (S \otimes \mathrm{id}) \circ \Delta = \mathrm{e} \circ \varepsilon.$$

A connected graded bialgebra $H = \bigoplus_{n\geq 0} H_n$ is *always* a connected graded Hopf algebra.

Suppose A is a commutative unital algebra. The subset $\mathfrak{g}_0 \subset L(H, A)$ of linear maps α satisfying $\alpha(\mathbf{1}) = 0$ forms a Lie algebra in $L(H, A)$. The exponential $\exp^\star(\alpha) = \sum_{j\geq 0} \frac{1}{j!}\alpha^{\star j}$ is well defined and gives a bijection from $\mathfrak{g}_0$ onto the group $G_0 = \iota + \mathfrak{g}_0$ of linear maps γ satisfying $\gamma(\mathbf{1}) = 1_A$. A map $\Phi \in L(H, A)$ is called a *character* if $\Phi(\mathbf{1}) = 1_A$ and $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in H$. The set of characters is denoted by $G_A \subset G_0$. The neutral element $\iota := \mathrm{e}_A \circ \varepsilon$ in G_A is given by $\iota(\mathbf{1}) = 1_A$ and $\iota(x) = 0$ for $x \in \mathrm{Ker}(\varepsilon) = H_+$. The inverse of $\Phi \in G_A$ is given by

$$\Phi^{\star -1} = \Phi \circ S. \quad (1)$$

Given an arbitrary group G, the set of real-valued functions defined on G is a commutative unital algebra. There is a subalgebra of functions known as the *representative functions*, $R(G)$, which can be endowed with a Hopf algebra, H. In this case, there is a group isomorphism relating G to the convolution group G_A, say, $\Phi : G \to G_A : g \mapsto \Phi_g$. A *coordinate map* is any $a : H \to \mathbb{R}$ satisfying

$$(\Phi_{g_1} \star \Phi_{g_2})(a) = a(g_1g_2), \quad \forall g_i \in G. \quad (2)$$

In some sense, the coordinates maps are the generators of H, though they cannot always be easily identified in general.

Example 1 ([9]). $(\mathbb{R}\langle X\rangle, \mathrm{cat}, \mathrm{e}, \mathrm{sh}^*, \varepsilon, S)$ is a Hopf algebra, where $\mathrm{e} : \mathbb{R} \to \mathbb{R}\langle X\rangle : k \mapsto k\mathbf{1}$, $\varepsilon : \mathbb{R}\langle X\rangle \to \mathbb{R} : p \mapsto (p, \emptyset)$,

$$f \star g : \mathbb{R}\langle X\rangle \to \mathbb{R}\langle X\rangle : p \mapsto \sum_{\eta,\xi\in X^*} (p, \eta ⧢ \xi) f(\eta)g(\xi),$$

for all $f, g \in L(\mathbb{R}\langle X\rangle, \mathbb{R}\langle X\rangle)$, and $S(x_{i_1}x_{i_2}\cdots x_{i_k}) = (-1)^k x_{i_k}x_{i_{k-1}} \cdots x_{i_1}$.

2.2. Fliess operators and their interconnections

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all

[2] Here $\mathbf{1}$ is the unit polynomial $1\emptyset$.

measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : \|u\|_{\mathfrak{p}} \le R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input–output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{3}$$

[1,13]. If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \le K_c M_c^{|\eta|} |\eta|!, \quad \forall \eta \in X^*, \tag{4}$$

then F_c constitutes a well defined mapping from $B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ into $B^\ell_{\mathfrak{q}}(S)[t_0,\ t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [14]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. In particular, when $\mathfrak{p} = 1$, the series (3) converges absolutely and uniformly if $\max\{R, T\} < 1/M_c(m + 1)$ [15,16]. It is important in applications to identify the *smallest* possible geometric growth constant, M_c, in order to avoid over restricting the domain of F_c. So let $\pi : \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}^+ \cup \{0\}$ take each series c to the infimum of all M_c satisfying (4). Therefore, $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ can be partitioned into equivalence classes, and the number $1/M_c(m + 1)$ will be referred to as the *radius of convergence* for the class $\pi^{-1}(M_c)$. This is in contrast to the usual situation where a radius of convergence is assigned to individual series. When c satisfies the more stringent growth condition

$$|(c, \eta)| \le K_c M_c^{|\eta|}, \quad \forall \eta \in X^*, \tag{5}$$

the series (3) defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $C[t_0, \infty)$, where

$$L^m_{\mathfrak{p},e}(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_{\mathfrak{p}}[t_0, t_1], \forall t_1 \in (t_0, \infty)\},$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ [14]. The set of all such *globally convergent* series is designated by $\mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$.

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup\, d}$, respectively [1]. When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(1)$$

[17,18]. Here ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$ uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with

$$\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup\, e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the ith component series of d ($d_0 := 1$). $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. This composition product is associative and $\mathbb{R}$-linear in its left argument. In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 1, it was shown in [2] that there always exists a unique generating series $c@d$ such that $y = F_{c@d}[u]$ whenever $c, d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. This so called *feedback product* of c and d can be viewed as the unique fixed point of a contractive iterated map on a complete ultrametric space, but to compute it explicitly requires Hopf algebraic tools such as those employed in [4,19] for SISO systems. The multivariable case is considered in the next section.

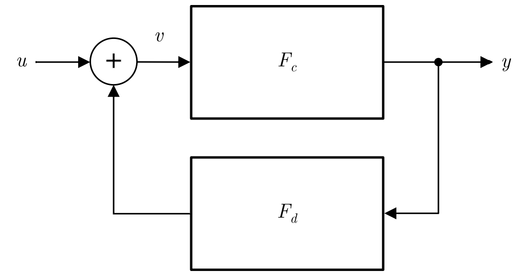

Fig. 1. Feedback connection.

3. Hopf algebra for multivariable output feedback group

Consider the set of operators $\mathscr{F}_\delta := \{I + F_c : c \in \mathbb{R}^m\langle\langle X\rangle\rangle\}$, where I denotes the identity operator. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}^m\langle\langle X_\delta\rangle\rangle$. The first theorem describes the multivariable output feedback group which is at the heart of all the analyses in this paper. The group product is described in terms of the *modified composition product* of $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, namely,

$$c \,\tilde{\circ}\, d = \sum_{\eta \in X^*} (c, \eta)\, \phi_d(\eta)(1),$$

where ϕ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$ uniquely specified by $\phi_d(x_i\eta) = \phi_d(x_i) \circ \phi_d(\eta)$ with

$$\phi_d(x_i)(e) = x_i e + x_0(d_i \sqcup\!\sqcup\, e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where $d_0 := 0$. Again, $\phi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$ [2]. It can be easily shown that $F_c \circ (I + F_d) = F_{c \tilde{\circ} d}$ and for any $x_i \in X$

$$(x_i c) \,\tilde{\circ}\, d = x_i(c \,\tilde{\circ}\, d) + x_0(d_i \sqcup\!\sqcup\, (c \,\tilde{\circ}\, d)). \tag{6}$$

The following (non-associativity) identity was proved in [20]

$$(c \,\tilde{\circ}\, d) \,\tilde{\circ}\, e = c \,\tilde{\circ}\, (d \,\tilde{\circ}\, e + e) \tag{7}$$

for all $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$. The lemma below will be also useful. Its proof is deferred until Section 3, when all the appropriate tools are available.

Lemma 1. *Let $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ be fixed. Then $c \,\tilde{\circ}\, d = K \in \mathbb{R}^\ell$ if and only if $c = K$.*[3]

The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group of operators under the composition

$$F_{c_\delta} \circ F_{d_\delta} := (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c \,\tilde{\circ}\, d =: \delta + c \circledcirc d$.[4] Given the uniqueness of generating series of Fliess operators, this assertion is equivalent to the following theorem.

Theorem 2. *The triple $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ is a group.*

Proof. By design, δ is the identity element of the group. The associativity of the product can be established in a manner similar to the SISO case addressed in [4]. (See [21] for an alternative approach.) The existence of an inverse will be handled differently

3 For notational convenience, $c = K\emptyset$ is written as $c = K$.

4 The same symbol will be used for composition on $\mathbb{R}^m\langle\langle X\rangle\rangle$ and $\mathbb{R}^m\langle\langle X_\delta\rangle\rangle$. As elements in these two sets have a distinct notation, i.e., c versus c_δ, respectively, it will always be clear which product is at play.

Table 1
Bases in the gradings $V = \bigoplus_{k\geq 0} V_k$ and $H = \bigoplus_{k\geq 0} H_k$. Here $i_\ell, j_\ell \neq 0$.

k	V_k	H_k	$\dim(V_k)$	$\dim(H_k)$
0	$\mathbf{1}$	$\mathbf{1}$	1	1
1	$a_\emptyset^{i_1}$	$a_\emptyset^{i_1}$	m	m
2	$a_{x_{j_1}}^{i_1}$	$a_{x_{j_1}}^{i_1}, a_\emptyset^{i_1} a_\emptyset^{i_2}$	m^2	$2m^2$
3	$a_{x_0}^{i_1}, a_{x_{j_1}x_{j_2}}^{i_1}$	$a_{x_0}^{i_1}, a_{x_{j_1}x_{j_2}}^{i_1}, a_\emptyset^{i_1} a_\emptyset^{i_2} a_\emptyset^{i_3}, a_\emptyset^{i_1} a_{x_{j_1}}^{i_2}$	$m + m^3$	$m + 3m^3$
4	$a_{x_0x_{j_1}}^{i_1}, a_{x_{j_1}x_0}^{i_1}, a_{x_{j_1}x_{j_2}x_{j_3}}^{i_1}$	$a_{x_0x_{j_1}}^{i_1}, a_{x_{j_1}x_0}^{i_1}, a_{x_{j_1}x_{j_2}x_{j_3}}^{i_1}, a_\emptyset^{i_1} a_\emptyset^{i_2} a_\emptyset^{i_3} a_\emptyset^{i_4}, a_\emptyset^{i_1} a_\emptyset^{i_2} a_{x_{j_1}}^{i_3}, a_\emptyset^{i_1} a_{x_0}^{i_2}, a_\emptyset^{i_1} a_{x_{j_1}x_{j_2}}^{i_2}, a_{x_{j_1}}^{i_1} a_{x_{j_2}}^{i_2}$	$2m^2 + m^4$	$3m^2 + 5m^4$

here (more directly) via Lemma 1. Specifically, for a fixed $c_\delta \in \mathbb{R}^m\langle\langle X_\delta\rangle\rangle$, the composition inverse, $c_\delta^{-1} = \delta + c^{-1}$, must satisfy $c_\delta \circ c_\delta^{-1} = \delta$ and $c_\delta^{-1} \circ c_\delta = \delta$, which reduce, respectively, to

$$c^{-1} = (-c)\tilde{\circ}\, c^{-1} \tag{8a}$$

$$c = (-c^{-1})\tilde{\circ}\, c. \tag{8b}$$

It was shown in [2] that $e \mapsto (-c)\tilde{\circ}\, e$ is always a contraction in the ultrametric sense on $\mathbb{R}^m\langle\langle X\rangle\rangle$ as a complete ultrametric space and thus has a unique fixed point. So it follows directly that c_δ^{-1} is a right inverse of c_δ, i.e., satisfies (8a). To see that this same series is also a left inverse, first observe that (8a) is equivalent to

$$c^{-1}\tilde{\circ}\, 0 + c\tilde{\circ}\, c^{-1} = 0, \tag{9}$$

using the identity $c^{-1}\tilde{\circ}\, 0 = c^{-1}$ and the left linearity of the modified composition product. Substituting (9) back into itself where zero appears and applying (7) gives

$$c^{-1}\tilde{\circ}\,(c\tilde{\circ}\, c^{-1} + c^{-1}) + c\tilde{\circ}\, c^{-1} = 0$$
$$(c^{-1}\tilde{\circ}\, c)\tilde{\circ}\, c^{-1} + c\tilde{\circ}\, c^{-1} = 0.$$

Again from left linearity of the modified composition product it follows that

$$(c^{-1}\tilde{\circ}\, c + c)\tilde{\circ}\, c^{-1} = 0.$$

Finally, Lemma 1 implies that $c^{-1}\tilde{\circ}\, c + c = 0$, which is equivalent to (8b). This concludes the proof.

A Faà di Bruno type Hopf algebra is now defined for the output feedback group. The coordinate maps for this algebra have the form

$$a_\eta^i : \mathbb{R}^m\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c_i, \eta),$$

where $\eta \in X^*$ and $i = 1, 2, \ldots, m$.[5] Let V denote the $\mathbb{R}$-vector space spanned by these maps. If the *degree* of a_η^i is defined as $\deg(a_\eta^i) = 2\,|\eta|_{x_0} + \sum_{j=1}^m |\eta|_{x_j} + 1$, then V is a connected graded vector space, that is, $V = \bigoplus_{n\geq 0} V_n$ with

$$V_n = \mathrm{span}_{\mathbb{R}}\{a_\eta^i : \deg(a_\eta^i) = n\}, \quad n > 0,$$

$V_0 = \mathbb{R}\mathbf{1}$, and $\mathbf{1}$ maps every $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$ to 1.

Consider next the free unital commutative $\mathbb{R}$-algebra, H, with product

$$\mu : a_\eta^i \otimes a_\xi^j \mapsto a_\eta^i a_\xi^j$$

and unit $\mathbf{1}$. This product is clearly associative. The graduation on V induces a connected graduation on H with $\deg(a_\eta^i a_\xi^j) = \deg(a_\eta^i) + \deg(a_\xi^j)$ and $\deg(\mathbf{1}) = 0$. Specifically, $H = \bigoplus_{n\geq 0} H_n$, where H_n is the set of all elements of degree n and $H_0 = \mathbb{R}\mathbf{1}$. Bases for these subspaces are given in Table 1.

Three coproducts are now introduced. The first coproduct is used to define the Hopf algebra on H. The remaining two coproducts provide a recursive manner in which to compute it. Recalling that $c_\delta \circ d_\delta = \delta + c \circledcirc d$, define Δ for any $a_\eta^i \in V_+$ such that

$$\Delta a_\eta^i(c, d) = a_\eta^i(c \circledcirc d) = (c_i \circledcirc d, \eta).$$

The coassociativity of Δ follows from the associativity of the product $c \circledcirc d$ [4]. Specifically, for any $c, d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$:

$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \Delta a_\eta^i(c, d, e) &= (c_i \circledcirc (d \circledcirc e), \eta) \\ &= ((c \circledcirc d)_i \circledcirc e, \eta) \\ &= (\Delta \otimes \mathrm{id}) \circ \Delta a_\eta^i(c, d, e).\end{aligned}$$

Therefore, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ as required. The following result motivates the primary interest in proving that (H, μ, Δ) is a Hopf algebra, namely, that it has an antipode corresponding to the group inverse for $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$.

Lemma 2. *The Hopf algebra (H, μ, Δ) has an antipode S satisfying $a_\eta^i(c^{-1}) = (Sa_\eta^i)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$.*

Proof. First observe that for each $c_\delta \in \mathbb{R}^m\langle\langle X_\delta\rangle\rangle$, one can identify a character map $\Phi_c \in L(H, \mathbb{R})$ as

$$\Phi_c : a_\eta^i \mapsto a_\eta^i(c) = (c_i, \eta),$$

whereby $\Phi_c(\mathbf{1}) = 1$ and

$$\Phi_c(a_\eta^i a_\xi^j) = a_\eta^i(c) a_\xi^j(c) = \Phi_c(a_\eta^i)\Phi_c(a_\xi^j).$$

Coordinate maps, therefore, should satisfy (2), specifically,

$$\begin{aligned}(\Phi_c \star \Phi_d)(a_\eta^i) &= \mu \circ (\Phi_c \otimes \Phi_d) \circ \Delta a_\eta^i \\ &= \sum \Phi_c\left(a_{\eta(1)}^i\right) \Phi_d\left(a_{\eta(2)}^i\right) \\ &= \sum a_{\eta(1)}^i(c) a_{\eta(2)}^i(d) \\ &= \Delta a_\eta^i(c, d) \\ &= a_\eta^i(c \circledcirc d) \\ &= a_\eta^i(c_\delta \circ d_\delta),\end{aligned}$$

where the summation is taken over all terms that appear in Δa_η^i (following the notation of Sweedler [12]). From this identification between the convolution of characters and the group product on $\mathbb{R}^m\langle\langle X_\delta\rangle\rangle$, it is clear that $\Phi_c^{\star -1} = \Phi_{c^{-1}}$. Using (1) then $\Phi_{c^{-1}} = \Phi_c^{\star -1} = \Phi_c \circ S$, where $c_\delta^{-1} = \delta + c^{-1}$, so that $a_\eta^i(c^{-1}) = (Sa_\eta^i)(c)$ as desired.

The second coproduct is $\Delta_{\sqcup\!\sqcup}^j(V_+) \subset V_+ \otimes V_+$, which is isomorphic to sh* via the coordinate maps. That is,

$$\Delta_{\sqcup\!\sqcup}^j a_\emptyset^i = a_\emptyset^i \otimes a_\emptyset^j \tag{10a}$$

$$\Delta_{\sqcup\!\sqcup}^j \circ \theta_k = (\theta_k \otimes \mathrm{id} + \mathrm{id} \otimes \theta_k) \circ \Delta_{\sqcup\!\sqcup}^j, \tag{10b}$$

where id is the identity map on V_+, and θ_k denotes the endomorphism on V_+ specified by $\theta_k a_\eta^i = a_{x_k\eta}^i$ for $k = 0, 1, \ldots, m$ and $i, j = 1, 2, \ldots, m$.

5 The use of this terminology will be justified in the proof of Lemma 2.

Example 2. The first few terms of $\Delta^j_{\sqcup\!\sqcup}$ are:

$$\Delta^j_{\sqcup\!\sqcup}\, a^i_{\emptyset} = a^i_{\emptyset} \otimes a^j_{\emptyset}$$

$$\Delta^j_{\sqcup\!\sqcup}\, a^i_{x_{i_1}} = a^i_{x_{i_1}} \otimes a^j_{\emptyset} + a^i_{\emptyset} \otimes a^j_{x_{i_1}}$$

$$\Delta^j_{\sqcup\!\sqcup}\, a^i_{x_{i_2}x_{i_1}} = a^i_{x_{i_2}x_{i_1}} \otimes a^j_{\emptyset} + a^i_{x_{i_2}} \otimes a^j_{x_{i_1}} + a^i_{x_{i_1}} \otimes a^j_{x_{i_2}} + a^i_{\emptyset} \otimes a^j_{x_{i_2}x_{i_1}}$$

$$\begin{aligned}\Delta^j_{\sqcup\!\sqcup}\, a^i_{x_{i_3}x_{i_2}x_{i_1}} = {} & a^i_{x_{i_3}x_{i_2}x_{i_1}} \otimes a^j_{\emptyset} + a^i_{x_{i_3}x_{i_2}} \otimes a^j_{x_{i_1}} \\ & + a^i_{x_{i_3}x_{i_1}} \otimes a^j_{x_{i_2}} + a^i_{x_{i_3}} \otimes a^j_{x_{i_2}x_{i_1}} \\ & + a^i_{x_{i_2}x_{i_1}} \otimes a^j_{x_{i_3}} + a^i_{x_{i_2}} \otimes a^j_{x_{i_3}x_{i_1}} \\ & + a^i_{x_{i_1}} \otimes a^j_{x_{i_3}x_{i_2}} + a^i_{\emptyset} \otimes a^j_{x_{i_3}x_{i_2}x_{i_1}}.\end{aligned}$$

The third coproduct is $\tilde{\Delta} a^i_\eta = \Delta a^i_\eta - \mathbf{1} \otimes a^i_\eta$ or, equivalently, the coproduct induced by the identity

$$\tilde{\Delta} a^i_\eta(c,d) = (c_i \,\tilde{\circ}\, d, \eta) = \sum a^i_{\eta(1)}(c) a^i_{\eta(2)}(d).$$

A key observation is that this coproduct can be computed recursively as described in the next lemma, which is the multivariable version of Proposition 3 in [7]. It is not difficult to show using (2) and (3) of this lemma that $a^i_{\eta(1)} \in V_+$ and $a^i_{\eta(2)} \in H$, and thus, $\tilde{\Delta} V_+ \subseteq V_+ \otimes H$.

Lemma 3. *The following identities hold:*

(1) $\tilde{\Delta} a^i_{\emptyset} = a^i_{\emptyset} \otimes \mathbf{1}$
(2) $\tilde{\Delta} \circ \theta_i = (\theta_i \otimes \mathrm{id}) \circ \tilde{\Delta}$
(3) $\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes \mathrm{id}) \circ \tilde{\Delta} + (\theta_i \otimes \mu) \circ (\tilde{\Delta} \otimes \mathrm{id}) \circ \Delta^i_{\sqcup\!\sqcup}$,

$i = 1, 2, \ldots, m$, *where* id *denotes the identity map on* H.[6]

Proof. (1) First note that any series c can be uniquely decomposed as $c = (c, \emptyset)\emptyset + x_i c^i$, $i = 0, 1, \ldots, m$, where the series c^i are arbitrary. In which case, using the left linearity of the modified composition product and (6), it follows that

$$\begin{aligned}\tilde{\Delta} a^i_{\emptyset}(c,d) &= a^i_{\emptyset}(c \,\tilde{\circ}\, d) = a^i_{\emptyset}\left((c,\emptyset)\emptyset + (x_j c^j) \,\tilde{\circ}\, d\right) \\ &= (c_i, \emptyset) + a^i_{\emptyset}(x_j(c^j \,\tilde{\circ}\, d) + x_0(d_j \sqcup\!\sqcup (c^j \,\tilde{\circ}\, d))) \\ &= (c_i, \emptyset) = (a^i_{\emptyset} \otimes \mathbf{1})(c,d).\end{aligned}$$

(2) For any $\eta \in X^*$ observe

$$\begin{aligned}(\tilde{\Delta} \circ \theta_i) a^j_\eta(c,d) &= \tilde{\Delta} a^j_{x_i\eta}(c,d) \\ &= a^j_{x_i\eta}(x_k(c^k \,\tilde{\circ}\, d) + x_0(d_k \sqcup\!\sqcup (c^k \,\tilde{\circ}\, d))) \\ &= a^j_\eta(c^i \,\tilde{\circ}\, d) \\ &= \tilde{\Delta} a^j_\eta(c^i, d) \\ &= \sum a^j_{\eta(1)} \otimes a^j_{\eta(2)}(c^i, d) \\ &= \sum \theta_i(a^j_{\eta(1)}) \otimes a^j_{\eta(2)}(c,d) \\ &= (\theta_i \otimes \mathrm{id}) \circ \tilde{\Delta} a^j_\eta(c,d).\end{aligned}$$

Note that since $a^j_{\eta(1)} \in V_+$, the operation $\theta_i(a^j_{\eta(1)})$ is well defined.

(3) Proceeding as in the previous item, it follows that

$$\begin{aligned}(\tilde{\Delta} \circ \theta_0) a^i_\eta(c,d) &= a^i_{x_0\eta}(c \,\tilde{\circ}\, d) \\ &= a^i_{x_0\eta}(x_j(c^j \,\tilde{\circ}\, d) + x_0(d_j \sqcup\!\sqcup (c^j \,\tilde{\circ}\, d))) \\ &= a^i_\eta(c^0 \,\tilde{\circ}\, d + d_j \sqcup\!\sqcup (c^j \,\tilde{\circ}\, d)) \\ &= a^i_\eta(c^0 \,\tilde{\circ}\, d) + \sum_{j=1}^m \Delta^j_{\sqcup\!\sqcup}\, a^i_\eta(c^j \,\tilde{\circ}\, d, d) \\ &= a^i_\eta(c^0 \,\tilde{\circ}\, d) + \sum_{j=1}^m \sum_{\xi,\nu \in X^*} (\eta, \xi \sqcup\!\sqcup \nu)\, a^i_\xi(c^j \,\tilde{\circ}\, d) a^j_\nu(d) \\ &= \tilde{\Delta} a^i_\eta(c^0, d) + \sum_{j=1}^m \sum_{\xi,\nu \in X^*} (\eta, \xi \sqcup\!\sqcup \nu)\, (\tilde{\Delta} a^i_\xi \otimes a^j_\nu)(c^j, d, d) \\ &= (\theta_0 \otimes \mathrm{id}) \circ \tilde{\Delta} a^i_\eta(c,d) + (\theta_j \otimes \mathrm{id}) \\ &\quad \circ \sum_{\xi,\nu \in X^*} (\eta, \xi \sqcup\!\sqcup \nu)\, (\tilde{\Delta} a^i_\xi \otimes a^j_\nu)(c,d,d) \\ &= (\theta_0 \otimes \mathrm{id}) \circ \tilde{\Delta} a^i_\eta(c,d) + (\theta_j \otimes \mu) \circ (\tilde{\Delta} \circ \mathrm{id}) \circ \Delta^j_{\sqcup\!\sqcup}\, a^i_\eta(c,d).\end{aligned}$$

The next theorem is a central result of the paper.

Theorem 3. *(H, μ, Δ) is a connected graded commutative unital Hopf algebra.*

Proof. From the development above, it is clear that (H, μ, Δ) is a bialgebra with unit $\mathbf{1}$ and counit ε defined by $\varepsilon(a_\eta) = 0$ for all $\eta \in X^*$ and $\varepsilon(\mathbf{1}) = 1$ (see also [4, Eq. (14)]). Here it is shown that this bialgebra is graded and connected. Therefore, H automatically has an antipode, and thus, is a Hopf algebra [8]. Specifically, since the algebra H is graded by H_n, $n \geq 0$ with $H_0 = \mathbb{R}\mathbf{1}$, it only needs to be shown for any $a^i_\eta \in V_+$ that

$$\tilde{\Delta} a^i_\eta \in (V_+ \otimes H)_n := \bigoplus_{\substack{j+k=n \\ j \geq 1, k \geq 0}} V_j \otimes H_k. \tag{11}$$

This fact is evident from the first few terms computed via Lemma 3:

$$n = 1 : \tilde{\Delta} a^i_{\emptyset} = a^i_{\emptyset} \otimes \mathbf{1}$$

$$n = 2 : \tilde{\Delta} a^i_{x_j} = a^i_{x_j} \otimes \mathbf{1}$$

$$n = 3 : \tilde{\Delta} a^i_{x_0} = a^i_{x_0} \otimes \mathbf{1} + a^i_{x_\ell} \otimes a^\ell_{\emptyset}$$

$$n = 3 : \tilde{\Delta} a^i_{x_j x_k} = a^i_{x_j x_k} \otimes \mathbf{1}$$

$$n = 4 : \tilde{\Delta} a^i_{x_0 x_j} = a^i_{x_0 x_j} \otimes \mathbf{1} + a^i_{x_\ell} \otimes a^\ell_{x_j} + a^i_{x_\ell x_j} \otimes a^\ell_{\emptyset}$$

$$n = 4 : \tilde{\Delta} a^i_{x_j x_0} = a^i_{x_j x_0} \otimes \mathbf{1} + a^i_{x_j x_\ell} \otimes a^\ell_{\emptyset}$$

$$n = 4 : \tilde{\Delta} a^i_{x_j x_k x_l} = a^i_{x_j x_k x_l} \otimes \mathbf{1}$$

$$\begin{aligned}n = 5 : \tilde{\Delta} a^i_{x_0^2} = {} & a^i_{x_0^2} \otimes \mathbf{1} + a^i_{x_\ell} \otimes a^\ell_{x_0} + a^i_{x_\ell x_0} \otimes a^\ell_{\emptyset} \\ & + a^i_{x_0 x_\ell} \otimes a^\ell_{\emptyset} + a^i_{x_\ell x_\nu} \otimes a^\ell_{\emptyset} a^\nu_{\emptyset},\end{aligned}$$

where $i, j, k, l = 1, 2, \ldots, m$. In which case, using the identities $\Delta(a^i_\eta a^j_\xi) = \Delta a^i_\eta \Delta a^j_\xi$ and $\Delta a^i_\eta = \tilde{\Delta} a^i_\eta + \mathbf{1} \otimes a^i_\eta$, it follows that $\Delta H_n \subseteq (H \otimes H)_n$, and this would complete the proof. To prove (11), the following facts are essential:

1. $\deg(\theta_l a^i_\eta) = \deg(a^i_\eta) + 1$, $l = 1, 2, \ldots, m$
2. $\deg(\theta_0 a^i_\eta) = \deg(a^i_\eta) + 2$
3. $\Delta^j_{\sqcup\!\sqcup}\, a^i_\eta \in (V_+ \otimes V_+)_{n+1}$, $n = \deg(a^i_\eta)$.

The proof is via induction on the length of η. When $|\eta| = 0$ then clearly $\tilde{\Delta} a^i_{\emptyset} = a^i_{\emptyset} \otimes \mathbf{1} \in V_1 \otimes H_0$ and $n = 1$. Assume now that (11) holds for words up to some fixed length $|\eta| \geq 0$. Let $n = \deg(a^i_\eta)$. There are two ways to increase the length of η. First consider $a^i_{x_l\eta}$ for some $l \neq 0$. From item 1 above $\deg(a^i_{x_l\eta}) = n + 1$, and from Lemma 3 $\tilde{\Delta} a^i_{x_l\eta} = (\theta_l \otimes \mathrm{id}) \circ \tilde{\Delta} a^i_\eta$. Therefore, using the induction hypothesis, $\tilde{\Delta} a^i_{x_l a_\eta} \in \bigoplus_{j+k=n} V_{j+1} \otimes H_k \subset (V \otimes H)_{n+1}$, which proves the assertion. Consider next $a^i_{x_0\eta}$. From item 2 above $\deg(a^i_{x_0\eta}) = n+2$. Lemma 3 is employed as in the first case. First note that from

[6] The Einstein summation notation is used in item (3) and throughout to indicate summations from either 0 or 1 to m, e.g., $\sum_{i=1}^m a_i b^i = a_i b^i$. It will be clear from the context which lower bound is applicable.

item 3 above $\Delta^j_{\sqcup\!\sqcup} a^i_\eta \in (V_+ \otimes V_+)_{n+1}$, and so using the induction hypothesis it follows that $(\tilde{\Delta} \otimes \mathrm{id}) \circ \Delta^j_{\sqcup\!\sqcup} a^i_\eta \in (V_+ \otimes H \otimes V_+)_{n+1}$. In which case, $(\theta_i \otimes \mu) \circ (\tilde{\Delta} \otimes \mathrm{id}) \circ \Delta^j_{\sqcup\!\sqcup} a^i_\eta \in (V_+ \otimes H)_{n+2}$. By a similar argument, $(\theta_0 \otimes \mathrm{id}) \circ \tilde{\Delta} a^i_\eta \in (V_+ \otimes H)_{n+2}$. Thus, $\tilde{\Delta} a^i_{x_0\eta} \in (V_+ \otimes H)_{n+2}$, which again proves the assertion and completes the proof.

The deferred proof from Section 2 is presented next.

Proof of Lemma 1. The only non trivial claim is that $c \tilde{\circ} d = K$ implies $c = K$. If $c \tilde{\circ} d = K$ then clearly $K_i = a^i_\emptyset(c \tilde{\circ} d) = \tilde{\Delta} a^i_\emptyset(c, d) = a^i_\emptyset c$, $i = 1, 2, \ldots, \ell$. Furthermore, for any $x_j \in X$ with $j \neq 0$, $0 = a^i_{x_j}(c \tilde{\circ} d) = \tilde{\Delta} a^i_{x_j}(c, d) = a^i_{x_j} c$, $i = 1, 2, \ldots, \ell$. Now suppose $a^i_\eta c = 0$, $i = 1, 2, \ldots, \ell$ for all $a^i_\eta \in V_k$ with $k = 1, 2, \ldots, n$. Then for any $x_j \in X$

$$0 = \tilde{\Delta} a^i_{x_j\eta}(c, d) = a^i_{x_j\eta} c + \sum_{a^i_{x_j\eta(2)} \neq 1} a^i_{x_j\eta(1)}(c)\, a^i_{x_j\eta(2)}(d),$$

where in general $a^i_{x_j\eta(1)} \neq a^i_\emptyset$. Therefore, $a^i_{x_j\eta} c = 0$, $i = 1, 2, \ldots, \ell$. In which, case $c = K$.

The antipode of any graded connected Hopf algebra can be computed as described in the following theorem. It can be viewed as being *partially* recursive in that the coproduct needs to be computed first before the antipode recursion can be applied.

Theorem 4 ([8]). *The antipode, S, of any graded connected Hopf algebra (H, μ, Δ) can be computed for any $a \in H_k$, $k \geq 1$ by*

$$Sa = -a - \sum (Sa'_{(1)})a'_{(2)} = -a - \sum a'_{(1)} Sa'_{(2)},$$

where the reduced coproduct is $\Delta' a = \Delta a - a \otimes \mathbf{1} - \mathbf{1} \otimes a = \sum a'_{(1)} a'_{(2)}$.

The next theorem provides a *fully* recursive algorithm to compute the antipode for the output feedback group.

Theorem 5. *The antipode, S, of any $a^i_\eta \in V_+$ for the output feedback group can be computed by the following algorithm:*

i. *Recursively compute $\Delta^j_{\sqcup\!\sqcup}$ via (10).*
ii. *Recursively compute $\tilde{\Delta}$ via Lemma 3.*
iii. *Recursively compute S via Theorem 4 with $\Delta' a^i_\eta = \tilde{\Delta} a^i_\eta - a^i_\eta \otimes \mathbf{1}$.*

Proof. In light of the previous results, the only detail is the minor observation that S is the antipode of the Hopf algebra with coproduct $\Delta a = \tilde{\Delta} a + \mathbf{1} \otimes a$. In which case, the corresponding reduced coproduct is as described in step iii.

Applying the algorithm above gives the antipode of the first few coordinate maps:

$$H_1 : Sa^i_\emptyset = -a^i_\emptyset$$

$$H_2 : Sa^i_{x_j} = -a^i_{x_j}$$

$$H_3 : Sa^i_{x_0} = -a^i_{x_0} + a^i_{x_\ell} a^\ell_\emptyset$$

$$H_3 : Sa^i_{x_j x_k} = -a^i_{x_j x_k}$$

$$H_4 : Sa^i_{x_0 x_j} = -a^i_{x_0 x_j} + a^i_{x_\ell} a^\ell_{x_j} + a^i_{x_\ell x_j} a^\ell_\emptyset$$

$$H_4 : Sa^i_{x_j x_0} = -a^i_{x_j x_0} + a^i_{x_j x_\ell} a^\ell_\emptyset$$

$$H_4 : Sa^i_{x_j x_k x_l} = -a^i_{x_j x_k x_l}$$

$$\begin{aligned} H_5 : Sa^i_{x_0^2} &= -a^i_{x_0^2} - (Sa^i_{x_\ell})a^\ell_{x_0} - (Sa^i_{x_\ell x_0})a^\ell_\emptyset \\ &\quad - (Sa^i_{x_0 x_\ell})a^\ell_\emptyset - (Sa^i_{x_\ell x_\nu})a^\ell_\emptyset a^\nu_\emptyset \\ &= -a^i_{x_0^2} - (-a^i_{x_\ell})a^\ell_{x_0} - (-a^i_{x_\ell x_0} + \cancel{a^i_{x_\ell x_\nu} a^\nu_\emptyset})a^\ell_\emptyset \\ &\quad - (-a^i_{x_0 x_\ell} + a^i_{x_\nu} a^\nu_{x_\ell} + a^i_{x_\nu x_\ell} a^\nu_\emptyset)a^\ell_\emptyset - \cancel{(-a^i_{x_\ell x_\nu})a^\ell_\emptyset a^\nu_\emptyset} \\ &= -a^i_{x_0^2} + a^i_{x_\ell} a^\ell_{x_0} + a^i_{x_\ell x_0} a^\ell_\emptyset + a^i_{x_0 x_\ell} a^\ell_\emptyset - a^i_{x_\nu} a^\nu_{x_\ell} a^\ell_\emptyset - a^i_{x_\nu x_\ell} a^\nu_\emptyset a^\ell_\emptyset, \end{aligned}$$

where $i, j, k, l = 1, 2, \ldots, m$. The explicit calculations for $Sa^i_{x_0^2}$ are shown above to demonstrate that this approach, not unexpectedly, involves some inter-term cancellation. This is consistent with what is known about the classical Faà di Bruno Hopf algebra and the Zimmermann formula, which provides a cancellation free approach to computing the antipode [22,23]. This hints at the possibility of even more efficient antipode algorithms, but this topic will not be pursued here. It should also be noted that when $m = 1$, i.e., the SISO case, all the summations above vanish, and the identities reduce to those given in [4].

Example 3. Suppose $c \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ has finite *Lie rank*, that is, the range space of the Hankel mapping for c defined on the $\mathbb{R}$-vector space of Lie polynomials is finite. Then $I + F_c$ has a finite dimensional control-affine state space realization of the form

$$\dot{z} = g_0(z) + \sum_{j=1}^m g_j(z) u_j, \qquad z(0) = z_0$$

$$y_i = h_i(z) + u_i, \quad i = 1, 2, \ldots, m,$$

where each g_j and h_i is an analytic vector field and function, respectively, on some neighborhood W of z_0 [13,24]. In which case,

$$(c_i, \eta) = L_{g_\eta} h_i(z_0), \quad \forall \eta \in X^*, \tag{12}$$

where

$$L_{g_\eta} h_i := L_{g_{j_1}} \cdots L_{g_{j_k}} h_i, \quad \eta = x_{j_k} \cdots x_{j_1},$$

the *Lie derivative* of h_i with respect to g_j is defined as

$$L_{g_j} h_i : W \to \mathbb{R} : z \mapsto \frac{\partial h_i}{\partial z}(z)\, g_j(z),$$

and $L_{g_\emptyset} h_i = h_i$. It is not difficult to see that the composition inverse of the return difference operator $I + F_c$, that is, $(I + F_c)^{-1} = I + F_{c^{-1}} : u' \mapsto y'$, is described by the feedback system in Fig. 2. A straightforward calculation gives a realization for $F_{c_i^{-1}}$, namely, $(\{g_0 - \sum_{j=1}^m g_j h_j, g_1, \ldots, g_m\}, -h_i, z_0)$. Using this realization and (12), it can be readily verified that Lemma 2 holds. For example,

$$\begin{aligned} (c_i^{-1}, x_0) &= L_{g_0 - \sum_j g_j h_j}(-h_i)(z_0) \\ &= -L_{g_0} h_i(z_0) + \sum_{j=1}^m (L_{g_j} h_i(z_0)) h_j(z_0) \\ &= -(c_i, x_0) + \sum_{j=1}^m (c_i, x_j)(c_j, \emptyset) \\ &= (-a^i_{x_0} + a^i_{x_j} a^j_\emptyset)c \\ &= (Sa^i_{x_0})c. \end{aligned}$$

In the special case of a linear time-invariant system with strictly proper $m \times m$ transfer function $H(s)$ and state space realization (A, B, C), the corresponding components of the linear generating series are $c_i = \sum_{k \geq 0} \sum_{j=1}^m (c_i, x_0^k x_j)\, x_0^k x_j$, where $(c_i, x_0^k x_j) = C_i A^k B_j$, $k \geq 0$, and C_i, B_j denote the ith row of C and the jth column of B, respectively. The composition inverse of the return difference matrix $I + H(s)$ is computed directly as

$$(I + C(sI - A)^{-1}B)^{-1} = I - C(sI - (A - BC))^{-1}B.$$

Therefore, it follows that

$$(c_i^{-1}, x_0^k x_j) = -C_i(A - BC)^k B_j, \quad k \geq 0,\ i, j = 1, 2, \ldots, m.$$

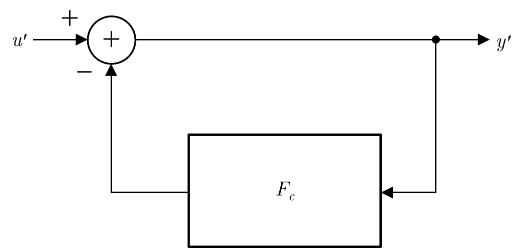

Fig. 2. Feedback implementation of the composition inverse of $I + F_c$.

Expanding this product gives the expected antipode formulas. For example,

$$\begin{aligned}(c_i^{-1}, x_0x_j) &= -C_i(A - BC)B_j \\ &= -C_iAB_j + C_iBCB_j \\ &= -C_iAB_j + \sum_{\ell=1}^{m} C_iB_\ell C_\ell B_j \\ &= -(c_i, x_0x_j) + \sum_{\ell=1}^{m}(c_i, x_\ell)(c_\ell, x_j) \\ &= (-a^i_{x_0x_j} + a^i_{x_\ell}a^\ell_{x_j} + a^i_{x_\ell x_j}a^\ell_{\emptyset})c \\ &= (Sa^i_{x_0x_j})c,\end{aligned}$$

where the fact that $(c, x_\ell x_j) = (c, \emptyset) = 0$ has been used in the second to the last line. It is worth repeating that the antipode formulas derived at the beginning of this section required no state space setting. Hence, they still apply even when c does *not* have finite Lie rank.

The next theorem establishes that local convergence is preserved by the composition inverse operation. This fact was proved for the SISO case in [4] using only a grading of H. But here a different approach is taken, one that produces the exact radius of convergence for the operation.

Theorem 6. *For any $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ it follows that*

$$|(c^{-1}, \eta)| \le K\,(\mathcal{A}(K_c)M_c)^{|\eta|}\,|\eta|!, \quad \forall \eta \in X^*, \tag{13}$$

for some $K > 0$ and

$$\mathcal{A}(K_c) = \frac{1}{1 - mK_c \ln\left(1 + \frac{1}{mK_c}\right)}.$$

Therefore, $c^{-1} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. Furthermore, no geometric growth constant smaller than $\mathcal{A}(K_c)M_c$ can satisfy (13), so the radius of convergence for this operator is $1/\mathcal{A}(K_c)M_c(m+1)$.

Proof. It was shown in [10, Corollary 2] that the generating series for the unity feedback system $c@\delta$ has exactly the properties described above, and therefore, so does $(-c)@\delta$. The present theorem is thus proved by showing that $c^{-1} = (-c)@\delta$. Recall that in the proof of Theorem 2 it was shown in general that $c^{-1} = (-c)\tilde{\circ}\,c^{-1}$. But it is also known that $(-c)@\delta$ satisfies the fixed point equation $(-c)@\delta = (-c)\tilde{\circ}\,((-c)@\delta)$ [2]. Therefore, since $e \mapsto (-c)\tilde{\circ}\,e$ is a contraction on a complete ultrametric space, the identity in question must hold.

It is worth noting that the growth constants determined in Theorem 6 must hold for every series c with growth constants K_c, M_c. Thus, it tends to be conservative for specific series in this class (see [10] for further discussion on this topic). A similar approach yields the global counterpart of this theorem.

Theorem 7. *For any $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ it follows that*

$$|(c^{-1}, \eta)| \le K\,(\mathcal{B}(K_c)M_c)^{|\eta|}\,|\eta|!, \quad \forall \eta \in X^*, \tag{14}$$

for some $K > 0$ and

$$\mathcal{B}(K_c) = \frac{1}{\ln\left(1 + \frac{1}{mK_c}\right)}.$$

Therefore, $c^{-1} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. Furthermore, no geometric growth constant smaller than $\mathcal{B}(K_c)M_c$ can satisfy (14), so the radius of convergence for this operator is $1/\mathcal{B}(K_c)M_c(m+1)$.

It is known that feedback does not in general preserve global convergence (see [25] for a specific example). Thus, there is no reason to expect that the composition inverse will do so either.

4. Feedback product

The goal of this section is to derive an explicit formula for the multivariable feedback product $c@d$ using the Faà di Bruno Hopf algebra described in the previous section. Given two Fliess operators F_c and F_d which are linear time-invariant systems with $\ell_c \times m_c$ transfer function G_c and $\ell_d \times m_d$ transfer function G_d, respectively, the closed-loop transfer function is clearly

$$G_c(I - G_dG_c)^{-1} = G_c\sum_{k=0}^{\infty}(G_dG_c)^k, \tag{15}$$

where necessarily $\ell_c = m_d$ and $\ell_d = m_c$. There is no a priori requirement that the systems be *square*, that is, $m_c = \ell_c$ or $m_d = \ell_d$. But to handle the most general case here, the series composition products introduced in Sections 2 and 3 have to be generalized to accommodate two alphabets, $X_c = \{x_0, x_1, \ldots, x_{m_c}\}$ and $X_d = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{m_d}\}$. This offers no serious technical issues as described in [26, Example 3.5], just a bit more bookkeeping. The inverse is computed easily in this special case because all the underlying series are rational. The next theorem gives the nonlinear generalization of (15).

Theorem 8. *For any $c \in \mathbb{R}^{m_d}\langle\langle X_c\rangle\rangle$ and $d \in \mathbb{R}^{m_c}\langle\langle X_d\rangle\rangle$, it follows that*

$$c@d = c\,\tilde{\circ}\,(-d\circ c)^{-1} = c\circ(\delta - d\circ c)^{-1}. \tag{16}$$

Proof. The proof is not significantly different from the SISO case presented in [4]. Since it is short, it is presented here for completeness. Clearly the function v in Fig. 1 must satisfy the identity

$$v = u + F_{d\circ c}[v].$$

Therefore,

$$(I + F_{-d\circ c})\,[v] = u,$$

where in the notation of Section 3, the operator on the left-hand side is an element of $\mathscr{F}_\delta$ with $m = m_c = \ell_d$. Applying the composition inverse $\left(I + F_{(-d\circ c)^{-1}}\right)$ on the left gives

$$v = \left(I + F_{(-d\circ c)^{-1}}\right)[u],$$

and thus,

$$F_{c@d}[u] = F_c[v] = F_{c\,\tilde{\circ}\,(-d\circ c)^{-1}}[u]$$

as desired. The second identity in the theorem is just a formal way of expressing the first identity since $c\,\tilde{\circ}\,(-d\circ c)^{-1} = c\circ(\delta + (-d\circ c)^{-1})$ and by definition $(\delta - d\circ c)^{-1} = \delta + (-d\circ c)^{-1}$.

As noted earlier, (16) also makes sense when either $c = \delta$ or $d = \delta$, namely, $\delta@d = (\delta - d)^{-1} = \delta + (-d)^{-1}$ and $c@\delta = c\circ(\delta - c)^{-1} = (-c)^{-1}$. In addition, it was shown in [2, Theorem 4.3] that $c@d$ satisfies the fixed point equation $c@d = c\circ(\delta + d\circ(c@d))$.

So if c is a linear series then

$$c@d = c + c \circ d \circ (c@d)$$
$$(\delta - c \circ d) \circ (c@d) = c$$
$$c@d = (\delta - c \circ d)^{-1} \circ c.$$

But in general, even in the SISO case, $c@d \neq (\delta - c \circ d)^{-1} \circ c$.

Next it is shown that feedback preserves local convergence. But the following preliminary result is needed first.

Theorem 9. *The triple* $(\mathbb{R}^m_{LC}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ *is a subgroup of* $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$.

Proof. The set of series $\mathbb{R}^m_{LC}\langle\langle X_\delta\rangle\rangle$ is closed under composition since the set $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ is closed under addition and modified composition [2,20]. In light of Theorem 6, $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ is also closed under inversion. Hence, the theorem is proved.

Theorem 10. *If* $c \in \mathbb{R}^{m_d}_{LC}\langle\langle X_c\rangle\rangle$ *and* $d \in \mathbb{R}^{m_c}_{LC}\langle\langle X_d\rangle\rangle$ *then* $c@d \in \mathbb{R}^{m_d}_{LC}\langle\langle X_c\rangle\rangle$.

Proof. Since the composition product, the modified composition product, and the composition inverse all preserve local convergence, the claim follows directly from Theorem 8.

Example 4. Consider the differential axle shown in Fig. 3. This device has zero mass and moves in the plane with independent angular velocities u_r and u_l corresponding to the right and left wheels, respectively. The dynamics of this system are

$$\dot{z}_1 = \frac{r}{2}(u_l + u_r)\cos(z_3)$$
$$\dot{z}_2 = \frac{r}{2}(u_l + u_r)\sin(z_3)$$
$$\dot{z}_3 = \frac{r}{L}(u_r - u_l).$$

In particular, if $u_l = u_r > 0$ then the axle moves forward in the direction the wheels are pointing, and if $u_l = -u_r > 0$ the axle rotates clockwise because the wheels are turning in opposite directions. For simplicity, define $u_1 = \frac{1}{2}(u_l + u_r)$ and $u_2 = (u_r - u_l)$, and let $L = r = 1$. Choosing outputs $y_i = z_i$, $i = 1, 2$, the corresponding two-input, two-output state space realization is

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{pmatrix} = \begin{pmatrix}\cos(z_3)\\ \sin(z_3)\\ 0\end{pmatrix} u_1 + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} u_2 \tag{17a}$$

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}z_1\\ z_2\end{pmatrix}. \tag{17b}$$

Its generating series, c, can be computed directly from (12) using the vector fields and output function given in (17). With the help of the Mathematica software package NCAlgebra [27], this calculation gives

$$c = \begin{pmatrix} x_1 - x_1x_2^2 + x_1x_2^4 - x_1x_2^6 + x_1x_2^8 - x_1x_2^{10} + \cdots \\ x_1x_2 - x_1x_2^3 + x_1x_2^5 - x_1x_2^7 + x_1x_2^9 - x_1x_2^{11} + \cdots \end{pmatrix}$$

when $z_1(0) = z_2(0) = z_3(0) = 0$. This series is clearly in $\mathbb{R}^2_{GC}\langle\langle X\rangle\rangle$ with $X = \{x_0, x_1, x_2\}$ and growth constants $K_c = 1$ and $M_c = 1$.

Consider now the problem of steering the differential axle around a circle. For this purpose, a two channel proportional–integral controller is used in the feedback path so that one obtains a closed-loop system as shown in Fig. 1. The dynamics of the controller are

$$\begin{pmatrix}\dot{z}_4\\ \dot{z}_5\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}\tilde{u}_1 + \begin{pmatrix}0\\ 1\end{pmatrix}\tilde{u}_2 \tag{18a}$$

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix} = \begin{pmatrix}k_1z_4\\ k_2z_5\end{pmatrix}. \tag{18b}$$

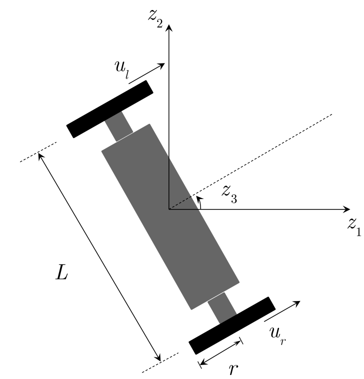

Fig. 3. Differential axle.

For gains $k_1 = 2$ and $k_2 = 10$, the corresponding generating series is

$$d = \begin{pmatrix} 4 + 2x_1 \\ 20 + 10x_2 \end{pmatrix}$$

when $z_4(0) = z_5(0) = 2$. Here $d \in \mathbb{R}^2_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_d = 20$ and $M_d = 0.5$.

From Theorem 8, the feedback product of c and d is given by the series

$$\begin{aligned}(c@d)_1 = {} & 4x_0 + x_1 - 1592x_0^3 + 2x_0^2x_1 - 80x_0^2x_2 - 80x_0x_2x_0 \\ & - 4x_0x_2^2 - 400x_1x_0^2 - 20x_1x_0x_2 - 20x_1x_0x_2 \\ & - 20x_1x_2x_0 - x_1x_2^2 + 617{,}616x_0^5 - 2396x_0^4x_1 \\ & + 31{,}360x_0^4x_2 - 1600x_0^3x_1x_0 - 80x_0^3x_1x_2 \\ & + 31{,}360x_0^3x_2x_0 - 80x_0^3x_2x_1 + 1584x_0^3x_2^2 \\ & - 1600x_0^2x_1x_0^2 - 80x_0^2x_1x_0x_2 - 80x_0^2x_1x_2x_0 - 4x_0^2x_1x_2^2 \\ & + 31{,}520x_0^2x_2x_0^2 - 80x_0^2x_2x_0x_1 + 1592x_0^2x_2x_0x_2 \\ & - 40x_0^2x_2x_1x_0 - 2x_0^2x_2x_1x_2 + 1592x_0^2x_2^2x_0 - 2x_0^2x_2^2x_1 \\ & + 80x_0^2x_2^3 + 31{,}520x_0x_2x_0^3 - 80x_0x_2x_0^2x_1 \\ & + 1592x_0x_2x_0^2x_2 - 40x_0x_2x_0x_1x_0 - 2x_0x_2x_0x_1x_2 \\ & + 1592x_0x_2x_0x_2x_0 - 2x_0x_2x_0x_2x_1 + 80x_0x_2x_0x_2^2 \\ & + 1592x_0x_2^2x_0^2 - 2x_0x_2^2x_0x_1 + 160{,}000x_1x_0^4 + \cdots\end{aligned}$$

and

$$\begin{aligned}(c@d)_2 = {} & 80x_0^2 + 4x_0x_2 + 20x_1x_0 + x_1x_2 - 31{,}520x_0^4 + 80x_0^3x_1 \\ & - 1592x_0^3x_2 + 40x_0^2x_1x_0 + 2x_0^2x_1x_2 - 1592x_0^2x_2x_0 \\ & + 2x_0^2x_2x_1 - 80x_0^2x_2^2 - 1592x_0x_2x_0^2 + 2x_0x_2x_0x_1 \\ & - 80x_0x_2x_0x_2 - 80x_0x_2^2x_0 - 4x_0x_2^3 - 8000x_1x_0^3 \\ & - 400x_1x_0^2x_2 - 400x_1x_0x_2x_0 - 20x_1x_0x_2^2 - 400x_1x_2x_0^2 \\ & - 20x_1x_2x_0x_2 - 20x_1x_2^2x_0 - x_1x_2^3 + 3200x_0^5 + 160x_0^4x_2 \\ & + 800x_0^3x_1x_0 + 40x_0^3x_1x_2 + 800x_1x_0^4 + 40x_1x_0^3x_2 \\ & + 200x_1x_0^2x_1x_0 + 10x_1x_0^2x_1x_2 + 11{,}841{,}600x_0^6 + \cdots.\end{aligned}$$

The outputs of the closed-loop system for a zero reference input are then

$$\begin{aligned}y_1(t) &= F_{(c@d)_1}[0](t) \\ &= 4t - \frac{796t^3}{3} + \frac{25{,}734t^5}{5} - \frac{4000t^6}{9} - \frac{13{,}528{,}798t^7}{315} \\ &\quad + \frac{1{,}653{,}800t^8}{63} + \frac{594{,}150{,}001t^9}{5670} + \cdots\end{aligned}$$

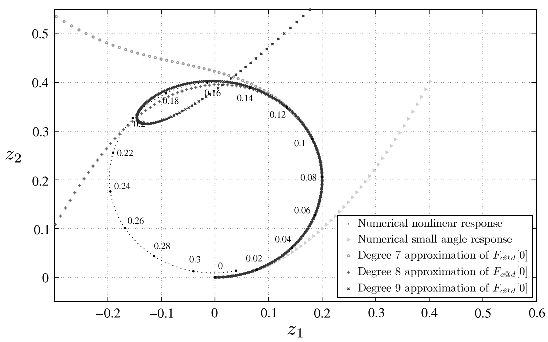

Fig. 4. Various estimates of the natural response of the closed-loop differential axle system.

and

$$y_2(t) = F_{(c@d)_2}[0](t)$$
$$= 40t^2 - \frac{3{,}940t^4}{3} + \frac{80t^5}{3} + \frac{49{,}340t^6}{3} - \frac{254{,}560t^7}{63}$$
$$- \frac{5{,}496{,}593t^8}{63} + \frac{25{,}525{,}060t^9}{189} + \cdots.$$

Various estimates of the natural response of the closed-loop differential axle system are shown in Fig. 4. The tick marks along the circle indicate time. Specifically, the numerically computed nonlinear closed-loop response of (17)–(18) is compared against Fliess operator responses whose generating series are computed from the feedback product truncated to degrees 7, 8 and 9. Also shown in Fig. 4 is the response of the small angle approximation system

$$\begin{pmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{pmatrix} = \begin{pmatrix}1\\ z_3\\ 0\end{pmatrix} u_1 + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} u_2$$
$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

steered by the same proportional–integral controller. It is evident that this system underestimates the correct position of the differential axle almost immediately. On the other hand, the Fliess operator approximations clearly improve as additional terms are added to the approximation.

One way to get some insight into the convergence characteristics of $F_{c@d}[0]$ is to empirically estimate the geometric growth constants for the natural response portion of the series $c@d$, i.e., the series $(c@d)_N := \sum_{k\geq 0}(c@d, x_0^k)x_0^k$, by plotting $\ln((c@d)_N/|\eta|!)$ versus $|\eta|$ for the local case and $\ln((c@d)_N)$ versus $|\eta|$ for the global case as shown in Fig. 5. To improve the quality of the estimates, coefficients above order nine were computed using (12). In each case, the corresponding growth constants can be estimated by linearly fitting the data (see [19] for more discussion concerning this methodology). The parameter R^2 is the square of Pearson's correlation coefficient, so the closer this statistic is to unity, the better the linear fit. In this case, the data appears to match better the global growth rate with $M_{(c@d)_N} = \exp(3.1157) = 22.549$. But as will be discussed shortly, the series can fall somewhere *in between* being locally convergent and globally convergent as defined by (4) and (5), respectively. There is also the option of constructing a piecewise analytic approximation of the response using a sequence of closed-loop generating series computed by brute force or via analytic extension [28]. This approach has the additional advantage that lower order approximations of each piece are likely to suffice. For example, it appears here that two degree nine approximations joined at the midpoint of the path could easily traverse the entire circle.

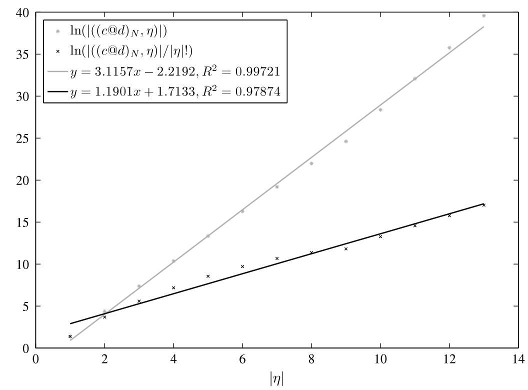

Fig. 5. Linear fits of the coefficients of $(c@d)_N$ on a logarithmic scale.

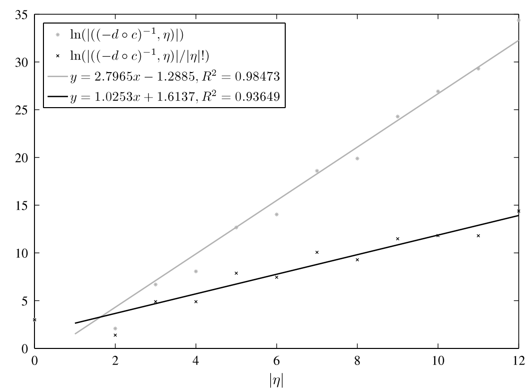

Fig. 6. Linear fits of the coefficients of $e = (-d \circ c)^{-1}$ on a logarithmic scale.

An alternative method to exploring the nature of the convergence of $F_{c@d}[0]$ is to use Theorem 6 or Theorem 7 in conjunction with what is known at present about the convergence of interconnected Fliess operators as reported in [10]. First observe that $(-d\circ c, \eta) = \pm\binom{2}{10}$ when $\eta \neq \emptyset$, and therefore, $-d\circ c$ is globally convergent with $K_{d\circ c} = 20$ and $M_{d\circ c} = 1$. Applying Theorem 7 provides an upper bound on the local geometric growth constant of $e := (-d\circ c)^{-1}$, specifically,

$$M_e = \frac{M_{d\circ c}}{\ln\left(1 + \frac{1}{2K_{d\circ c}}\right)} = 40.50.$$

Repeating the empirical method used above for e gives the data shown in Fig. 6. It indicates that e is more globally convergent in nature than locally convergent, but if it were assumed to be locally convergent, the corresponding geometric growth constant would be $\exp(1.0253) = 2.7879 < 40.50$. On the other hand, if it were taken to be globally convergent then (since $F_{c\tilde{\circ}e}[0] = F_{c\circ e}[0]$) it follows that $(c@d)_N = c\circ e$ is the composition of two globally convergence series. As discussed in [10, p. 2800], the resulting series can lie strictly in between locally and globally convergent. But independent of this fact, it is still known in this instance that the series $F_{c\circ e}[0]$ will converge over any finite interval [10, Theorem 9].

5. Conclusions

The main thrust of this paper was to provide the full multivariable extension of a theory to explicitly compute the generating series of a feedback interconnection of two systems represented as Fliess operators. This was largely facilitated by utilizing a new type of grading for the underlying Hopf algebra. This grading also provided a fully recursive algorithm to compute the antipode of the algebra and thus, the corresponding feedback product can be computed much more efficiently. Finally, an improved convergence analysis of the antipode operation was presented, one that gives the radius of convergence for this operation.

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