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A Faà di Bruno Hopf algebra for a group of Fliess operators with applications to feedback

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ABSTRACT

A Faà di Bruno type Hopf algebra is developed for a group of integral operators known as Fliess operators, where operator composition is the group product. Such operators are normally written in terms of generating series over a noncommutative alphabet. Using a general series expansion for the antipode, an explicit formula for the generating series of the compositional inverse operator is derived. The result is applied to analytic nonlinear feedback systems to produce an explicit formula for the feedback product, that is, the generating series for the Fliess operator representation of the closed-loop system written in terms of the generating series of the Fliess operator component systems. This formula is employed to provide a proof that local convergence is preserved under feedback.

1. Introduction

Let f and g be two functions with convergent Taylor series about $x = 0$ which leave the origin invariant, say $f(x) = \sum_{n\geq 1} f_n x^n/n!$ and $g(x) = \sum_{n\geq 1} g_n x^n/n!$ The composition $h = f \circ g$ has the same nature as f and g, and the well-known Faà di Bruno formula provides its Taylor series coefficients, specifically,

$$h_n = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{j} \frac{n!k!}{j_1!j_2!\cdots j_n!} \frac{g_1^{j_1} g_2^{j_2} \cdots g_n^{j_n}}{(1!)^{j_1}(2!)^{j_2}\cdots(n!)^{j_n}}, \tag{1}$$

where the second sum is over all $j_1, j_2, \ldots, j_n \geq 0$ such that $j_1+j_2+\cdots+j_n = k$. In the event that the series are not convergent, the functions involved can be interpreted as formal functions rather than as analytic functions. In either case, if $f_1 \neq 0$ then f has a compositional inverse, f^{-1}, and therefore, the corresponding set of functions forms a group under composition. In the special case where $f_1 = 1$, the coordinate functions $a_n : f \mapsto f_n$, $n \geq 1$ on the corresponding subgroup form a graded connected Hopf algebra, a so-called Faà di Bruno Hopf algebra [1–5]. The antipode of this Hopf algebra acts on each coordinate function to produce a polynomial expression for the coordinates of the compositional inverse. It turns out that this algebra has great utility in quantum field theory and related areas [3,5].

In this paper, an analogous Faà di Bruno Hopf algebra is developed for a group of integral operators known as Fliess operators. Such an operator, F_c, is normally written in terms of a generating series c over a noncommutative alphabet $X = \{x_0, x_1, \ldots, x_m\}$ [6–8]. It was shown in [9] that a noncommutative version of (1) describes the input–output map $F_c : u \mapsto y$ when u is described by a Taylor series (in one variable). In contrast, the focus here is on system interconnections. First, it is shown that the set of operators

$$\mathscr{F}_\delta := \{I + F_c : c \in \mathbb{R}\langle\langle X \rangle\rangle\},$$

where I denotes the identity operator, and $\mathbb{R}\langle\langle X \rangle\rangle$ is the set of all formal power series over X, forms a group under operator composition when $m = 1$. It is worth noting that the elements of $\mathscr{F}_\delta$ bear some resemblance to the group of diffeomorphisms on $\mathbb{R}$ having the form $f(x) = x+O(x^2)$, as well as to the noncommutative compositional groups that appear in [2,10]. In the latter case, however, composition refers to the direct composition of power series, a notion which is entirely distinct from the composition product used here to describe Fliess operator composition [11–15]. Furthermore, an element like $I + F_c$ is *not*, strictly speaking, a Fliess operator since I has no integral representation. Nevertheless, tools already exist for handling this modest generalization of the Fliess operator concept in the context of operator composition since $\mathscr{F}_\delta$ naturally arises in the study of analytic nonlinear feedback systems [13,14]. Next, a graded Faà di Bruno bialgebra

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is systematically constructed for the coordinate functions of $\mathscr{F}_\delta$. Since the generating series are completely arbitrary, F_c may only be a formal Fliess operator and not necessarily convergent in any sense [6,16]. It will be shown subsequently that convergent operators form a subgroup of $\mathscr{F}_\delta$. Next, the existence of an antipode is addressed. It is shown that while the bialgebra under consideration is *not* connected, a well-defined antipode *does* exist so as to render a graded Faà di Bruno Hopf algebra. This class of combinatorial Hopf algebras is quite distinct from those normally associated with the Cauchy product and shuffle product [17–21], which for the most part involve a finite alphabet. Finally, it is shown that the subgroup of operators having proper generating series, i.e., generating series with a zero constant term, leads to a connected Faà di Bruno Hopf subalgebra. This structure is most similar to the one in the classical case. As an application, it is demonstrated that the antipode formula naturally appears in the context of feedback theory for Fliess operators. Specifically, it was shown via a fixed point argument in [13,16] that any feedback connection involving two Fliess operators F_c and F_d always produces a closed-loop system with a Fliess operator representation. The fixed point, represented by the generating series $c@d$, defines a formal series product of c and d referred to as the *feedback product*. Such an approach, however, does not provide an explicit formula for computing this product. It will be shown here that a suitable formula can be derived in terms of the Faà di Bruno Hopf antipode associated with $\mathscr{F}_\delta$. Aside from the obvious computational benefits, it will be used to provide a proof of the fact that feedback preserves convergence. That is, if F_c and F_d are convergent Fliess operators, then their feedback connection, $F_{c@d}$, is also convergent. This result was recently proved in [22,23] for the special case of unity feedback systems, that is, when F_d is replaced with I, by computing the radius of convergence. Here the general case is addressed.

The paper is organized as follows. In the next section, a brief overview is given of Fliess operator theory, and the notation is established. Similarly, the essential elements of Hopf algebra theory employed in the paper are summarized. In Section 3, the Faà di Bruno Hopf algebra of interest is constructed. Its application to feedback systems is described in the subsequent section. The main conclusions are summarized in the final section.

2. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is written as X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η). Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. A series c is called *proper* when $(c, \emptyset) = 0$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation (Cauchy) product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, that is, the $\mathbb{R}$-bilinear mapping $\mathbb{R}^\ell\langle\langle X\rangle\rangle \times \mathbb{R}^\ell\langle\langle X\rangle\rangle \to \mathbb{R}^\ell\langle\langle X\rangle\rangle$ uniquely specified by the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \eta$.

2.1. Fliess operators and their interconnections

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. $C[t_0, t_1]$ is the set of all real-valued continuous functions on $[t_0, t_1]$. Define recursively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input–output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0)$$

[6–8]. If there exists real numbers $K_c, M_c > 0$ such that $|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!$ for all $\eta \in X^*$, then F_c constitutes a well-defined mapping from $B^m_\mathfrak{p}(R)[t_0, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [24]. The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. It was shown in [15, Corollary 2.2] that if $F_c = F_d$ on any $B^m_\mathfrak{p}(R)[t_0, t_0 + T]$ then $c = d$. A similar uniqueness result for the formal case is described in [16].

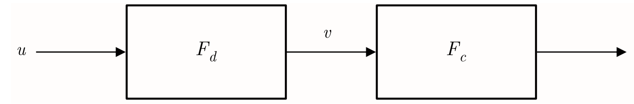


Fig. 1. Cascade connection of two Fliess operators.

When two Fliess operators are interconnected in a cascade fashion as shown in Fig. 1, the new system always has a Fliess operator representation, and the composition product can be used to describe its generating series. It is convenient to first define a family of mappings for any $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, namely,

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Let $D_\emptyset$ be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_i x_j} := D_{x_i} D_{x_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the catenation product.

Definition 1 (*[11–13,25]*). The *composition product* of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$(\underbrace{x_{i_k} x_{i_{k-1}} \cdots x_{i_1}}_{\eta}) \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ define

$$c \circ d = \sum_{\eta \in X^*} (c, \eta) D_\eta(1).$$

The composition product is linear in its left argument, distributes to the left over the shuffle product, and has the key property that $F_c \circ F_d = F_{c \circ d}$ for any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ [11,12]. In addition, the composition product preserves local convergence [13], and the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ in the ultrametric sense [11,13].

In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 2, the output y must satisfy the feedback equation

$$y = F_c[u + F_d[y]]$$

for every admissible input u. It was shown in [13,16] that there always exists a generating series e so that $y = F_e[u]$. In which case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d \circ e}[u]] = F_{c \tilde{\circ} (d \circ e)}[u],$$

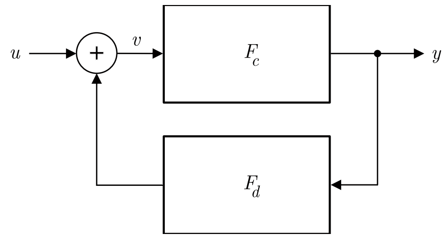


Fig. 2. Feedback connection of two Fliess operators.

where $\tilde{\circ}$ denotes the *modified* composition product. That is, the product

$$c\tilde{\circ}d = \sum_{\eta\in X^*}(c,\eta)\tilde{D}_\eta(1),$$

where

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup e)$$

with $d_0 := 0$. The *feedback product* of c and d, namely $c@d$, is thus defined as the unique fixed point of the contractive iterated map

$$\tilde{S} : e_i \mapsto e_{i+1} = c\tilde{\circ}(d \circ e_i).$$

Specifically, $c@d = e$, where $e = c\tilde{\circ}(d \circ e)$. In the case of a unity feedback system, this equation reduces to $e = c\tilde{\circ}e$. Given arbitrary c and d, there is no general method for computing $c@d$ explicitly.

2.2. Hopf algebra fundamentals

The basic elements of Hopf algebra theory used in the paper are now summarized. The treatment is based on [1,3,26–28]. The starting point is a systematic statement of what it means for a set A to be an associative $\mathbb{R}$-algebra. Let A be an $\mathbb{R}$-vector space and consider an $\mathbb{R}$-bilinear map and an $\mathbb{R}$-linear map,

$$\mu : A \otimes A \to A, \qquad \sigma : \mathbb{R} \to A,$$

respectively, which satisfy the associative property and unitary property as described by the commutative diagrams in Fig. 3. Here id denotes the identity map on A, and the symbol $\sim$ denotes the canonical isomorphism between the vector spaces A and $A \otimes \mathbb{R}$. These diagrams are equivalent to, respectively, the identities

$$(ab)c = a(bc), \quad a, b, c \in A$$
$$1_A a = a = a 1_A, \quad a \in A,$$

where $\mu(a \otimes b) = ab$ and $\sigma(1) = 1_A$ is the unit of A. Traditionally, μ is called the *multiplication map*, and σ is called the *unit map*. The triple (A, μ, σ) is an associative $\mathbb{R}$-algebra. Next suppose there exist two $\mathbb{R}$-linear maps

$$\Delta : A \to A \otimes A, \qquad \epsilon : A \to \mathbb{R},$$

which satisfy the coassociative property and the counitary property as illustrated in Fig. 4. These commutative diagrams are the same as the ones depicted in Fig. 3 except that the directions of the arrows have been reversed. In this case, Δ is called the *comultiplication map*, and ϵ is the *counit map*. The triple (A, Δ, ϵ) is called an $\mathbb{R}$-*coalgebra*. In this setting, consider the following definition.

Definition 2. A *morphism* between two $\mathbb{R}$-algebras (A_1, μ_1, σ_1) and (A_2, μ_2, σ_2) is any $\mathbb{R}$-linear map $\psi : A_1 \to A_2$ such that

$$\psi \circ \mu_1 = \mu_2 \circ (\psi \otimes \psi)$$
$$\psi \circ \sigma_1 = \sigma_2.$$

An analogous definition can be given for a morphism between two $\mathbb{R}$-coalgebras. Using either concept, one can produce the notion of a bialgebra as described next.

Definition 3. The five-tuple $(A, \mu, \sigma, \Delta, \epsilon)$ is called an $\mathbb{R}$-*bialgebra* when Δ and ϵ are both $\mathbb{R}$-algebra morphisms.

Specifically this means that the mapping $\Delta : A \to A \otimes A$ must be an $\mathbb{R}$-algebra morphism between the $\mathbb{R}$-algebras (A, μ, σ) and $(A \otimes A, \mu_{A\otimes A}, \sigma_{A\otimes A})$, where

$$\mu_{A\otimes A} : (A \otimes A) \otimes (A \otimes A) \to A \otimes A$$
$$: (a_1 \otimes a_2) \otimes (a_3 \otimes a_4) \mapsto \mu(a_1 \otimes a_3) \otimes \mu(a_2 \otimes a_4)$$
$$\sigma_{A\otimes A} : \mathbb{R} \to A \otimes A$$
$$: k \mapsto \sigma(k) \otimes \sigma(1).$$

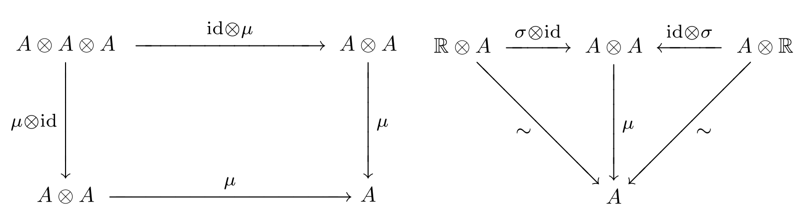
In which case, it follows directly that

1. $\Delta \circ \mu = \mu_{A\otimes A} \circ (\Delta \otimes \Delta) = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$
2. $\Delta \circ \sigma = \sigma_{A\otimes A} = \sigma \otimes \sigma$,

where $\tau : A \otimes A \to A \otimes A : a \otimes a' \mapsto a' \otimes a$. Similarly, $\epsilon : A \to \mathbb{R}$ must be an $\mathbb{R}$-algebra morphism between the $\mathbb{R}$-algebras (A, μ, σ) and $(\mathbb{R}, \mu_\mathbb{R}, \sigma_\mathbb{R})$. Therefore,

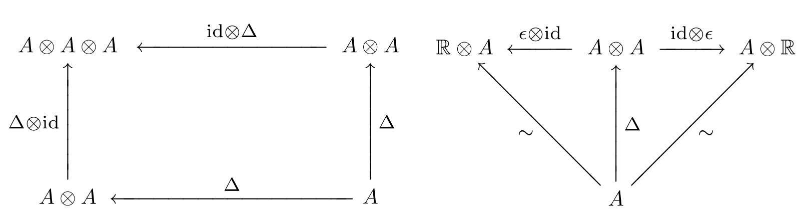
3. $\epsilon \circ \mu = \mu_\mathbb{R} \circ (\epsilon \otimes \epsilon) = \epsilon^2$
4. $\epsilon \circ \sigma = \sigma_\mathbb{R} = 1$.

Note that properties 1 and 2 can be expressed in terms of the commutative diagrams shown in Fig. 5, and, likewise, properties 3 and 4 are shown in Fig. 6. If instead one introduces the notion of a $\mathbb{R}$-coalgebra morphism as suggested above, then an equivalent characterization of a bialgebra is one where μ and σ are both



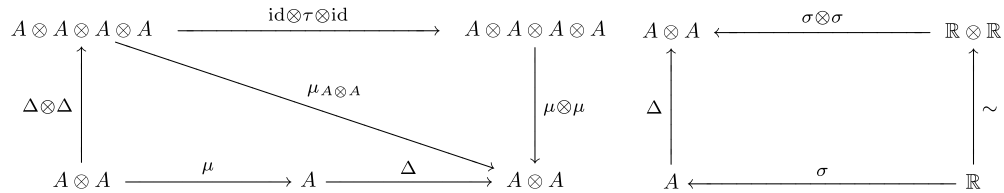
(a) The associative property. (b) The unitary property.

Fig. 3. The defining properties of an $\mathbb{R}$-algebra (A, μ, σ).



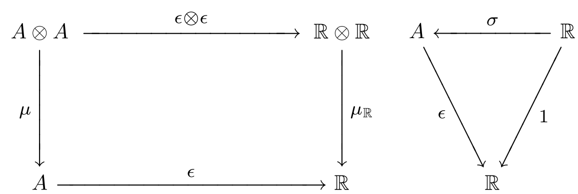
(a) The coassociative property. (b) The counitary property.

Fig. 4. The defining properties of an $\mathbb{R}$-coalgebra (A, Δ, ϵ).



(a) Property 1. (b) Property 2.

Fig. 5. The commutative diagrams describing Δ as an $\mathbb{R}$-algebra morphism.



(a) Property 3. (b) Property 4.

Fig. 6. The commutative diagrams describing ϵ as an $\mathbb{R}$-algebra morphism.

$\mathbb{R}$-coalgebra morphisms, yielding properties 1 and 3, and properties 2 and 4, respectively.

To complete the development of the Hopf algebra definition, consider the set of all $\mathbb{R}$-endomorphisms on A, denoted by $\mathrm{End}(A)$. Given two arbitrary $f, g \in \mathrm{End}(A)$, the *Hopf convolution product*,

$$f * g := \mu \circ (f \otimes g) \circ \Delta,$$

defines another element of $\mathrm{End}(A)$. The following theorem is central to the theory.

Theorem 1. *The triple* $(\mathrm{End}(A), *, \vartheta)$ *forms an associative* $\mathbb{R}$*-algebra with unit* $\vartheta = \sigma \circ \epsilon$.

Finally, an element $\alpha \in \mathrm{End}(A)$ is called an *antipode* of the bialgebra if

$$\mathrm{id} * \alpha = \alpha * \mathrm{id} = \vartheta.$$

Clearly, this implies that an antipode is a convolution inverse of the identity map id. When an antipode exists, it is unique and described by the series

$$\alpha = \mathrm{id}^{*-1} = (\vartheta - (\vartheta - \mathrm{id}))^{*-1} = \sum_{k=0}^{\infty} (\vartheta - \mathrm{id})^{*k}. \tag{2}$$

For any $a, a' \in A$ it follows that $\alpha(aa') = \alpha(a')\alpha(a)$. This final bit of structure culminates in the definition below.

Definition 4. The six-tuple $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ is called an $\mathbb{R}$-*Hopf algebra*.

The following definitions concerning bialgebras will be important.

Definition 5. An $\mathbb{R}$-bialgebra $(A, \mu, \sigma, \Delta, \epsilon)$ is *filtered* if there exists a nested sequence of $\mathbb{R}$-vector subspaces of A, say $A_0 \subsetneq A_1 \subsetneq \cdots$, such that $A = \cup_{n\geq 0} A_n$ and

$$\Delta A_n \subseteq \sum_{i=0}^{n} A_i \otimes A_{n-i}.$$

The collection $\{A_n\}_{n\geq 0}$ is called a *filtration* of A.

Definition 6. An $\mathbb{R}$-bialgebra that is filtered such that $A_0 = \sigma(\mathbb{R})$ is said to be *connected*.

Definition 7. An $\mathbb{R}$-bialgebra is *graded* if there exists a set of $\mathbb{R}$-vector subspaces of A, say $\{A_{(n)}\}_{n\geq 0}$, such that $A = \oplus_{n\geq 0} A_{(n)}$ with

$$A_{(i)}A_{(j)} \subseteq A_{(i+j)}, \qquad \Delta A_{(n)} \subseteq \bigoplus_{i=0}^{n} A_{(i)} \otimes A_{(n-i)},$$

and $\epsilon(A_{(n)}) = 0$, $n > 0$.

Definition 8. Let A be an $\mathbb{R}$-bialgebra. An element $g \in A$ is *group-like* if $\epsilon(g) = 1$ and $\Delta g = g \otimes g$. If A has only one group-like element, then any other element $a \in A$ is *primitive* if $\Delta a = a \otimes g + g \otimes a$.

A number of useful results follow from these definitions. For example, if A has a grading $\{A_{(n)}\}_{n\geq 0}$, then a natural filtration of A is $\{A_n\}_{n\geq 0}$, where

$$A_n = \bigoplus_{i=0}^{n} A_{(i)}.$$

Furthermore, if $A_{(0)} = \sigma(\mathbb{R})$ then A has only one group-like element. Perhaps the most important result in the present context concerns a key property of the coalgebra. If $A^+ := \ker \epsilon$ and $A_n^+ := A^+ \cap A_n$ then for any $a \in A_n^+$ it follows that

$$\Delta a = a \otimes 1 + 1 \otimes a + \Delta' a, \tag{3}$$

where $\Delta' a \in A_{n-1}^+ \otimes A_{n-1}^+$. From this property, it can be shown that $A = A_0 \oplus A^+$ and that the following theorem holds.

Theorem 2. *Let* $(A, \mu, \sigma, \Delta, \epsilon)$ *be a connected* $\mathbb{R}$*-bialgebra. Then* $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ *is an* $\mathbb{R}$*-Hopf algebra, where the antipode is given on* A^+ *by*

$$\alpha = -\mathrm{id} + \sum_{k=1}^{\infty} (-1)^{k+1} \mu^k \Delta'^k \tag{4}$$

with

$$\mu^k : A \otimes A \otimes \cdots \otimes A \to A : a_1 \otimes a_2 \otimes \cdots \otimes a_{k+1} \mapsto a_1 a_2 \cdots a_{k+1}$$
$$\Delta'^{n+1} = (\mathrm{id} \otimes \Delta')\Delta'^n = (\Delta' \otimes \mathrm{id})\Delta'^n, \quad n \geq 1.$$

Furthermore,

$$(\vartheta - \mathrm{id})^{*k+1} a = (-1)^{k+1} \mu^k \Delta'^k a = 0, \quad a \in A_n^+,\ k \geq n \geq 1, \tag{5}$$

and thus, (4) *evaluated at* a *has at most* n *nonzero terms. Otherwise, on* A_0, $\alpha = \mathrm{id}$.

It is easy to show that the *reduced* coproduct, Δ', inherits its coassociativity property from that of Δ.

3. A Faà di Bruno Hopf algebra for a group of Fliess operators

3.1. Group of Fliess operators

For brevity the presentation henceforth is restricted to the single-input, single-output case, i.e., $m = \ell = 1$. Let $X = \{x_0, x_1\}$

and define the set of operators

$$\mathscr{F}_\delta = \{I + F_c : c \in \mathbb{R}\langle\langle X\rangle\rangle\}.$$

It is convenient to introduce the Dirac symbol δ and the definition $F_\delta = I$ such that $I + F_c = F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. The transformation $\omega : c \mapsto \delta + c$ can be viewed as a type of Magnus transformation. That is, ω maps the free semigroup $(\mathbb{R}\langle\langle X\rangle\rangle, \circ, \delta)$ to a free group with generators $\delta + x_i$, $i = 0, 1$ [29, Theorem 5.6]. This suggests that $\mathscr{F}_\delta$ will also form a group under composition. Consider the composition of two elements in $\mathscr{F}_\delta$:

$$\begin{aligned} F_{c_\delta} \circ F_{d_\delta} &= (I + F_c) \circ (I + F_d) \\ &= I + F_d + F_c(I + F_d) \\ &= I + F_d + F_{c\tilde{\circ}d} \\ &= F_{\delta+d+c\tilde{\circ}d} \\ &= F_{c_\delta \circ d_\delta}, \end{aligned}$$

where $c_\delta \circ d_\delta := \delta + d + c\tilde{\circ}d$. It was shown in [14] that the modified composition product on $\mathbb{R}\langle\langle X\rangle\rangle$ is not associative. The following lemma describes precisely *how* nonassociative this product is. This fact is used subsequently to show that the composition product on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ *is* associative.

Lemma 1. *For any $c, d, e \in \mathbb{R}\langle\langle X\rangle\rangle$, it follows that*

$$(c\tilde{\circ}d)\tilde{\circ}e = c\tilde{\circ}(e + d\tilde{\circ}e).$$

Proof. Observe that

$$\begin{aligned} F_{(c\tilde{\circ}d)\tilde{\circ}e} &= F_{c\tilde{\circ}d}(I + F_e) \\ &= F_c((I + F_e) + F_d(I + F_e)) \\ &= F_c(I + F_e + F_{d\tilde{\circ}e}) \\ &= F_{c\tilde{\circ}(e+d\tilde{\circ}e)}, \end{aligned}$$

and the lemma is proved. □

Lemma 2. *The composition product on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ is associative.*

Proof. Using the various notions of compositions introduced above, observe

$$\begin{aligned} (F_{c_\delta} \circ F_{d_\delta}) \circ F_{e_\delta} &= (I + F_d + F_{c\tilde{\circ}d}) \circ (I + F_e) \\ &= I + F_e + F_d(I + F_e) + F_{c\tilde{\circ}d}(I + F_e) \\ &= I + F_e + F_{d\tilde{\circ}e} + F_{(c\tilde{\circ}d)\tilde{\circ}e} \\ &= F_{\delta+e+d\tilde{\circ}e+(c\tilde{\circ}d)\tilde{\circ}e}. \end{aligned}$$

Now apply Lemma 1,

$$\begin{aligned} (F_{c_\delta} \circ F_{d_\delta}) \circ F_{e_\delta} &= F_{\delta+e+d\tilde{\circ}e+c\tilde{\circ}(e+d\tilde{\circ}e)} \\ &= I + F_e + F_{d\tilde{\circ}e} + F_{c\tilde{\circ}(e+d\tilde{\circ}e)} \\ &= I + F_e + F_{d\tilde{\circ}e} + F_c(I + F_{e+d\tilde{\circ}e}) \\ &= (I + F_c) \circ (I + F_e + F_{d\tilde{\circ}e}) \\ &= F_{c_\delta} \circ (F_{d_\delta} \circ F_{e_\delta}). \quad \square \end{aligned}$$

In light of the uniqueness of generating series, the semigroups $(\mathscr{F}_\delta, \circ, I)$ and $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ are clearly isomorphic. That is,

$$\mathscr{L}_f(F_{c_\delta} \circ F_{d_\delta}) = \mathscr{L}_f(F_{c_\delta}) \circ \mathscr{L}_f(F_{d_\delta})$$

and $\mathscr{L}_f(I) = \delta$, where $\mathscr{L} : F_{c_\delta} \mapsto c_\delta$ is the formal Laplace transform [14,15]. The next theorem establishes that $(\mathscr{F}_\delta, \circ, I)$ is indeed a group.

Theorem 3. *The triple $(\mathscr{F}_\delta, \circ, I)$, or equivalently $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, forms a group.*

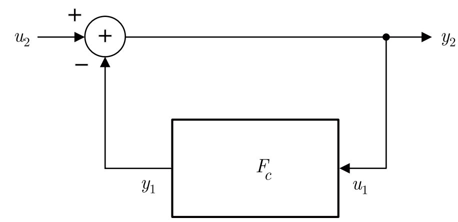


Fig. 7. Compositional inverse of $I + F_c$.

Proof. The only open issue concerns the existence of an inverse. Let $c \in \mathbb{R}\langle\langle X\rangle\rangle$. A compositional inverse element, say $c_\delta^{-1} \in \mathbb{R}\langle\langle X_\delta\rangle\rangle$, has to satisfy the identity

$$c_\delta \circ c_\delta^{-1} = \delta + c^{-1} + c\tilde{\circ}c^{-1} = \delta,$$

where $c_\delta^{-1} = \delta + c^{-1}$ for some $c^{-1} \in \mathbb{R}\langle\langle X\rangle\rangle$. In which case, c^{-1} must satisfy the identity $c^{-1} = (-c)\tilde{\circ}c^{-1}$. It was shown in [13] that $e \mapsto (-c)\tilde{\circ}e$ is always a contraction in the ultrametric sense and thus has a unique fixed point. So it follows directly that c_δ always has a right inverse. To be a left inverse, it follows similarly that this same c^{-1} must also satisfy the identity $c = (-c^{-1})\tilde{\circ}c$. Observe from Lemma 1 that

$$\begin{aligned} c^{-1} &= (-c)\tilde{\circ}c^{-1} \\ c^{-1}\tilde{\circ}c &= ((-c)\tilde{\circ}c^{-1})\tilde{\circ}c \\ &= (-c)\tilde{\circ}(c + c^{-1}\tilde{\circ}c). \end{aligned}$$

Thus, the series $d = c^{-1}\tilde{\circ}c$ satisfies the identity

$$d = (-c)\tilde{\circ}(c + d). \tag{6}$$

It can be seen by direct substitution that $d = -c$ is another solution to (6) using the identity $(-c)\tilde{\circ}0 = -c$. But the mapping $e \mapsto (-c)\tilde{\circ}(c+e)$ is also a contraction. So (6) has a unique solution. Therefore, any $c_\delta \in \mathbb{R}\langle\langle X\rangle\rangle$ is invertible. □

Example 1. Suppose $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ has finite *Lie rank*, that is, the range space of the Hankel mapping for c defined on the $\mathbb{R}$-vector space of Lie polynomials is finite. Then $I + F_c : u_1 \mapsto y_1$ has a finite dimensional input-affine state space realization of the form

$$\begin{aligned} \dot{z} &= g_0(z) + g_1(z)u_1, \qquad z(0) = z_0 \\ y_1 &= h(z) + u_1, \end{aligned}$$

where g_i and h is an analytic vector field and function, respectively, on some neighborhood W of z_0 [7,30]. In which case,

$$(c, \eta) = L_{g_\eta}h(z_0), \quad \forall \eta \in X^*, \tag{7}$$

where

$$L_{g_\eta}h := L_{g_{i_1}} \cdots L_{g_{i_k}}h, \quad \eta = x_{i_k} \cdots x_{i_1},$$

the *Lie derivative* of h with respect to g_i is defined as

$$L_{g_i}h : W \to \mathbb{R} : z \mapsto \frac{\partial h}{\partial z}(z)g_i(z),$$

and $L_{g_\emptyset}h = h$. It is not difficult to see that the compositional inverse $(I+F_c)^{-1} = I+F_{c^{-1}} : u_2 \mapsto y_2$ is described by the feedback system in Fig. 7. A straightforward calculation gives a realization for $F_{c^{-1}}$, namely, $(g_0 - g_1h, g_1, -h, z_0)$. Using this realization and (7), one can compute as many coefficients of c^{-1} as desired. The first few are:

$$\begin{aligned} (c^{-1}, \emptyset) &= -(c, \emptyset) \\ (c^{-1}, x_0) &= -(c, x_0) + (c, \emptyset)(c, x_1) \\ (c^{-1}, x_1) &= -(c, x_1) \end{aligned}$$

$$(c^{-1}, x_0^2) = -(c, x_0^2) + (c, \emptyset)(c, x_0x_1) + (c, x_0)(c, x_1) + (c, \emptyset)(c, x_1x_0) - (c, \emptyset)(c, x_1)^2 - (c, \emptyset)^2(c, x_1^2)$$
$$(c^{-1}, x_0x_1) = -(c, x_0x_1) + (c, x_1)^2 + (c, \emptyset)(c, x_1^2)$$
$$(c^{-1}, x_1x_0) = -(c, x_1x_0) + (c, \emptyset)(c, x_1^2)$$
$$(c^{-1}, x_1^2) = -(c, x_1^2)$$
$$\vdots$$

Example 2. For a single-input, single-output linear time-invariant system with transfer function $H(s)$ and state space realization (A, B, C), the corresponding generating series is $c = \sum_{i\geq 0}(c, x_0^ix_1)\, x_0^ix_1$, where $(c, x_0^ix_1) = CA^iB$, $i \geq 0$. In light of the previous example, it follows that

$$(c^{-1}, x_0^ix_1) = -C(A - BC)^iB, \quad i \geq 0.$$

Simply expanding these matrix powers gives

$$(c^{-1}, x_1) = -(c, x_1)$$
$$(c^{-1}, x_0x_1) = -(c, x_0x_1) + (c, x_1)^2$$
$$(c^{-1}, x_0^2x_1) = -(c, x_0^2x_1) + 2(c, x_1)(c, x_0x_1) - (c, x_1)^3$$
$$\vdots$$

3.2. *Construction of the Faà di Bruno Hopf algebra*

The goal of this section is to describe a Faà di Bruno Hopf algebra associated with the group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, where the antipode, α, satisfies the identity

$$c_\delta^{-1} = \delta + c^{-1} = \delta + \sum_{\eta\in X^*}(\alpha a_\eta)(c)\eta \tag{8}$$

with

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta)$$

denoting the coordinate function for $\eta \in X^*$. Formally extend such mappings to $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ by letting $a_\delta(c_\delta) = 1$. Next define a commutative $\mathbb{R}$-algebra of polynomials denoted by

$$A = \mathbb{R}[a_\eta : \eta \in X^* \cup \delta],$$

where the product is defined by

$$a_\eta a_\xi(c_\delta) = a_\eta(c_\delta)a_\xi(c_\delta)$$

for all $\eta, \xi \in X^*\cup\delta$ and any given $c_\delta \in \mathbb{R}\langle\langle X_\delta\rangle\rangle$. The first objective is to produce a bialgebra having commutative product and noncommutative coproduct

$$\mu : A \otimes A \to A : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi \tag{9}$$
$$\Delta : A \to A \otimes A : a_\nu \mapsto \Delta a_\nu, \tag{10}$$

respectively, such that

$$\mu(\Delta a_\nu(d_\delta \otimes c_\delta)) = a_\nu(c_\delta \circ d_\delta) = (c_\delta \circ d_\delta, \nu). \tag{11}$$

It is clear that the associativity of the composition product on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ supplies the required coassociativity property for Δ. But it seems quite difficult to produce a general combinatorial formula for Δa_ν as is done for the Hopf algebra derived from function composition via the Faà di Bruno formula (1) [3, see p. 918]. This deeper problem will not be pursued here. However, the defining expression for the composition product, namely,

$$(c_\delta \circ d_\delta, \nu) = (\delta, \nu) + (d, \nu) + \sum_{\eta\in X^*}(c, \eta)(\eta\tilde{\circ}d, \nu),$$

can be used directly to compute as many terms of Δa_ν as desired. For example, the first eight terms are:

$$\Delta 1 = 1 \otimes 1 \tag{12a}$$
$$\Delta a_\emptyset = a_\emptyset \otimes 1 + 1 \otimes a_\emptyset \tag{12b}$$
$$\Delta a_{x_0} = a_{x_0} \otimes 1 + 1 \otimes a_{x_0} + a_\emptyset \otimes a_{x_1} \tag{12c}$$
$$\Delta a_{x_1} = a_{x_1} \otimes 1 + 1 \otimes a_{x_1} \tag{12d}$$
$$\Delta a_{x_0^2} = a_{x_0^2} \otimes 1 + 1 \otimes a_{x_0^2} + a_\emptyset \otimes a_{x_0x_1} + a_{x_0} \otimes a_{x_1} + a_\emptyset \otimes a_{x_1x_0} + a_\emptyset^2 \otimes a_{x_1^2} \tag{12e}$$
$$\Delta a_{x_0x_1} = a_{x_0x_1} \otimes 1 + 1 \otimes a_{x_0x_1} + a_{x_1} \otimes a_{x_1} + a_\emptyset \otimes a_{x_1^2} \tag{12f}$$
$$\Delta a_{x_1x_0} = a_{x_1x_0} \otimes 1 + 1 \otimes a_{x_1x_0} + a_\emptyset \otimes a_{x_1^2} \tag{12g}$$
$$\Delta a_{x_1^2} = a_{x_1^2} \otimes 1 + 1 \otimes a_{x_1^2} \tag{12h}$$
$$\vdots$$

Continuing the construction, define the unit and counit, respectively, as

$$\sigma : \mathbb{R} \to A : \lambda \mapsto \lambda 1 \tag{13}$$
$$\epsilon : A \to \mathbb{R} : a_{\eta_1}a_{\eta_2}\cdots a_{\eta_\ell} \mapsto a_{\eta_1}(\delta)a_{\eta_2}(\delta)\cdots a_{\eta_\ell}(\delta). \tag{14}$$

As required by the definition of a bialgebra, $\sigma(1) = 1$, which is the unit of A, and $\epsilon \circ \sigma = 1$. Furthermore, since $\epsilon(a_\delta) = 1$, it follows that $a_\delta \sim 1$ is group-like, and hence, the terms $\Delta a_{x_1^i}$, $i = 0, 1, 2$ are all primitive.

A central result of the paper now follows.

Theorem 4. *The five-tuple $(A, \mu, \sigma, \Delta, \epsilon)$ described by (9) – (10) and (13)–(14) is a graded $\mathbb{R}$-bialgebra.*

Proof. The properties required of a bialgebra are verified straightforwardly. It will be shown here that the set of $\mathbb{R}$-vector subspaces of A

$$A_{(n)} = \text{span}_{\mathbb{R}}\left\{a_{\eta_1}a_{\eta_2}\cdots a_{\eta_l} \in A : \sum_{i=1}^{l}|\eta_i| = n\right\}, \quad n \geq 0 \tag{15}$$

provide a grading for A. (By definition, $|\delta| = 0$.) These subspaces clearly decompose A in the required fashion, i.e., $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$. It is also trivial to verify that $\epsilon(A_{(n)}) = 0$ when $n > 0$. Concerning the coproduct, it follows directly from (11) that

$$\mu(\Delta a_\nu(d_\delta \otimes c_\delta)) = (d, \nu) + \sum_{k=0}^{\infty}\sum_{\eta\in X^k}(\tilde{D}_\eta(1), \nu)(c, \eta), \quad \nu \in X^*.$$

Since $\tilde{D}_{x_i} : e \mapsto x_ie + x_0(d_i ⧢ e)$, it is immediate that the $\text{ord}(\tilde{D}_{x_i}\tilde{D}_\eta(1)) \geq \text{ord}(\tilde{D}_\eta(1)) + 1$ for any letter $x_i \in X$. (The *order* of a series c, $\text{ord}(c)$, is taken as the length of the smallest word in the support of c.) Therefore, $\text{ord}(\tilde{D}_\eta(1)) \geq |\eta|$ for any $\eta \in X^*$, and one can write instead the finite sum

$$\mu(\Delta a_\nu(d_\delta \otimes c_\delta)) = (d, \nu) + \sum_{k=0}^{|\nu|}\sum_{\eta\in X^k}(\tilde{D}_\eta(1), \nu)(c, \eta).$$

For a fixed k in the indicated range, the coefficients of d come from $\tilde{D}_\eta(1)$ and correspond to products of coordinate functions whose associated words have exactly length $|\nu| - k$, i.e., $a_{\eta_1}a_{\eta_2}\cdots a_{\eta_l} \in A_{(|\nu|-k)}$, while the only coefficient from c corresponds to $a_\eta \in$

$A_{(k)}$. Thus, it follows that the bialgebra is graded. The grading is evident (12). □

It is important to observe that this bialgebra is not connected, that is, using the natural filtration associated with the given grading, $A_0 \neq \sigma(\mathbb{R})$. For example, $a_\emptyset \in A_0$ but $a_\emptyset \notin \sigma(\mathbb{R})$. It is also clear that the coproduct terms computed above do not satisfy (3). Despite this fact, the following theorem still holds.

Theorem 5. *The six-tuple $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ described by (9)–(10), (13)–(14) and*

$$\alpha\, a_\nu = -a_\nu + \sum_{k=1}^{n} (-1)^{k+1} \mu^k \Delta'^k a_\nu, \quad \nu \in X^n,\ \nu \neq \delta \tag{16}$$

with $\alpha 1 = 1$ is a $\mathbb{R}$-Hopf algebra with a grading given by (15).

Proof. Similar to the connected case, (16) is equivalent to (2), however, it is truncated so as to include one additional term when compared to (4)–(5). Thus, it is necessary to show that

$$(\vartheta - \mathrm{id})^{*n+2} a_\nu = 0, \quad \nu \in X^n,\ n \geq 0, \tag{17}$$

assuming $\nu \neq \delta$. First, it is shown for any such ν that

$$\Delta a_\nu = a_\nu \otimes 1 + 1 \otimes a_\nu + \Delta' a_\nu,$$

where $\Delta' a_\nu \in A_n^+ \otimes A_n^+$. Define $\Delta' a_\nu = \Delta a_\nu - a_\nu \otimes 1 - 1 \otimes a_\nu$, and observe that from the counit property shown in Fig. 4(b)

$$\begin{aligned}(\mathrm{id} \otimes \epsilon)\Delta' a_\nu &= (\mathrm{id} \otimes \epsilon)\Delta a_\nu - a_\nu \otimes \epsilon(1) - 1 \otimes \epsilon(a_\nu) \\ &= a_\nu \otimes 1 - a_\nu \otimes 1 = 0.\end{aligned}$$

Likewise, $(\epsilon \otimes \mathrm{id})\Delta' a_\nu = 0$. Therefore, $\Delta' a_\nu \in A^+ \otimes A^+$. Since $\Delta A_n \subseteq \sum_{i=0}^{n} A_i \otimes A_{n-i} \subset A_n \otimes A_n$, it thus follows that $\Delta' a_\nu \in A_n^+ \otimes A_n^+$. Using this fact, identity (17) is proved by induction. For $n = 0$, observe that

$$\begin{aligned}(\vartheta - \mathrm{id})^{*2} a_\emptyset &= ((\vartheta - \mathrm{id}) * (\vartheta - \mathrm{id})) a_\emptyset \\ &= \mu \circ ((\vartheta - \mathrm{id}) \otimes (\vartheta - \mathrm{id})) \circ \Delta a_\emptyset \\ &= \mu(((\vartheta - \mathrm{id}) \otimes (\vartheta - \mathrm{id}))(a_\emptyset \otimes 1 + 1 \otimes a_\emptyset)) \\ &= 0,\end{aligned}$$

since $(\vartheta - \mathrm{id})1 = 0$. Now assume the claim holds for some fixed $n \geq 0$. Then

$$\begin{aligned}(\vartheta - \mathrm{id})^{*n+3} a_\nu &= ((\vartheta - \mathrm{id}) * (\vartheta - \mathrm{id})^{*n+2}) a_\nu \\ &= \mu(((\vartheta - \mathrm{id}) \otimes (\vartheta - \mathrm{id})^{*n+2})(a_\nu \otimes 1 + 1 \otimes a_\nu + \Delta' a_\nu)) \\ &= 0,\end{aligned}$$

in light of the induction hypothesis and the fact that $\Delta' a_\nu \in A_n^+ \otimes A_n^+$. This completes the proof. □

Observe that the first few reduced coproduct terms in this case are:

$$\begin{aligned}
&\Delta' a_\emptyset = 0 \\
&\Delta' a_{x_0} = a_\emptyset \otimes a_{x_1} \\
&\Delta' a_{x_1} = 0 \\
&\Delta' a_{x_0^2} = a_\emptyset \otimes a_{x_0 x_1} + a_{x_0} \otimes a_{x_1} + a_\emptyset \otimes a_{x_1 x_0} + a_\emptyset^2 \otimes a_{x_1^2} \\
&\Delta' a_{x_0 x_1} = a_{x_1} \otimes a_{x_1} + a_\emptyset \otimes a_{x_1^2} \\
&\Delta' a_{x_1 x_0} = a_\emptyset \otimes a_{x_1^2} \\
&\Delta' a_{x_1^2} = 0 \\
&\vdots
\end{aligned}$$

The corresponding antipode terms are then found from (16) to be:

$$\alpha 1 = 1 \tag{18a}$$
$$\alpha a_\emptyset = -a_\emptyset \tag{18b}$$
$$\alpha a_{x_0} = -a_{x_0} + a_\emptyset a_{x_1} \tag{18c}$$
$$\alpha a_{x_1} = -a_{x_1} \tag{18d}$$
$$\alpha a_{x_0^2} = -a_{x_0^2} + a_\emptyset a_{x_0 x_1} + a_{x_0} a_{x_1} + a_\emptyset a_{x_1 x_0} - a_\emptyset a_{x_1}^2 - a_\emptyset^2 a_{x_1^2} \tag{18e}$$
$$\alpha a_{x_0 x_1} = -a_{x_0 x_1} + a_{x_1}^2 + a_\emptyset a_{x_1^2} \tag{18f}$$
$$\alpha a_{x_1 x_0} = -a_{x_1 x_0} + a_\emptyset a_{x_1^2} \tag{18g}$$
$$\alpha a_{x_1^2} = -a_{x_1^2} \tag{18h}$$
$$\vdots$$

These terms agree exactly with those for c^{-1} computed from Lie derivatives in Example 1, where it was assumed that c had finite Lie rank. In the present context, however, no such assumption is required.

The following corollary establishes a direct analogy to the classical Faà di Bruno Hopf algebra.

Corollary 1. *The set of proper series forms a subgroup of $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, and the corresponding Faà di Bruno Hopf subalgebra is connected and graded.*

Proof. The first claim follows directly from the identities $(c_\delta \circ d_\delta, \emptyset) = (c, \emptyset) + (d, \emptyset)$ and (18b). The second claim is evident from the fact that under the properness assumption, $A_0 \sim \mathbb{R}$. □

Example 3. In the state space setting employed in Example 1, c is proper if and only if $z_0 = 0$. This is precisely the case for the linear system described in Example 2.

4. An explicit formula for the feedback product

Given two Fliess operators F_c and F_d which are linear time-invariant systems with transfer functions G and H, respectively, the closed-loop system in Fig. 2 has the transfer function

$$G(I - HG)^{-1} = G \sum_{k=0}^{\infty} (HG)^k.$$

When c is a linear series (i.e., its support is a subset of the language $L := \{x_0^{n_1} x_1 x_0^{n_0} : n_i \geq 0\}$) and d is arbitrary, it was shown in [14, p. 72] that

$$c@d = c \circ \sum_{k=0}^{\infty} (d \circ c)^{\circ k}.$$

The next theorem gives the full nonlinear generalization of this type of closed-loop system representation.

Theorem 6. *For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that*
$c@d = c\tilde{\circ}((-d) \circ c)^{-1} = c \circ (\delta - d \circ c)^{-1}$.

Proof. Clearly the function v in Fig. 2 must satisfy the identity

$$v = u + F_{d \circ c}[v].$$

Therefore,

$$\left(I + F_{(-d) \circ c}\right)[v] = u.$$

Applying the compositional inverse $\left(I + F_{((-d) \circ c)^{-1}}\right)$ on the left gives

$$v = \left(I + F_{((-d) \circ c)^{-1}}\right)[u],$$

and thus,

$$F_{c@d}[u] = F_c[v] = F_{c \tilde{\circ} ((-d) \circ c)^{-1}}[u]$$

as desired. The second identity in the theorem is just a formal way of expressing the first identity since $F_\delta[u] = u$. □

Next it is shown that feedback preserves local convergence. But the following preliminary result is needed first.

Theorem 7. *The triple* $(\mathbb{R}_{LC}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ *is a subgroup of* $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$.

Proof. The set of series $\mathbb{R}_{LC}\langle\langle X_\delta\rangle\rangle$ is closed under composition since the set $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ is closed under composition. To show that $\mathbb{R}_{LC}\langle\langle X_\delta\rangle\rangle$ is closed under inversion, suppose c is locally convergent with growth constants K, M. Without loss of generality, assume $K \geq 1$. From (8) and (16) it follows that

$$\left|(c^{-1}, \eta)\right| \leq |(c, \eta)| + \sum_{k=1}^{|\eta|} \left|(\mu^k \Delta'^k c, \eta)\right|, \quad \eta \in X^*.$$

In light of the grading, it clear that

$$\begin{aligned}\left|(\mu^k \Delta'^k c, \eta)\right| &\leq \sum_{\substack{\xi_1,\ldots,\xi_k \in X^* \\ |\xi_1|+\cdots+|\xi_k|=|\eta|}} |(c, \xi_1)| \cdots |(c, \xi_k)| \\ &\leq \sum_{\substack{\xi_1,\ldots,\xi_k \in X^* \\ |\xi_1|+\cdots+|\xi_k|=|\eta|}} KM^{|\xi_1|}\, |\xi_1|! \cdots KM^{|\xi_k|}\, |\xi_k|! \\ &= K^k M^{|\eta|} \sum_{\substack{\xi_1,\ldots,\xi_k \in X^* \\ |\xi_1|+\cdots+|\xi_k|=|\eta|}} |\xi_1|! \cdots |\xi_k|! \\ &\leq K^k M^{|\eta|}\, |\eta|! \sum_{\substack{\xi_1,\ldots,\xi_k \in X^* \\ |\xi_1|+\cdots+|\xi_k|=|\eta|}} 1 \\ &= K^k M^{|\eta|}\, |\eta|! \binom{|\eta|+k-1}{k-1} 2^{|\eta|} \\ &\leq K^k M^{|\eta|}\, |\eta|! 2^{|\eta|+k-1} 2^{|\eta|} \\ &\leq K^k (4M)^{|\eta|}\, |\eta|! 2^{k-1},\end{aligned}$$

where the fact has been used that the number of compositions of a positive integer i into j positive parts is $\binom{i-1}{j-1}$. Therefore, since $K \geq 1$,

$$\begin{aligned}\left|(c^{-1}, \eta)\right| &\leq KM^{|\eta|}\, |\eta|! + (4M)^{|\eta|}\, |\eta|! \sum_{k=1}^{|\eta|} (2K)^k \\ &\leq KM^{|\eta|}\, |\eta|! + (8KM)^{|\eta|}\, |\eta|\, |\eta|! \\ &\leq K(8KM)^{|\eta|}(|\eta|+1)!,\end{aligned}$$

which is sufficient to conclude that c^{-1} is locally convergent. It should be noted, however, that in most cases this bound will be conservative. □

Theorem 8. *If* $c, d \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ *then* $c@d \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$.

Proof. Since the composition product, the modified composition product, and the compositional inverse all preserve local convergence, the claim follows directly from Theorem 6. □

Example 4. Consider the operator F_c in the feedback configuration shown in Fig. 2, where $c = \sum_{k\geq 0} x_1^k$ and $F_d = F_\delta = I$. This unity feedback system has a generating series given by

$$c@\delta = c\tilde{\circ}(-c)^{-1} = \sum_{k=0}^{\infty} x_1^k \tilde{\circ}(-c)^{-1}.$$

The first few terms of $(-c)^{-1}$, as shown in Table 1, were computed using the antipode formulas (18). After which, $c@\delta$ can be computed term-by-term using the above expression. To the authors' knowledge, this sequence has not been computed by any other means, however, it was shown in [22, Example 4] that the subsequence corresponding to the self-excited case (i.e., setting $u = 0$) is $(c@\delta)_0 = \sum_{k\geq 0} k! x_0^k$, which is consistent with the results given in Table 1. That is, $(c@\delta, x_0^k) = ((c@\delta)_0, x_0^k),\ k \geq 0$.

Table 1
Coefficients for the sequences in Example 4.

Sequence	Coefficients						
	$\emptyset$	x_0	x_1	x_0^2	x_0x_1	x_1x_0	x_1^2
c	1	0	1	0	0	0	1
$(-c)^{-1}$	1	1	1	2	2	1	1
$c@\delta$	1	1	1	2	2	1	1
$(c@\delta)_0$	1	1	0	2	0	0	0

5. Conclusions

In this paper, a connected Faà di Bruno Hopf algebra was constructed for a group of Fliess operators. Then a known series expansion for the antipode was used to produce an explicit formula for the generating series of the compositional inverse operator. Using this result, an explicit formula for the feedback product of two formal power series was derived, which up to now had only been described implicitly in terms of a fixed point equation. It was then shown that this expression could be used to prove that local convergence is preserved under feedback.

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