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Bilinear system interconnections and generating series of weighted Petri nets

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ABSTRACT

This paper has three objectives. The first objective is to provide a simpler proof concerning a sufficient condition due to Ferfera under which bilinearity is preserved for cascade interconnections. The next objective is to provide a specific counterexample to show that this condition does not apply to the feedback connection. The final objective is to show that the well-known correspondence between rational series and formal power series recognized by weighted finite-state automata can be generalized to produce a correspondence between the generating series of cascaded and feedback connected bilinear systems and a class of weighted Petri nets.

1. Introduction

Consider a bilinear state space system of the form

$$\dot{z}(t) = Az(t) + \sum_{j=1}^{m} N_j z(t) u_j(t), \quad z(0) = z_0$$

$$y(t) = Cz(t),$$

where $z(t) \in \mathbb{R}^n$; $u_j(t) \in \mathbb{R}$; $y(t) \in \mathbb{R}^\ell$; and A, N_j and C are real matrices of appropriate dimensions. It is easily verified that if two bilinear state space systems $(A_i, N_{\cdot,i}, C_i, z_{i,0})$, $i = 1, 2$ are interconnected in a cascade fashion, that is, if $m = \ell$ and one feeds the outputs of one system into the inputs of the other, then one possible state space realization for the input–output mapping $u_1 \mapsto y_2$ is

$$\dot{z}_1(t) = A_1 z_1(t) + \sum_{j=1}^{m} N_{j,1} z_1(t) u_{j,1}(t), \quad z_1(0) = z_{1,0} \tag{1}$$

$$\dot{z}_2(t) = A_2 z_2(t) + \sum_{j=1}^{m} N_{j,2} z_2(t) (C_1 z_1(t))_j, \quad z_2(0) = z_{2,0} \tag{2}$$

$$y_2(t) = C_2 z_2(t), \tag{3}$$

which is an input-affine nonlinear system (f, g, h, z_0) [1]. (Here $(v)_j$ denotes the jth component of $v \in \mathbb{R}^m$.) Clearly some components of the drift vector field f are quadratic polynomials in the state. In 1972, Brockett indicated that it was an open problem to determine under what conditions bilinearity is preserved for cascade interconnections [2]. One trivial sufficient condition can be identified immediately from the state space system above: when a bilinear system is followed by a linear system. The composite system is bilinear since in this case $N_{j,2} = 0$, $j = 1, 2, \ldots, m$. But this condition is very restrictive and not necessary. In 1979, Ferfera provided in [3,4] a much less restrictive sufficient condition using formal power series representations of the input–output mappings, namely, $F_{c_i} : u_i \mapsto y_i$, $i = 1, 2$, where c_i is a generating series written in terms of a noncommutative alphabet $X = \{x_0, x_1, \ldots, x_m\}$ [5–7]. In this setting, system composition can be described by $F_{c_2} \circ F_{c_1} = F_{c_2 \circ c_1}$, where $c_2 \circ c_1$ denotes the composition product of two formal power series [3,4,8,9]. The existence of a bilinear state space system is then equivalent to its input–output map having a *rational* or *regular* generating series [10,11]. An operation on formal power series is said to be rational if it preserves rationality. Ferfera introduced the notion of an *input-limited* rational series and showed that the composition product is rational if its left argument is restricted to input-limited rational series. It is easily demonstrated that this condition is not necessary. For parallel connections, i.e., when either $F_{c_1}[u] + F_{c_2}[u] = F_{c_1 + c_2}[u]$ or $F_{c_1}[u] F_{c_2}[u] = F_{c_1 ⧢ c_2}[u]$, where ⧢ denotes the shuffle product, there is no need to restrict either c_1 or c_2 to be an input-limited rational series. It is trivial to show that addition is a rational operation, and the rationality of the shuffle product was proved

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in [5, Proposition I.3]. On the other hand, very little is known concerning the rationality of the feedback connection, except for a claim made by Ferfera that rationality is not preserved even under the input-limited restriction [3, Remark III.4.4]. No specific example or rationale was given.

This paper has three objectives. The first objective is to provide a much simpler proof for the rationality of the composition product under the input-limited assumption. The original proof of this result, which appears (only) in [3], is a relatively complex argument relying extensively on the theory of rational transductions [12–14]. A re-interpretation of this approach appeared in [15]. Here a completely different approach is presented, one that relies only on elementary properties of formal power series and is in fact much shorter. The second objective is to provide a specific counterexample to show that the feedback connection does not constitute a rational operation, even with the input-limited restriction. This means in particular that for composite systems from these two classes of interconnections, the well-known correspondence between rational series and formal power series recognized by weighted finite-state automata is not applicable [11,16,17]. The final objective of the paper is to show that one can recover this correspondence if Petri nets are substituted for automata. In particular, it will be shown that the generating series of any cascade or feedback connection of two bilinear systems can always be put in correspondence with the generating series of a certain weighted Petri net. This result is an application of recent work by Foursov and Hespel [18] and promises to provide a more complete characterization of bilinear system interconnections when combined with their notion of multiset weighted grammars [19]. It should also be noted that these results are related to the graph-theoretic/combinatoric analysis of formal differential equations presented in [20,21].

The paper is organized as follows. First some preliminaries are summarized in Section 2 to better frame the problems and introduce the notation. In Section 3, the new results concerning the composition product are presented. In Section 4, the non-rationality of the feedback connection is addressed. In Section 5, the correspondence between the generating series of cascaded and feedback connected bilinear systems and weighted Petri nets is presented. Conclusions and suggestions for future research are summarized in Section 6.

2. Preliminaries: Rational series and Fliess operators

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_k} \cdots x_{i_1}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k will be denoted by X^k. The set of all words including the empty word, $\emptyset$, will be denoted by X^*. It forms a monoid under catenation. A *language* is any subset of X^*. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η). Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an $\mathbb{R}$-algebra under the Cauchy product and a commutative $\mathbb{R}$-algebra under the shuffle product, that is, the $\mathbb{R}$-bilinear mapping $\mathbb{R}^\ell\langle\langle X\rangle\rangle \times \mathbb{R}^\ell\langle\langle X\rangle\rangle \to \mathbb{R}^\ell\langle\langle X\rangle\rangle$ uniquely specified by the shuffle product of two words $(x_i\nu) ⧢ (x_j\xi) = x_i(\nu ⧢ (x_j\xi)) + x_j((x_i\nu) ⧢ \xi)$ and $\nu ⧢ \emptyset = \nu$ for all $\nu, \xi \in X^*$. Given $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the subset of X^* defined by $\mathrm{supp}(c) = \{\eta : (c, \eta) \neq 0\}$ is called the *support* of c. The subset of $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ consisting of all the series with finite support is denoted by $\mathbb{R}^\ell\langle X\rangle$, and its elements are called *polynomials*. A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is called *proper* if $\emptyset \notin \mathrm{supp}(c)$ and *invertible* if there exists a series $c^{-1} \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $cc^{-1} = c^{-1}c = 1$. In the event that c is not proper, it is always possible to write $c = (c, \emptyset)(1 - c')$, where $c' \in \mathbb{R}\langle\langle X\rangle\rangle$ is proper. It then follows that

$$c^{-1} = \frac{1}{(c, \emptyset)}(1 - c')^{-1} = \frac{1}{(c, \emptyset)}(c')^*,$$

where $(c')^* := \sum_{i \geq 0} (c')^i$. It can be shown that c is invertible if and only if c is not proper. Now let S be any subalgebra of the $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the Cauchy product. S is said to be *rationally closed* when every invertible $c \in S$ has $c^{-1} \in S$. The *rational closure* of any set $E \subset \mathbb{R}\langle\langle X\rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ containing E.

Definition 1 (*[22]*). A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is **rational** if it belongs to the rational closure of $\mathbb{R}\langle X\rangle$.

Thus, a given rational series can be obtained from a finite set of polynomials by performing a finite number of additions, scalar products, Cauchy products and inversions (or star operations), the fundamental *rational operations*. The following definitions and theorem provide another characterization of rational series which can be used to establish the precise connection between rational series and series recognized by weighted finite-state automaton [11].

Definition 2. A **linear representation** of a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is any triple (μ, γ, λ), where $\mu : X^* \to \mathbb{R}^{n\times n}$ is a monoid morphism, γ, $\lambda^T \in \mathbb{R}^{n\times 1}$, and $(c, \eta) = \lambda\mu(\eta)\gamma$ for all $\eta \in X^*$.

Definition 3. A series is called **recognizable** if it has a linear representation.

Theorem 1 (*[11]*). *A formal power series is rational if and only if it is recognizable.*

For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1 < \infty$ be given. For a measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$ norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Define recursively for each $\eta \in X^*$ the mapping $E_\eta : L^m_1[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by setting $E_\emptyset[u] = 1$, and

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, \mathrm{d}\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The causal input–output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator* [1,5–9,23,24]. When there exist real numbers $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!$ for all $\eta \in X^*$, where $|z| := \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ for $z \in \mathbb{R}^\ell$, then F_c constitutes a well-defined operator from $B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ into $B^\ell_{\mathfrak{q}}(S)[t_0, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [23]. Such a power series c is said to be *locally convergent*. Generating series are unique in the sense that if $F_c = F_d$ on any $B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ then $c = d$ (i.e., $(c, \eta) = (d, \eta), \forall \eta \in X^*$) [1,5,25].

A Fliess operator F_c defined on $B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ is said to be *realized* by a state space realization when there exists a system of n analytic differential equations and ℓ analytic output equations

$$\dot{z} = f(z) + \sum_{i=1}^{m} g_i(z)u_i, \quad z(t_0) = z_0 \tag{4}$$

$$y = h(z) \tag{5}$$

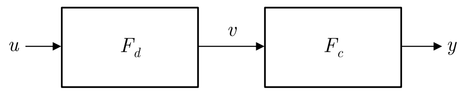


Fig. 1. Cascade connection of two Fliess operators.

such that (4) has a well-defined solution $z(t)$, $t \in [t_0, t_0 + T]$ in a neighborhood $\mathcal{W} \subseteq \mathbb{R}^n$ of z_0 for any given input $u \in B_{\mathrm{p}}^m(R)[t_0, t_0 + T]$, and

$$F_c[u](t) = h(z(t)), \quad t \in [t_0, t_0 + T].$$

It is well-known in this situation that the generating series c is related to the realization (f, g, h, z_0) by

$$(c, \eta) = L_{g_\eta} h(z_0), \quad \forall \eta \in X^*, \tag{6}$$

where the iterated Lie derivatives are defined by

$$L_{g_\eta} h = L_{g_{i_1}} \cdots L_{g_{i_k}} h, \qquad \eta = x_{i_k} \cdots x_{i_1} \in X^* \tag{7}$$

with $L_{g_i} : h \mapsto \partial h/\partial z \cdot g_i, g_0 = f$ and $L_\emptyset h = h$ [1,5–7]. The analyticity of the mappings f, g and h ensure that c is locally convergent. F_c will be referred to as a *rational operator* whenever c is rational. It can be easily shown via Theorem 1 that every rational series satisfies a stricter growth condition of the form $|(c, \eta)| \le KM^{|\eta|}$, $\forall \eta \in X^*$. Given any linear representation (μ, γ, λ) of c, it follows that

$$c = \sum_{k=0}^{\infty} \sum_{i_1,\ldots,i_k=0}^{m} (\lambda N_{i_k} \cdots N_{i_1} \gamma) x_{i_k} \cdots x_{i_1},$$

where $N_i = \mu(x_i)$. Thus, the corresponding rational operator is realized by the bilinear realization

$$\dot{z} = N_0 z + \sum_{i=1}^{m} N_i z\, u_i, \quad z(t_0) = \gamma$$
$$y = \lambda z.$$

The stronger growth condition on c guarantees that $z(t)$ is well-defined for all $t \ge t_0$ and any $u \in L_{\mathrm{p},e}^m(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_0+T]} \in L_{\mathrm{p}}^m[t_0, t_0 + T], \quad 0 < T < \infty\}$ [23, Corollary 4.1].

3. Cascaded systems

The cascade connection of two Fliess operators F_c and F_d as shown in Fig. 1, where $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, can be described in terms of the composition product defined below.

Definition 4 (*[3,4]*). For any $\eta \in X^*$ and series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the **composition** of η with d is defined in a recursive manner by

$$\eta \circ d = \begin{cases} \eta & : \quad |\eta|_{x_i} = 0, \quad \forall\, i \neq 0 \\ x_0^{k+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \quad \eta = x_0^k x_i \eta', \quad k \in \mathbb{N}, \\ & \quad\;\; i \neq 0, \quad \eta' \in X^*, \end{cases}$$

where $|\eta|_{x_i}$ is the number of times the letter x_i appears in η, and $d_i : \xi \mapsto (d, \xi)_i$ with $(d, \xi)_i$ being the ith component of the coefficient (d, ξ). The **composition** of any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ with d is

$$c \circ d = \sum_{\eta \in X^*} (c, \eta) \eta \circ d.$$

Theorem 2 (*[3,8]*). *Let $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$. The composition $F_c \circ F_d$ has generating series $c \circ d$, i.e., $F_c \circ F_d = F_{c \circ d}$. In addition, if c and d are locally convergent then $c \circ d$ is also locally convergent.*

The following example is due to Ferfera [3,4]. It shows that the composition product does not preserve rationality. Therefore, the composite system does not have a bilinear realization of any dimension in any coordinate frame. The computational approach taken here is distinct from the existing one in that it can be completely generalized as demonstrated at the end of this section.

Example 1. Suppose $X = \{x_0, x_1\}$ and consider the rational series $c = (1 - x_1)^{-1} = x_1^*$. The claim is that c composed with itself is not rational. The main goal is to show that

$$(c \circ c, x_0^{k_0} x_1^{k_1}) = (k_0)^{k_1}, \quad k_0 \ge 0, \; k_1 \ge 0,$$

or equivalently,

$$(x_1^{-k_1} x_0^{-k_0}(c \circ c), \emptyset) = (k_0)^{k_1}. \tag{8}$$

Here the left-shift operator $\xi^{-1}(\cdot)$ is defined for any $\xi \in X^*$ by

$$\xi^{-1} : X^* \to \mathbb{R}\langle X\rangle : \eta \mapsto \begin{cases} \eta' & : \quad \eta = \xi \eta' \\ 0 & : \quad \text{otherwise}, \end{cases}$$

and $\xi^{-1}(c) := \sum_{\eta \in X^*} (c, \eta)\, \xi^{-1}(\eta)$. (The left-shift $(x_0^k)^{-1}(\cdot)$ is denoted by $x_0^{-k}(\cdot)$.) The following identities are useful:

$$x_i^{-1}(c^{\sqcup\!\sqcup k}) = kc^{\sqcup\!\sqcup (k-1)} \sqcup\!\sqcup x_i^{-1}(c)$$

$$x_0^{-1}(c \circ d) = x_0^{-1}(c) \circ d + \sum_{i=1}^{m} d_i \sqcup\!\sqcup [x_i^{-1}(c) \circ d]$$

$$x_i^{-1}(c \circ d) = 0, \quad i = 1, 2, \ldots, m,$$

where the shuffle power of c is defined as

$$c^{\sqcup\!\sqcup k} = \underbrace{c \sqcup\!\sqcup c \sqcup\!\sqcup \cdots \sqcup\!\sqcup c}_{c \text{ appears } k \text{ times}}, \quad k > 1$$

and $c^{\sqcup\!\sqcup 0} = 1$. The claim is trivial when $k_0 = k_1 = 0$ provided that $0^0 := 1$. If $k_0 = 1$ and $k_1 = 0$, observe that

$$x_0^{-1}(c \circ c) = \underbrace{x_0^{-1}(c)}_{0} \circ c + c \sqcup\!\sqcup (\underbrace{x_1^{-1}(c)}_{c} \circ c) = c \sqcup\!\sqcup (c \circ c).$$

The intermediate claim then is that

$$x_0^{-k_0}(c \circ c) = c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c), \quad k_0 \ge 1.$$

If the identity above holds up to some fixed $k_0 \ge 1$ then

$$\begin{aligned} x_0^{-k_0-1}(c \circ c) &= x_0^{-1}(c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c)) \\ &= x_0^{-1}(c^{\sqcup\!\sqcup k_0}) \sqcup\!\sqcup (c \circ c) + c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup x_0^{-1}(c \circ c) \\ &= \Big[k_0 c^{\sqcup\!\sqcup (k_0-1)} \sqcup\!\sqcup \underbrace{x_0^{-1}(c)}_{0}\Big] \sqcup\!\sqcup (c \circ c) \\ &\quad + c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \sqcup\!\sqcup (c \circ c)) \\ &= c^{\sqcup\!\sqcup (k_0+1)} \sqcup\!\sqcup (c \circ c). \end{aligned}$$

Hence, the intermediate identity in question holds for $k_0 \ge 0$. Observe that

$$\begin{aligned} x_1^{-1} x_0^{-k_0}(c \circ c) &= x_1^{-1}(c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c)) \\ &= x_1^{-1}(c^{\sqcup\!\sqcup k_0}) \sqcup\!\sqcup (c \circ c) + c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup \underbrace{x_1^{-1}(c \circ c)}_{0} \\ &= k_0 c^{\sqcup\!\sqcup (k_0-1)} \sqcup\!\sqcup \underbrace{x_1^{-1}(c)}_{c} \sqcup\!\sqcup (c \circ c) \\ &= k_0 c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c). \end{aligned}$$

The next proposition is that

$$x_1^{-k_1} x_0^{-k_0}(c \circ c) = (k_0)^{k_1} c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c).$$

If this is the case up to some fixed $k_1 \geq 1$ then

$$\begin{aligned} x_1^{-k_1-1} x_0^{-k_0}(c \circ c) &= x_1^{-1}((k_0)^{k_1} c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c)) \\ &= (k_0)^{k_1} \left[x_1^{-1}(c^{\sqcup\!\sqcup k_0}) \sqcup\!\sqcup (c \circ c) \right. \\ &\quad \left. + \; c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup \underbrace{x_1^{-1}(c \circ c)}_{0} \right] \\ &= (k_0)^{k_1} \left[k_0 c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c) \right] \\ &= (k_0)^{k_1+1} c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c). \end{aligned}$$

Hence, the proposition holds for all $k_1, k_0 \geq 0$. To validate (8), simply compare the constant coefficients in the above identity:

$$(x_1^{-k_1} x_0^{-k_0}(c \circ c), \emptyset) = ((k_0)^{k_1} c^{\sqcup\!\sqcup k_0} \sqcup\!\sqcup (c \circ c), \emptyset)$$
$$(c \circ c, x_0^{k_0} x_1^{k_1}) = (k_0)^{k_1}.$$

Setting $k_0 = k_1$ reduces the expression to

$$(c \circ c, x_0^k x_1^k) = k^k, \quad k \geq 0.$$

The key observation is that these coefficients of $c \circ c$ are growing faster than any sequence of coefficients from a rational series can possibly grow, that is, at a rate exceeding $KM^{|\eta|}$ for any real numbers $K, M > 0$. Therefore, the series $c \circ c$ cannot be rational.

The next definition and theorem describe Ferfera's sufficient condition for preserving rationality under composition.

Definition 5. A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is **limited relative to $\mathbf{x_i}$** if there exists an integer $\mathcal{N}_i \geq 0$ such that

$$\max_{\eta \in \text{supp}(c)} |\eta|_{x_i} = \mathcal{N}_i < \infty.$$

If c is limited relative to x_i for every $i = 1, \ldots, m$ then c is said to be **input-limited**. In such cases, let $\mathcal{N}_c := \max_i \mathcal{N}_i$. A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is input-limited if each component series, c_j, is input-limited for $j = 1, \ldots, \ell$. In this case, $\mathcal{N}_c := \max_j \mathcal{N}_{c_j}$.

Theorem 3 ([3,4]). *Let $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ be two rational series. If c is input-limited then the series $c \circ d$ is rational.*

The proof presented here relies on the following lemma, which was ultimately motivated by the calculations presented in the previous example.

Lemma 1. *Let $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ be a rational series with a linear representation (μ, γ, λ). Let $N_i := \mu(x_i) \in \mathbb{R}^{n \times n}$, $i = 0, 1, \ldots, m$. Then for any $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ it follows that*

$$c \circ d = \sum_{\eta \in \hat{X}^*} \lambda D_\eta((N_0 x_0)^*)\gamma,$$

where $\hat{X} := \{x_1, x_2, \ldots, x_m\}$, and the set of operators $\{D_\eta : \eta \in \hat{X}^*\}$ is the monoid under composition uniquely specified by

$$\begin{aligned} D_{x_i} &: \mathbb{R}^{n \times n}\langle\langle X \rangle\rangle \to \mathbb{R}^{n \times n}\langle\langle X \rangle\rangle \\ &: E \mapsto x_0(N_0 x_0)^* N_i (d_i \sqcup\!\sqcup E) \end{aligned}$$

with $D_\emptyset$ equivalent to the identity map. (The shuffle product above is defined componentwise.)

Proof. Without lost of generality, assume $\ell = 1$. Directly from the definition of the composition product observe that

$$\begin{aligned} c \circ d &= \sum_{k \geq 0} \sum_{i_1, \ldots, i_k = 1}^{m} \sum_{n_0, \ldots, n_k \geq 0} \lambda N_0^{n_k} N_{i_k} N_0^{n_{k-1}} N_{i_{k-1}} \\ &\quad \cdots N_0^{n_1} N_{i_1} N_0^{n_0} \gamma \, (x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \cdots x_0^{n_1} x_{i_1} x_0^{n_0}) \circ d \\ &= \sum_{k \geq 0} \sum_{i_1, \ldots, i_k = 1}^{m} \sum_{n_0, \ldots, n_k \geq 0} \lambda N_0^{n_k} N_{i_k} N_0^{n_{k-1}} N_{i_{k-1}} \\ &\quad \cdots N_0^{n_1} N_{i_1} N_0^{n_0} \gamma \, x_0^{n_k+1} \left[d_{i_k} \sqcup\!\sqcup \left[x_0^{n_{k-1}+1} \left[d_{i_{k-1}} \right.\right.\right. \\ &\quad \left.\left.\left. \sqcup\!\sqcup \ldots \sqcup\!\sqcup \left[x_0^{n_1+1} \left[d_{i_1} \sqcup\!\sqcup x_0^{n_0} \right] \right] \cdots \right] \right] \right]. \end{aligned}$$

From the bilinearity and continuity of the shuffle product (in the ultrametric sense), it follows that

$$\begin{aligned} c \circ d &= \sum_{k \geq 0} \sum_{i_1, \ldots, i_k = 1}^{m} \lambda x_0 \left(\sum_{n_k \geq 0} (N_0 x_0)^{n_k} \right) N_{i_k} \left[d_{i_k} \right. \\ &\quad \sqcup\!\sqcup \left[x_0 \left(\sum_{n_{k-1} \geq 0} (N_0 x_0)^{n_{k-1}} \right) N_{i_{k-1}} \left[d_{i_{k-1}} \right.\right. \\ &\quad \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left[x_0 \left(\sum_{n_1 \geq 0} (N_0 x_0)^{n_1} \right) N_{i_1} \left[d_{i_1} \right.\right. \\ &\quad \left.\left.\left.\left.\left.\left. \sqcup\!\sqcup \left(\sum_{n_0 \geq 0} (N_0 x_0)^{n_0} \right) \right] \right] \cdots \right] \right] \right] \right] \gamma \\ &= \sum_{k \geq 0} \sum_{i_1, \ldots, i_k = 1}^{m} \lambda x_0 (N_0 x_0)^* N_{i_k} \left[d_{i_k} \right. \\ &\quad \sqcup\!\sqcup \left[x_0 (N_0 x_0)^* N_{i_{k-1}} \left[d_{i_{k-1}} \sqcup\!\sqcup \ldots \right.\right. \\ &\quad \left.\left.\left. \sqcup\!\sqcup \left[x_0 (N_0 x_0)^* N_{i_1} \left[d_{i_1} \sqcup\!\sqcup (N_0 x_0)^* \right] \right] \cdots \right] \right] \right] \gamma. \end{aligned}$$

Finally, applying the definition of D_η,

$$\begin{aligned} c \circ d &= \sum_{k \geq 0} \sum_{x_{i_k} \cdots x_{i_1} \in \hat{X}^k} \lambda D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}} ((N_0 x_0)^*)\gamma \\ &= \sum_{\eta \in \hat{X}^*} \lambda D_\eta((N_0 x_0)^*)\gamma, \end{aligned}$$

and the lemma is proved.

Proof of Theorem 3. Since c is input-limited, it follows from Lemma 1 that

$$c \circ d = \sum_{k=0}^{\mathcal{N}_c} \sum_{\eta \in \hat{X}^k} \lambda D_\eta((N_0 x_0)^*)\gamma.$$

Clearly, each operator D_η is mapping a rational series to another rational series as it involves only a finite number of rational operations. Therefore, for any integer $k \geq 0$ the formal power series

$$\sum_{\eta \in \hat{X}^k} \lambda D_\eta((N_0 x_0)^*)\gamma$$

is again rational since the summation is finite. Thus, $c \circ d$ must be rational.

Example 2. Reconsider the series $c \circ c$, where $c = x_1^*$ as in Example 1. The nested inductive argument used there can be directly extended to establish the identity

$$\begin{aligned} (c \circ c, x_0^{k_0} x_1^{k_1} \cdots x_0^{k_{l-1}} x_1^{k_l}) &= (k_0)^{k_1}(k_0 + k_2)^{k_3} \\ &\quad \cdots (k_0 + k_2 + \cdots + k_{l-1})^{k_l} \qquad (9) \end{aligned}$$

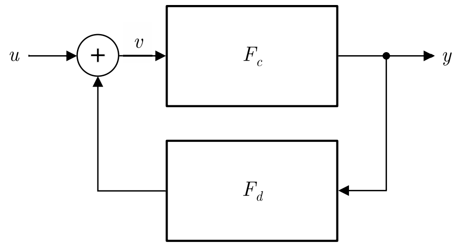


Fig. 2. Feedback connection of two Fliess operators.

for all $l \geq 0$ and $k_i \geq 0$, $i = 0, 1, \ldots, l$. In which case,

$$(c \circ c, x_0^{n_0} x_1 x_0^{n_1} x_1 \cdots x_0^{n_{j-1}} x_1 x_0^{n_j}) = n_0(n_0 + n_1) \cdots (n_0 + n_1 + \cdots + n_{j-1}) \quad (10)$$

$$(c \circ c, x_1^{m_0} x_0 x_1^{m_1} \cdots x_0 x_1^{m_k}) = 0^{m_0} 1^{m_1} 2^{m_2} \cdots k^{m_k} \quad (11)$$

for all $j \geq 0$ and $n_i \geq 0$, $i = 0, 1, \ldots, j$; and all $k \geq 0$ and $m_i \geq 0$, $i = 0, 1, \ldots, k$. Using identity (11), observe that

$$\begin{aligned} c \circ c &= \sum_{m_0 \geq 0} (c \circ c, x_1^{m_0})\, x_1^{m_0} + \sum_{k \geq 1} \sum_{m_0, \ldots, m_k \geq 0} (c \circ c, \\ &\quad x_1^{m_0} x_0 x_1^{m_1} \cdots x_0 x_1^{m_k}) x_1^{m_0} x_0 x_1^{m_1} \cdots x_0 x_1^{m_k} \\ &= 1 + \sum_{k \geq 1} \sum_{m_1, \ldots, m_k \geq 0} 1^{m_1} 2^{m_2} \cdots k^{m_k} x_0 x_1^{m_1} x_0 x_1^{m_2} \cdots x_0 x_1^{m_k} \\ &= 1 + \sum_{k \geq 1} x_0 \left(\sum_{m_1 \geq 0} x_1^{m_1} \right) x_0 \left(\sum_{m_1 \geq 0} (2x_1)^{m_2} \right) \\ &\quad \cdots x_0 \left(\sum_{m_k \geq 0} (k x_1)^{m_k} \right) \\ &= 1 + \sum_{k \geq 1} x_0 x_1^* x_0 (2x_1)^* \cdots x_0 (k x_1)^*. \end{aligned} \quad (12)$$

Alternatively, observe that x_1^* has a linear representation with $N_0 = 0$, $N_1 = 1$ and $\lambda = \gamma = 1$. Thus, $D_{x_1} : e \mapsto x_0(x_1^* \sqcup\!\sqcup\, e)$, and from Lemma 1

$$\begin{aligned} c \circ c &= \sum_{\eta \in \hat{X}^*} \lambda D_\eta((N_0 x_0)^*)\gamma = \sum_{k \geq 0} D_{x_1^k}(1) \\ &= 1 + \sum_{k \geq 1} x_0(x_1^* \sqcup\!\sqcup\, (x_0(x_1^* \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, (x_0(x_1^* \sqcup\!\sqcup\, 1)) \cdots))) \\ &= 1 + \sum_{k \geq 1} x_0 x_1^* x_0 (2x_1)^* \cdots x_0 (k x_1)^*, \end{aligned}$$

which is consistent with (12). Clearly, if the left argument of $c \circ c$ is truncated to $\tilde{c} = \sum_{k=0}^{\mathcal{N}} x_1^k$ for an integer $\mathcal{N} > 0$, then the resulting series composition produces a rational series as expected from Theorem 3.

4. Feedback systems

Consider the feedback system shown in Fig. 2. Given any $c, d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, the output y must satisfy the feedback equation

$$y = F_c[u + F_d[y]]$$

for any admissible input u. In particular, if there exists a generating series e so that $y = F_e[u]$, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d \circ e}[u]],$$

and the *feedback product* of c and d is defined by $c@d = e$. It was shown in [24] via a fixed point argument that $c@d$ is always well-defined if F_c and F_d are interpreted, for example, as operators in a formal sense, i.e., mappings between two formal series in one variable (one series for the input and one series for the output). If c and d are both locally convergent, then $c@d$ is *input–output* locally convergent [8]. That is, given any real analytic u, the corresponding output $y = F_{c@d}[u]$ is also real analytic. (It is unknown at present whether this condition is equivalent to $c@d$ being locally convergent in the sense described in Section 2.) In the event that $c@d$ is rational, one additional fact is easily deduced in the following lemma.

Lemma 2. *If $c@d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ is a rational series then $y = F_{c@d}[0]$ is well-defined on $[t_0, \infty)$.*

Proof. It follows directly that the generating series for y is $\sum_{k \geq 0}(c@d, x_0^k)x_0^k$. Since $c@d$ is rational, there exists constants $K, M > 0$ such that $\left|(c@d, x_0^k)\right| \leq KM^k$ for all $k \geq 0$. Thus, $|y(t)| \leq \sum_{k \geq 0} KM^k (t - t_0)^k / k! = K e^{M(t - t_0)} < \infty$ for all $t \geq t_0$.

The following example utilizes this result to show that the feedback product is not a rational operation, even if c is input-limited.

Example 3. Consider the feedback connection involving the rational series $c = x_1$ and $d = x_1^*$. Note that c is input-limited, and in fact, represents a linear operator. $F_{c@d}$ has a state space realization

$$\dot{z}_1 = z_1 z_2, \quad z_{1,0} = 1 \quad (13)$$

$$\dot{z}_2 = z_1 + u, \quad z_{2,0} = 0 \quad (14)$$

$$y = z_2. \quad (15)$$

Setting $u = 0$, the natural response y satisfies the initial value problem $\ddot{y} - \dot{y}y = 0$, $y(0) = 0$, $\dot{y}(0) = 1$, which has the solution

$$\begin{aligned} y(t) &= \sqrt{2} \tan\left(\frac{t}{\sqrt{2}}\right) \\ &= \sum_{k \geq 1} (-1)^{k-1} 2^k (2^{2k-1}) \frac{B_{2k}}{k} \frac{t^{2k-1}}{(2k-1)!} \\ &= t + \frac{t^3}{3!} + 4\frac{t^5}{5!} + 34\frac{t^7}{7!} + 496\frac{t^9}{9!} + \cdots \end{aligned} \quad (16)$$

for $0 \leq t < \pi/\sqrt{2}$, where B_k denotes the kth Bernoulli number. Since y is not defined for all $t \geq 0$, $c@d$ is not rational.

5. Generating series of weighted Petri nets

A wide variety of Petri net definitions appear in the literature [26,27]. The focus here is on a class of marked Petri nets as described in [18].

Definition 6. A **marked Petri net** (P, T, A, W, M_0) is a weighted bipartite directed graph, where

$P = \{z_1, z_2, \ldots, z_n\}$ is a set of places

$T = \{u_0, u_1, \ldots, u_m\}$ is a set of transitions

$A \subseteq (P \times T) \cup (T \times P)$ is a set of arcs from places to transitions and from transitions to places

$W : A \to \mathbb{N}$ is an arc weight function

$M_0 \in \mathbb{N}^n$ is an initial marking of the places.

A given transition $u_j \in T$ is said to be *enabled* if every input place z_i to u_j has a marking, say k_i, which satisfies $k_i \geq W(z_i, u_j)$. The dynamics of a marked Petri net corresponding to an enabled transition u_j are described by the transition function

$$F_j : \mathbb{N}^n \to \mathbb{N}^n : k_i \mapsto k_i - W(z_i, u_j) + W(u_j, z_i),$$

where $i = 1, 2, \ldots, n$. The transition u_j is said to have *fired* when the mapping F_j is applied to a given marking $M =$

$(k_1, k_2, \ldots, k_n)$. It is possible for more than one transition to be enabled simultaneously. For an initial marking M_0, a sequence of transitions $(u_{j_1}, u_{j_2}, \ldots, u_{j_r})$ is said to be *admissible* if u_{j_1} is enabled and transition $u_{j_{\ell+1}}$ is enabled after the firing of u_{j_ℓ} for every $\ell = 1, \ldots, r-1$. The transition labels in T need not be unique. In this situation, it is necessary to treat T as a *multiset*. That is, a set-like object where multiplicity is significant. So, for example, the multiset $T' = \{u_0, u_0, u_1\}$ is distinct from $T = \{u_0, u_1\}$. The collection of transitions labeled u_0 is said be enabled when it contains at least one enabled transition. It is possible, for example, that both transitions labeled u_0 in T' could be enabled. In which case, the admissible sequence (u_0, u_1) could refer to two distinct *execution traces* in the Petri net: the firing of the first transition labeled u_0 followed by the firing of u_1 or the firing of the second transition labeled u_0 followed by the firing u_1. In general, the set of all possible execution traces for a given admissible sequence $(u_{j_1}, u_{j_2}, \ldots, u_{j_r})$ is denoted by $\mathcal{E}(u_{j_1}, u_{j_2}, \ldots, u_{j_r})$. If T is not a multiset then $\mathcal{E}(u_{j_1}, u_{j_2}, \ldots, u_{j_r})$ is a singleton.

Definition 7. A **weighted Petri net** is a marked Petri net (P, T, A, W, M_0) with a token value function $V : P \to \mathbb{R}$ and a transition weight function $K : T \to \mathbb{R}$.

With any weighted Petri net, one can associate a generating series in $\mathbb{R}\langle\langle X\rangle\rangle$, where $X = \{x_0, x_1, \ldots, x_m\}$. This is analogous to the way in which rational series are generated from weighted finite-state automata [11,16,17]. It is also remarkably similar to the way vertex weights are assigned in a colored weighted increasing tree derived from the differential equation (4) [21, Section 6].

Definition 8. The **generating series** for a weighted Petri net (P, T, A, W, M_0, V, K), where T is a set, is defined to be $c_P \in \mathbb{R}\langle\langle X\rangle\rangle$, where

$$(c_P, x_{j_1}x_{j_2}\cdots x_{j_r}) = v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n}K_1(u_{j_1})K_2(u_{j_2})\cdots K_r(u_{j_r}), \quad (17)$$

$(u_{j_1}, u_{j_2}, \ldots, u_{j_r})$ is an admissible sequence, and the resulting terminal marking is $(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$. Here $v_i = V(z_i)$, $i = 1, 2, \ldots, n$, and the real numbers $K_l(u_{j_l})$ are computed according to the expression

$$K_l(u_{j_l}) = \binom{\tilde{k}_1}{w_1}\binom{\tilde{k}_2}{w_2}\cdots\binom{\tilde{k}_s}{w_s}K(u_{j_l}), \quad (18)$$

where the enabled transition u_{j_l} has s inputs with arc weights $w_1, w_2, \ldots, w_s$, and $\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_s$ denote the number of tokens in each place to which these inputs are connected at the instant before $u_{j_{l+1}}$ fires. If a firing sequence is not admissible then the corresponding series coefficient is zero. For the empty word let $(c_P, \emptyset) = v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n}$. If T is a multiset then

$$(c_P, x_{j_1}x_{j_2}\cdots x_{j_r}) = \sum_{\mathcal{E}(u_{j_1}, u_{j_2}, \ldots, u_{j_r})} v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n} \cdot K_1(u_{j_1})K_2(u_{j_2})\cdots K_r(u_{j_r}). \quad (19)$$

The importance of weighted Petri nets in connection with nonlinear dynamical systems can be explained in the context of the following definition and theorem. For brevity, the focus is on single-input, single-output systems.

Definition 9. A **polynomial state space system** is a state space system (f, g, h, z_0) as in (4)–(5) with the additional property that the mappings f, g and h have only component functions which are polynomial in z. Without loss of generality, it is assumed that $h(z) = z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}$, where each $\alpha_i \in \mathbb{N}$.

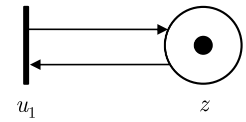


Fig. 3. The weighted Petri net for x_1^* with initial marking $M_0 = 1$.

Theorem 4 ([18]). *The generating series c for a given single-input, single-output polynomial system (f, g, h, z_0) as computed by (6) is equivalent to the generating series c_P of a weighted Petri net (P, T, A, W, M_0, V, K) as computed by (17), where A and W have the property that each transition has exactly one input and its arc weight is 1. Specifically, if:*

- *The places of the Petri net correspond to the n states of the polynomial system.*
- *The transitions of the Petri net correspond to the inputs $\{u_0, u_1\}$ of the polynomial system, and thus, to the letters of the alphabet $X = \{x_0, x_1\}$. (Note $u_0 = 1$ is used exclusively for drift terms.)*
- *Each term in the summand on the right-hand side of the equation for $\dot{z}_i$, that is, $K(u_j)u_jz_1^{w_1}z_2^{w_2}\cdots z_n^{w_n}$, corresponds to a transition labeled u_j with transition weight $K(u_j)$ and having a single input from place z_i with arc weight 1 and outputs to places z_s with arc weights w_s for $s = 1, 2, \ldots, n$.*
- *The initial marking, M_0, is $(k_{1,0}, k_{2,0}, \ldots, k_{n,0})$, where the output function is $h(z) = z_1^{k_{1,0}}z_2^{k_{2,0}}\cdots z_n^{k_{n,0}}$.*
- *The value v_i of a token at place z_i is taken to be the initial state $z_{i,0}$.*

then for any $x_{j_1}x_{j_2}\cdots x_{j_r} \in X^*$

$$(c_P, x_{j_1}x_{j_2}\cdots x_{j_r}) = z_{1,0}^{k_1}z_{2,0}^{k_2}\cdots z_{n,0}^{k_n}K_1(u_{j_1})K_2(u_{j_2})\cdots K_r(u_{j_r}) = (c, x_{j_1}x_{j_2}\cdots x_{j_r}). \quad (20)$$

This theorem provides a graph-theoretic interpretation of the usual process of computing generating series via (6)–(7). The following lemma makes the essential link between a class of weighted Petri nets and the cascade and feedback connections of bilinear systems.

Lemma 3. *Suppose F_{c_1} and F_{c_2} are realizable on $L_{\mathfrak{p},e}(t_0)$ by bilinear state space systems $(A_1, N_1, C_1, z_{1,0})$ and $(A_2, N_2, C_2, z_{2,0})$, respectively. Then $F_{c_2\circ c_1}$ and $F_{c_1@c_2}$ are each realizable by a quadratic polynomial state space system on $B_{\mathfrak{p}}(R)[t_0+T]$ for $R, T > 0$ sufficiently small.*

Proof. The claim concerning the cascade connection is immediately evident from (1)–(3). Introducing the additional state $\tilde{z} = C_2z_2$ yields a quadratic polynomial system where the output y_2 can be written in the desired form, in this case, $y_2 = \tilde{h}(z_1, z_2, \tilde{z}) = \tilde{z}$. The feedback connection is handled in a similar fashion.

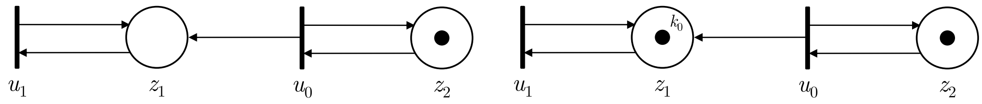
The following theorem is an immediate consequence of this lemma, Theorems 1 and 4. It is illustrated using the examples from Sections 3 and 4.

Theorem 5. *Let $X = \{x_0, x_1\}$ and $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ be rational series. Then there exists weighted Petri nets with generating series c_{P_1} and c_{P_2}, each corresponding to a quadratic polynomial state space system, such that $c \circ d = c_{P_1}$ and $c@d = c_{P_2}$.*

Example 4. Reconsider the series $x_1^* \circ x_1^*$ in Example 2. $F_{x_1^*}$ is realized by

$$\dot{z} = zu_1, \quad z_0 = 1$$
$$y = z.$$

The corresponding weighted Petri net is shown in Fig. 3. In this case, $P = \{z\}$, $T = \{u_0, u_1\}$, A and W are evident from the diagram,



(a) The initial marking.

(b) After the firing sequence $(u_0^{k_0})$ occurs.

Fig. 4. The weighted Petri net for $x_1^* \circ x_1^*$.

Fig. 5. The tree of weighted Petri nets for $x_1 @ x_1^*$ after the firing sequence (u_0^5) occurs. The transition weight factor $\binom{\tilde{k}_i}{w_i}$ is shown next to the transition which has most recently fired.

$M_0 = 1, K(u_0) = 0, K(u_1) = 1$ and $v = z_0 = 1$. (The transition for u_0 is not shown in this case since its weight is zero.) The admissible sequences are of the form $(u_1^r) := (\underbrace{u_1, u_1, \ldots, u_1}_{r \text{ times}}), r \geq 0$. In which case, $K_i(u_1) = 1$ for all $i \geq 0$. From (20) then the generating series for the Petri net is $c_P = 1 + x_1 + x_1^2 + \cdots = x_1^*$ as expected. For the cascade connection $F_{x_1^*} \circ F_{x_1^*} = F_{x_1^* \circ x_1^*}$, a corresponding quadratic polynomial state space system is

$$\dot{z}_1 = z_1 u_1, \quad z_{1,0} = 1$$
$$\dot{z}_2 = z_1 z_2, \quad z_{2,0} = 1$$
$$y = z_2.$$

The associated weighted Petri net is shown in Fig. 4a. Here $P = \{z_1, z_2\}$, $T = \{u_0, u_1\}$, A and W are as shown, $M_0 = (0, 1)$, $K(u_i) = 1$ for $i = 0, 1$ and $v_j = 1$ for $j = 1, 2$. Clearly, any firing sequence of the form $(u_1, u_{i_2}, u_{i_3}, \ldots)$ is *not* admissible, so the coefficient $(c_P, x_1\eta) = 0$ for any $\eta \in X^*$. All other firing sequences are admissible. This is consistent with the coefficients of $x_1^* \circ x_1^*$ as determined by (9). Consider the firing sequence $(u_0^{k_0})$. The resulting marking is shown in Fig. 4b, and it is easily verified that $K_i(u_0) = 1$ for $i = 1, 2, \ldots, k_0$. This marking will not change if u_1 is then fired k_1 times, but in this case

$$K_{k_0+i}(u_1) = \binom{k_0}{1} K(u_1), \quad i = 1, 2, \ldots, k_1.$$

Thus,

$$\begin{aligned}(c_P, x_0^{k_0} x_1^{k_1}) &= K_1(u_0)K_2(u_0)\cdots K_{k_0}(u_0) \\ &\quad \cdot K_{k_0+1}(u_1)K_{k_0+2}(u_1)\cdots K_{k_0+k_1}(u_1) \\ &= \binom{k_0}{1}\binom{k_0}{1}\cdots\binom{k_0}{1} \\ &= (k_0)^{k_1}.\end{aligned}$$

Continuing the process of firing u_0 in succession followed by firing u_1 in succession will directly generate all the coefficients of $x_1^* \circ x_1^*$ as given in (9).

Example 5. Reconsider the series $x_1 @ x_1^*$ in Example 3. The zero-input system has realization given by (13)–(15) with $u = 0$. The corresponding Petri net is shown at the root of the tree in Fig. 5. In this case, $P = \{z_1, z_2\}$, $T = \{u_0, u_0\}$, A and W are evident from the diagram, $M_0 = (0, 1)$, $K(u_0) = 1$ (in both state equations), and $v_1 = 1, v_2 = 0$. The only admissible sequences are of the form (u_0^r), $r \geq 0$. Since T is a multiset, the coefficient (c_P, u_0^r) is computed via (19), specifically

$$(c_P, x_0^r) = \sum_{\mathcal{E}(u_0^r)} 1^{k_1} 0^{k_2} \cdot K_1(u_0)K_2(u_0)\cdots K_r(u_0).$$

The set $\mathcal{E}(u_0^r)$ is determined using the binary tree shown in Fig. 5. The root corresponds to the initial marking. Each level of the tree corresponds to a specific value of r. At a given vertex it is possible that either one or both transitions, each having the same label u_0, may be enabled. In the latter case, it is necessary to produce a child at this vertex for each enabled transition. The transition weight factor $\binom{\tilde{k}_i}{w_i}$ generated by such a firing (see (18)) is placed next to the corresponding transition in the figure. The set $\mathcal{E}(u_0^r)$ corresponds to the set of paths that lead from the root to the rth level. When $r = 5$, for example, $\mathcal{E}(u_0^5)$ consists of six paths. The numerical value of (c_P, x_0^r) is computed by summing