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Expanding the Class of Globally Convergent Fliess Operators

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***Abstract*— A common representation of an input-output system in nonlinear control theory is the Chen-Fliess functional series or Fliess operator. Such a functional series is said to be *globally convergent* when there is no a priori upper bound on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined. It is known that every Fliess operator having a generating series with Gevrey order $0 \leq s < 1$ is globally convergent. In this paper it is shown that there exists a subset of series with Gevrey order $s = 1$ which also exhibit global convergence. In particular, the example of Ferfera, which arises in the context of system interconnections, is shown to be one such example.**

***Index Terms*— Nonlinear control systems, Chen-Fliess series, locally convex topological vector spaces.**

***AMS Subject Classifications*—93C10, 47H30, 46A99**

I. Introduction

A common representation of an input-output system in nonlinear control theory is the Chen-Fliess functional series or Fliess operator [8], [16]. It can be viewed as a noncommutative generalization of a Taylor series, and its algebraic nature is especially well suited for describing system interconnections [13], feedback invariants [10], [14] and solving system inversion problems [11] in a nonlinear setting. Such a functional series is said to be *globally convergent* when there is no a priori upper bound on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined. If such bounds are imposed to ensure convergence then the series is said to be only *locally convergent*. As Fliess operators have coefficients which are indexed by words, it is natural to describe their asymptotic behavior (in magnitude) via Gevrey order, that is, by a growth rate of the form $KM^n(n!)^s$ for some real $K, M > 0$ and $s \in \mathbb{R}$, where n is word length. In [15] it was shown that $s = 0$ ensures that a Fliess operator is globally convergent, while $s = 1$ provides for at least local convergence. It was implicitly observed in the early work of Ferfera that Gevrey order is not always preserved under system interconnection [6], [7]. For example, a bilinear system always has Gevrey order $s = 0$. Ferfera provided a specific example of two bilinear systems cascaded together which yields a composite system which is not bilinear. But since it does have an input-affine analytic state space realization, the cascaded system must have a generating series with Gevrey order satisfying $0 < s \leq 1$ [19]. Later it was proved in [20] that *every* cascade of two Fliess operators with generating series of Gevrey order $s = 0$ has a Fliess operator representation that converges globally. It was therefore conjectured that some cascades can somehow fall strictly *in-between* the cases $s = 0$ and $s = 1$. These observations were later partially explained in [22] by proving that $0 \leq s < 1$ is a sufficient condition for global convergence.

Despite all the progress described above, an interesting open question still remains: Can there exist a generating series with Gevrey order $s = 1$ for which the corresponding Fliess operator is not only locally convergent, but *also* globally convergent? In this paper, it is shown that the answer to this question is *yes*. In fact, it will be shown that Ferfera's original example yields precisely such a series. It turns out that there is a bifurcation in the class of locally convergence generating series. Namely, such series can be either *weakly* locally convergent or *strongly* locally convergent. Their corresponding Fliess operators have very different convergence behavior. In general, the former have no singularities in their defining functional series, and thus they converge globally, while the latter always have a singularity which renders a finite radius of convergence. Ferfera's example provides a specific instance of a weakly locally convergent generating series, which is the more subtle case. The distinction requires one to view the set of generating series as a topological vector space with a family of semi-norms instead of the more common ultrametric space setting found, for example, in [2]. The bottom line is that the set of generating series which render globally convergent Fliess operators is the *closure* in this semi-norm topology of the set of all generating series with Gevrey order $0 \leq s < 1$. In this way, the class of globally convergent Fliess operators is expanded.

The presentation is organized as follows. In the next section, a few preliminaries concerning spaces of formal power series and Fliess operators are summarized in order to make the paper more self-contained and to establish the notation. In the subsequent section, the Gevrey order of the Ferfera series is considered in detail. Then, in Section IV, the issue of strong versus weak local convergence is developed. This leads to the larger class of globally convergent Fliess operators. The final section provides the conclusions of the paper.

II. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*}(c, \eta)\eta$. The subset of X^* defined by $\mathrm{supp}(c) = \{\eta : (c, \eta) \neq 0\}$ is called the *support* of c. A series $\hat{c}$ is said to be a *subseries* of c if $\mathrm{supp}(\hat{c}) \subseteq \mathrm{supp}(c)$ and $(\hat{c}, \eta) = (c, \eta),\ \forall \eta \in \mathrm{supp}(\hat{c})$. The collection of all formal power

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series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product defined in terms of the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ [8], [18].

A. Spaces of Formal Power Series

A commonly used metric on $\mathbb{R}\langle\langle X\rangle\rangle$ is the ultrametric metric $\mathrm{dist} : (c, d) \mapsto \sigma^{\mathrm{ord}(c-d)}$, where the *order* of a series c, $\mathrm{ord}(c)$, is taken as the length of the smallest word in the support of c ($\mathrm{ord}(c) := \infty$ when $c = 0$), and σ is any real number $0 < \sigma < 1$ [2]. Alternatively, one can define for any real number $R > 0$ the norm

$$\|c\|_{\infty,R} := \sup_{\eta\in X^*} \left\{ |(c, \eta)| \frac{R^{|\eta|}}{|\eta|!} \right\}.$$

It is easy to verify that $S_\infty(R) := \{c \in \mathbb{R}\langle\langle X\rangle\rangle : \|c\|_{\infty,R} < \infty\}$ is a normed linear subspace of the $\mathbb{R}$-vector space $\mathbb{R}\langle\langle X\rangle\rangle$ [19]. Given $0 < R < R'$, it is clear for any $c \in \mathbb{R}\langle\langle X\rangle\rangle$ that $\|c\|_{\infty,R} \le \|c\|_{\infty,R'}$, and thus, $S_\infty(R') \subset S_\infty(R)$. Furthermore, if $c_i \to c$ as a sequence in $S_\infty(R)$ and $c_i \to c'$ as a sequence in $S_\infty(R')$ then necessarily $c = c'$.

It will be useful throughout to consider the spaces $S_{\infty,e} := \cup_{R>0} S_\infty(R)$ and $S_\infty := \cap_{R>0} S_\infty(R)$. The extended space $S_{\infty,e}$ can not in any obvious way be viewed as a normed linear space, but it is a locally convex topological vector space equipped with a family of semi-norms $\|\cdot\|_{\infty,R}$, $R > 0$. The semi-norm topology is that generated by the open sets

$$B_{c,R}(\epsilon) := \{d \in S_{\infty,e} : \|c - d\|_{\infty,R} < \epsilon\},$$

where $c \in S_{\infty,e}$ and $\epsilon, R > 0$. It is not difficult to show that this semi-norm topology is second countable, and thus first countable. The space is Hausdorff since for each $R > 0$, the norm property ensures that if $c \neq 0$ then $\|c\|_{\infty,R} \neq 0$ for every $R > 0$ [9, Proposition 5.16]. In which case, sequentially continuous maps are continuous [17, p. 20]. A sequence $\{c_i\}_{i\in\mathbb{N}}$ in $S_{\infty,e}$ converges to a (unique) $c \in S_{\infty,e}$ in the semi-norm topology if and only if $\|c_i - c\|_{\infty,R} \to 0$ as $i \to \infty$ for all $R > 0$. Given a series $c \in S_{\infty,e}$, define $\bar{R}_c$ as the supreme of all R for which $c \in S_\infty(R)$, i.e,

$$\bar{R}_c := \sup_{\substack{\|c\|_{\infty,R}<\infty \\ R>0}} R.$$

In particular, if $\bar{R}_c = \infty$ then $c \in S_\infty$. The various spaces are nested as shown in Figure 1.

The following examples illustrate that convergence in the semi-norm topology is in general *unrelated* to convergence in the ultrametric sense.

Example 1: Consider the sequence of constants $\{c_i = 1/i\}_{i\ge1}$ as polynomials in $\mathbb{R}\langle\langle X\rangle\rangle$. Clearly, $\|c_i - 0\|_{\infty,R} = 1/i$ for all $R > 0$. Thus, $c_i \to 0$ as $i \to \infty$ in the semi-norm topology. On the other hand, this sequence does *not* approach zero in the ultrametric sense because $\mathrm{dist}(c_i, 0) = 1$ for every $i \ge 1$. In fact, this sequence is not even Cauchy because $\mathrm{dist}(c_i, c_{i+1}) = 1$ for every $i \ge 1$. □

Example 2: Consider the sequence of polynomials

$$c_i = 1 + M1!\,x_0 + M^2 2!\,x_0^2 + \cdots + M^i i!\,x_0^i, \quad i \ge 0,$$

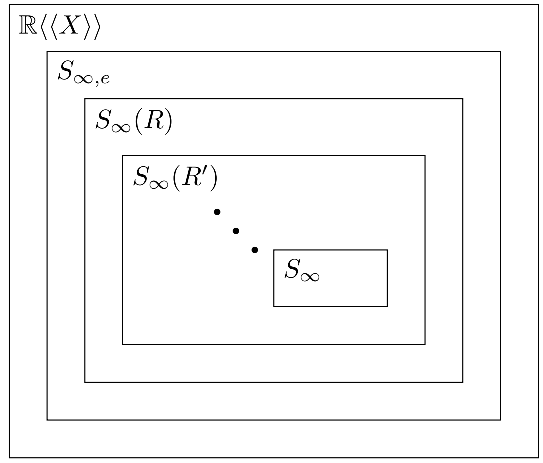

Fig. 1. The spaces $S_{\infty,e}, S_\infty(R)$ and S_∞ are nested.

where $M > 0$ is fixed. It is easily verified that $\mathrm{dist}(c_i, c) \to 0$ as $i \to \infty$ when

$$c = \sum_{n=0}^{\infty} M^n n!\, x_0^n.$$

Therefore, $c_i \to c$ in the ultrametric sense. Next observe that

$$\|c_i\|_{\infty,R} = \begin{cases} (MR)^i & : MR > 1 \\ 1 & : MR \le 1, \end{cases}$$

and thus, each $c_i \in S_{\infty,e}$. Similarly, $c \in S_{\infty,e}$ because $\|c\|_{\infty,R} < \infty$ when $MR \le 1$. On the other hand,

$$\|c_i - c\|_{\infty,R} = \sup_{n>i} (MR)^n = \begin{cases} (MR)^{i+1} & : MR < 1 \\ 1 & : MR = 1 \\ \infty & : MR > 1, \end{cases}$$

which implies that

$$\lim_{i\to\infty} \|c_i - c\|_{\infty,R} = \lim_{i\to\infty} (MR)^{i+1} = 0,$$

only when $MR < 1$. Therefore, the sequence $\{c_i\}_{i\ge1}$ converges to c in the normed linear space $S_\infty(R)$ when $R < 1/M$, but *not* to c in the semi-norm topology. In fact, $\|c_i - c_{i-1}\|_{\infty,R} = (MR)^i$ can not be made arbitrarily small for sufficient large i when $MR \ge 1$. So the sequence is not Cauchy in the semi-norm topology. □

B. Fliess Operators and Their Interconnections

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \ge 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \le i \le m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R_u)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : \|u\|_{\mathfrak{p}} \le R_u\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{1}$$

[8]. The generating series c is said to be of *Gevrey order* $s \in \mathbb{R}$ if there exists constants $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|} (|\eta|!)^s, \quad \forall \eta \in X^*, \tag{2}$$

and s is the smallest number having this property [1], [22]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all generating series with Gevrey order s is denoted by $\mathbb{R}_s\langle\langle X \rangle\rangle$. If $0 \leq s \leq 1$ then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R_u)[t_0,\, t_0 + T]$ into $B_{\mathfrak{q}}^\ell(S_u)[t_0,\, t_0 + T]$ for sufficiently small $R_u, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [15]. The set of all such *locally convergent* generating series is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$. The least upper bound on $R := \max\{R_u, T\}$, say $\rho(F_c)$, is called the *radius of convergence* of the operator. It was shown in [5] that $0 < 1/M(m+1) \leq \rho(F_c)$. Note that if $R \leq 1/M \leq \rho(F_c)(m+1)$ then

$$\|c\|_{\infty,R} \leq \sup_{\eta \in X^*} K(MR)^{|\eta|} = K < \infty,$$

otherwise, $\|c\|_{\infty,R}$ is unbounded. Thus, $\bar{R}_c = 1/M$, $c \in S_\infty(1/M)$ and $\mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle \subseteq S_{\infty,e}$. When $0 \leq s < 1$, the series (1) defines an operator from the extended space $L_{\mathfrak{p},e}^m(t_0)$ into $C[t_0, \infty)$, where

$$\begin{aligned} L_{\mathfrak{p},e}^m(t_0) :=& \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L_{\mathfrak{p}}^m[t_0, t_1], \\ & \forall t_1 \in (t_0, \infty)\}, \end{aligned}$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ [22]. The set of all such *globally convergent* series is designated by $\mathbb{R}_{GC}^\ell\langle\langle X \rangle\rangle$. In this case, it is not hard to see that for every $R > 0$

$$\|c\|_{\infty,R} \leq \sup_{\eta \in X^*} \frac{K(MR)^{|\eta|}}{(|\eta|!)^{1-s}} < \infty,$$

thus $\bar{R}_c = \infty, c \in S_\infty$ and $\mathbb{R}_{GC}^\ell\langle\langle X \rangle\rangle \subseteq S_\infty \subset S_{\infty,e}$.

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup\, d}$, respectively [8]. When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(1) \tag{3}$$

[6], [7]. Here ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X \rangle\rangle$ to the set of vector space endomorphism on $\mathbb{R}\langle\langle X \rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X \rangle\rangle)$, uniquely specified by $\psi_d(x_i \eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with

$$\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup\, e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := 1$). $\psi_d(\emptyset)$ is defined to be the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$. This composition product is associative and $\mathbb{R}$-linear in its left argument.

III. Gevrey Order of the Ferfera Series

Let $X = \{x_0, x_1\}$ and consider the rational series $x_1^* := \sum_{k \geq 0} x_1^k$. The series considered by Ferfera in [6], [7] is $c_F := x_1^* \circ x_1^*$ using the notion of formal power series composition defined in (3). In this section, the Gevrey order of c_F is shown to be exactly $s = 1$.

A general formula for the coefficients of c_F is

$$\begin{aligned} (c_F, x_0^{k_0} x_1^{k_1} \cdots x_0^{k_{l-1}} x_1^{k_l}) &= (k_0)^{k_1} (k_0 + k_2)^{k_3} \\ & \cdots (k_0 + k_2 + \cdots + k_{l-1})^{k_l} \end{aligned} \tag{4}$$

for all $l \geq 0$ and $k_i \geq 0$, $i = 0, 1, \ldots, l$ [12]. The following two subseries of c_F are also of interest here:

$$c_F^{1/2} := \sum_{k=0}^{\infty} (c_F, x_0^k x_1^k)\, x_0^k x_1^k$$

$$c_F^1 := \sum_{k_0, k_1 = 0}^{\infty} (c_F, x_0^{k_0} x_1^{k_1})\, x_0^{k_0} x_1^{k_1}.$$

Ferfera's central argument in showing that rationality is not preserved under composition was the observation that the coefficients

$$(c_F^{1/2}, x_0^k x_1^k) = k^k, \quad k \geq 0$$

grow too fast to satisfy (2) when $s = 0$. Therefore, c_F can not be rational. The following theorem gives the exact Gevrey order of $c_F^{1/2}$.

Theorem 1: The series $c_F^{1/2}$ has Gevrey order $s = 1/2$.

Proof: Let $n = |x_0^k x_1^k| = 2k \geq 0$ and define the sequences

$$a_n = (c_F^{1/2}, x_0^{n/2} x_1^{n/2}) = (n/2)^{(n/2)}$$

$$b_n(s) = K M^n (n!)^s$$

for any fixed $K, M > 0$. Also define the function

$$\begin{aligned} f_n(s) &= \ln\left(\frac{a_n}{b_n(s)}\right) \\ &= (n/2)\ln(n/2) - \ln(K) - n\ln(M) - s\ln(n!). \end{aligned}$$

Using Stirling's approximation

$$n! \approx \sqrt{2\pi n}(n/e)^n, \quad n \gg 1, \tag{5}$$

it follows directly that

$$\begin{aligned} f_n(s) \approx& -\ln(K) - \frac{n}{2}\ln(2) - n\ln(M) \\ & + n\left(\frac{1}{2} - s\right)\ln(n) + ns - \frac{s}{2}\ln(2\pi) - \frac{s}{2}\ln(n). \end{aligned}$$

Consider the following cases:

1) If $s < 1/2$ then $\lim_{n \to \infty} f_n(s) = +\infty$.

2) If $s > 1/2$ then $\lim_{n \to \infty} f_n(s) = -\infty$.

3) If $s = 1/2$ then

$$\begin{aligned} \lim_{n \to \infty} f_n(1/2) =& -\ln(K) - \frac{1}{4}\ln(2\pi) - \frac{1}{4}\lim_{n \to \infty}\ln(n) \\ & + \lim_{n \to \infty} n\left(\frac{1}{2} - \frac{\ln(2)}{2} - \ln(M)\right). \end{aligned}$$

Therefore, if $\frac{1}{2} - \frac{\ln(2)}{2} - \ln(M) \leq 0$ then $\lim_{n \to \infty} f_n(1/2) = -\infty$. In summary, when $s \geq 1/2$

$$\lim_{n \to \infty} \frac{a_n}{b_n(s)} = 0,$$

and this implies in particular that $b_n(1/2)$ is growing faster than a_n. On the other hand, if $s < 1/2$ then a_n can not be bounded by a sequence of the form $b_n(s)$. Hence, the coefficients $(c_F^{1/2}, x_0^{n/2} x_1^{n/2})$ must be upper bounded for all

$n \geq 0$ by $KM(n!)^{1/2}$ for some $K, M > 0$, which implies that the series $c_F^{1/2}$ has Gevrey order $s = 1/2$. ■

As a check, an estimate of the Gevrey order of $c_F^{1/2}$ was computed numerically using the nonlinear fitting capabilities of *Mathematica* via the code:

```
nmax=300;
data=Table[{n,Log[(n/2)^(n/2)]},{n,1,nmax,2}];
nlm=NonlinearModelFit[data,Log[K*M^n*(n!)^s],{K,M,s},n]
Show[ListPlot[data],Plot[nlm[n],{n,1,nmax}],Frame->True]
```

The corresponding growth parameters estimates are $K = 0.39102$, $M = 1.14373$, and $s = 0.503423$. The quality of the fit for the first 30 coefficients is shown in Figure 2. It is representative of the fit for the entire data set of 300 points.

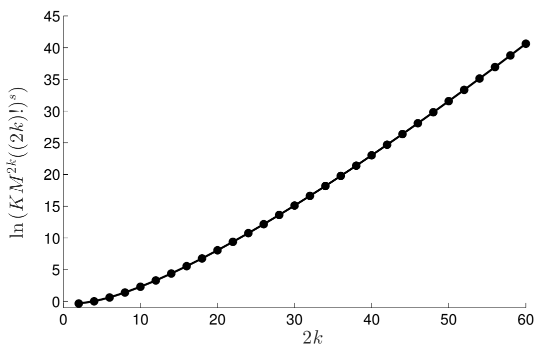

Fig. 2. Empirical fit of $KM^{2k}((2k)!)^s$ (solid line) to the first 30 coefficients (word length $n = 2k \leq 60$) of the series $c_F^{1/2}$ (dots).

The next theorem is the main result of this section.

Theorem 2: The series c_F has Gevrey order $s = 1$.

Proof: It is sufficient to show that c_F^1 has Gevrey order $s = 1$ since it is known that c_F has a Gevrey order $s \leq 1$ (because it has an analytic state space realization [19]), and it contains c_F^1 as a subseries. Let $n = \left|x_0^{k_0} x_1^{k_1}\right| = k_0 + k_1 \geq 0$ and

$$d_n := \sum_{k_0=0}^{n} (c_F, x_0^{k_0} x_1^{n-k_0})\, x_0^{k_0} x_1^{n-k_0}.$$

Define the sequences

$$\begin{aligned} a_n(k_0) &= (d_n, x_0^{k_0} x_1^{n-k_0}) = (k_0)^{(n-k_0)},\ 0 \leq k_0 \leq n, \\ b_n(s) &= KM^n (n!)^s,\ K, M > 0, \end{aligned}$$

using (4) with $l = 1$. Also, define

$$\begin{aligned} f_n(k_0, s) &= \ln\left(\frac{a_n(k_0)}{b_n(s)}\right) \\ &= (n - k_0)\ln(k_0) - \ln(K) - n\ln(M) - s\ln(n!), \end{aligned}$$

so that when $n \gg 1$ (5) yields

$$\begin{aligned} f_n(k_0, s) \approx{}& (n - k_0)\ln(k_0) - \ln(K) - n\ln(M) \\ &+ sn - \frac{s}{2}\ln(2\pi n) - sn\ln(n). \end{aligned}$$

Observe that $f_n(k_0, s)$ has a maximum over $\mathbb{R}$ if and only if

$$k_0 = \hat{k}_0 := \exp(W(ne) - 1),$$

since

$$\left.\frac{\partial f_n(k_0, s)}{\partial k_0}\right|_{k_0=\hat{k}_0} = -\ln(\hat{k}_0) + \frac{(n - \hat{k}_0)}{\hat{k}_0} = 0$$

and

$$\left.\frac{\partial^2 f_n(k_0, s)}{\partial k_0^2}\right|_{k_0=\hat{k}_0} = -\frac{1}{\hat{k}_0} - \frac{n}{\hat{k}_0^2} < 0,\ \ \forall\, 0 \leq \hat{k}_0 \leq n,$$

where W denotes the *Lambert W–function*, namely, the inverse of the function $g(z) = z\exp(z)$ [3]. Therefore, the goal is to compute $\lim_{n\to\infty} f_n(\hat{k}_0, s)$. Observe that

$$\frac{\partial}{\partial s}\lim_{n\to\infty} f_n(\hat{k}_0, s) = \lim_{n\to\infty}\left(n - \frac{1}{2}\ln(2\pi n) - n\ln(n)\right) < 0,$$

which implies that $\lim_{n\to\infty} f_n(\hat{k}_0, s)$ is a non-increasing function of s. A direct calculation gives

$$\begin{aligned} \lim_{n\to\infty} f_n(\hat{k}_0, s) = \Big(&\lim_{n\to\infty} nW(ne) + W(ne)\exp(W(ne) - 1) \\ &+ \exp(W(ne) - 1) + sn - n - n\ln(M) \\ &- \frac{s}{2}\ln(2\pi n) - sn\ln(n)\Big). \end{aligned}$$

Using the fact that $W(ne)\exp(W(ne) - 1) = n$ gives

$$\begin{aligned} \lim_{n\to\infty} f_n(\hat{k}_0, s) = \lim_{n\to\infty}\Big(&nW(ne) + \exp(W(ne) - 1) + sn \\ &- n\ln(M) - \frac{s}{2}\ln(2\pi n) - sn\ln(n)\Big). \end{aligned}$$

This reduces to computing the limit

$$\lim_{n\to\infty} nW(ne) - sn\ln(n).$$

But since $\lim_{n\to\infty} W(ne)/\ln(n) = 1$, it follows that:

1) If $s < 1$ then $\lim_{n\to\infty} f_n(\hat{k}_0, s) = +\infty$.
2) If $s \geq 1$ then $\lim_{n\to\infty} f_n(\hat{k}_0, s) = -\infty$.

Thus, if $s \geq 1$ then

$$\lim_{n\to\infty} \frac{a_n(\hat{k}_0)}{b_n(s)} = 0,$$

which implies that $b_n(1)$ is growing faster than $a_n(\hat{k}_0)$, and thus faster than $a_n(k_0)$ for all $0 \leq k_0 \leq n$. On the other hand, if $s < 1$ then $a_n(k_0)$ can not be bounded by a sequence of the form $b_n(s)$. Hence, the coefficients of c_F^1 for words of length n must be upper bounded by $KM^n n!$ for some $K, M > 0$, and no smaller Gevrey type bound applies. Therefore, the series c_F^1 has Gevrey order equal to $s = 1$. ■

An estimate of the Gevrey order of c_F^1 was also computed numerically via *Mathematica* as shown in Figure 3. A sample of the corresponding data is shown in Table I. The asymptotic behavior of the estimates of the Gevrey order of c_F^1 as a function of maximum word length is shown on a semi-logarithmic scale in Figure 4. The estimates are monotonic increasing towards $s = 1$ but at an extremely slow rate.

IV. Strong Versus Weak Local Convergence

In this section, the notions of strong and weak local convergence are described for generating series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. There is no loss of generality in assuming $\ell = 1$. Let $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ denote the closure of $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ in the semi-norm topology. The first two theorems describe some relationships between the spaces $S_{\infty,e}$, S_∞, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$, and $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$.

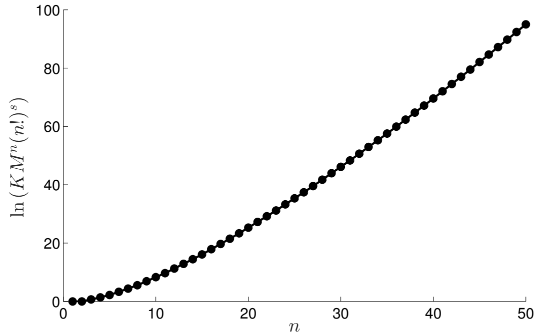

Fig. 3. Empirical fit of $KM^n(n!)^s$ (solid line) to the first 30 coefficients of the series c_F^1 (dots).

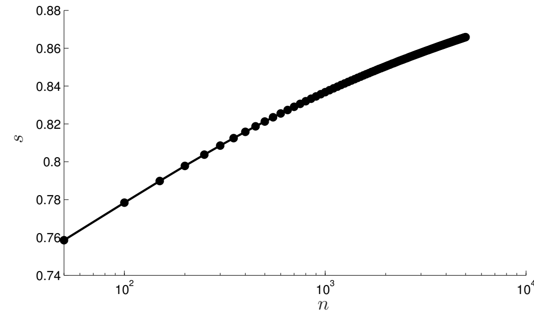

Fig. 4. Gevrey order estimates of c_F^1 as a function of maximum word length, n.

TABLE I
GROWTH PARAMETERS ESTIMATES FOR THE SERIES c_F^1.

maximum word length	K	M	s
50	1.49671	0.696282	0.758571
300	6.60041	0.579581	0.808544
500	19.496	0.545318	0.821234
5000	1.04761×10^9	0.414991	0.865870

Theorem 3: $\mathbb{R}_{LC}\langle\langle X\rangle\rangle = S_{\infty,e}$.

Proof: In Section II it was shown that $\mathbb{R}_{LC}\langle\langle X\rangle\rangle \subseteq S_{\infty,e}$. Thus, it only needs to be shown that $S_{\infty,e} \subseteq \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. The proof is by contradiction. If $c \in S_{\infty,e}$ then there exists $\bar{R}_c > 0$ such that $c \in S_\infty(R)$ for all $0 < R < \bar{R}_c$. Assume $c \notin \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. Then for any constants $K, M > 0$ there is a subseries $\hat{c}$ of c and some $\epsilon > 0$ such that

$$|(\hat{c},\eta)| > KM^{|\eta|}(|\eta|!)^{1+\epsilon}, \ \ \forall \eta \in \mathrm{supp}(\hat{c}).$$

On the other hand, for all $0 < R < \bar{R}_c$

$$\begin{aligned}\|c\|_{\infty,R} &= \sup_{\eta\in X^*}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\}\\ &\geq \sup_{\eta\in\mathrm{supp}(\hat{c})} K(MR)^{|\eta|}(|\eta|!)^{\epsilon}.\end{aligned}$$

Thus, $\|c\|_{\infty,R} = \infty$ for any $R > 0$. Which contradicts the fact $c \in S_\infty(R)$ for all $0 < R < \bar{R}_c$. Therefore, $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, and the theorem is proved. ■

Theorem 4: $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \subseteq S_\infty$.

Proof: If $c \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ then there exists a sequence $\{c_i\}_{i\geq 0}$ in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle \subseteq S_\infty \subset S_\infty(R)$ which converges to c in the semi-norm topology. Therefore, $\{c_i\}_{i\geq 0}$ also converges to c as a sequence in the complete normed linear space $S_\infty(R)$ for every $R > 0$. This implies that $c \in S_\infty(R)$ for every $R > 0$. Thus, $c \in S_\infty := \cap_{R>0} S_\infty(R)$. ■

The following definitions are essential.

Definition 1: A series $c \in \mathbb{R}_1\langle\langle X\rangle\rangle$ is said to be *strongly locally convergent* if $c \notin \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$.

Definition 2: A series $c \in \mathbb{R}_1\langle\langle X\rangle\rangle$ is said to be *weakly locally convergent* if $c \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \setminus \mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

The next theorem ensures that the set of weakly locally series is non-empty.

Theorem 5: The series c_F^1 is weakly locally convergent.

Proof: In light of Theorem 2, it only needs to be shown that $c_F^1 \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \setminus \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. Consider the truncation of c_F^1

$$d_N := \sum_{n=0}^{N} d_n = \sum_{n=0}^{N}\sum_{k_0=0}^{n}(c_F, x_0^{k_0}x_1^{n-k_0})\, x_0^{k_0}x_1^{n-k_0}.$$

Clearly, the polynomial d_N has Gevrey order equal to zero. Observe that for any $R > 0$

$$\|d_N\|_{\infty,R} = \sup_{\substack{n\leq N\\ 0\leq k_0\leq n}}\left\{k_0^{n-k_0}\,\frac{R^n}{n!}\right\} < \infty,$$

and

$$\|d_N - c_F^1\|_{\infty,R} = \sup_{\substack{n>N\\ 0\leq k_0\leq n}}\left\{k_0^{n-k_0}\,\frac{R^n}{n!}\right\}.$$

Since $k_0 = \hat{k}_0 := \exp(W(ne)-1)$ maximizes $k_0^{n-k_0}$ over $0 \leq k_0 \leq n$,

$$\|d_N - c_F^1\|_{\infty,R} \leq \sup_{n>N}\left\{\exp(W(ne)-1)^{n-\exp(W(ne)-1)}\,\frac{R^n}{n!}\right\}. \tag{6}$$

Now define

$$f(n) = \exp(W(ne)-1)^{n-\exp(W(ne)-1)}\,\frac{R^n}{n!}.$$

Applying the logarithm to both sides of this equation and using (5), it follows that

$$\begin{aligned}\ln(f(n)) &= (W(ne)-1)(n-\exp(W(ne)-1))\\ &\quad + n\ln(R) + n - \frac{1}{2}\ln(2\pi n) - n\ln(n).\end{aligned}$$

The identity $W(ne)\exp(W(ne)-1)) = n$ then yields

$$\begin{aligned}\ln(f(n)) &= nW(ne) + \frac{n}{W(ne)} + n(\ln(R)-1)\\ &\quad - \frac{1}{2}\ln(2\pi n) - n\ln(n).\end{aligned} \tag{7}$$

Observe that $f(n)$ has a maximum over $\mathbb{R}$ if and only if $n = \hat{n}$, where

$$\left.\frac{\mathrm{d}\ln(f(n))}{\mathrm{d}n}\right|_{n=\hat{n}} = W(\hat{n}e) - \frac{1}{2\hat{n}} - \ln(\hat{n}) + \ln(R) - 1 = 0$$

since

$$\left.\frac{\mathrm{d}^2 \ln(f(n))}{\mathrm{d}n^2}\right|_{n=\hat{n}} = \frac{W(\hat{n}e) - 2\hat{n} + 1}{2\hat{n}^2(W(\hat{n}e) + 1)} < 0.$$

Therefore,

$$\sup_{n>N} f(n) = \sup_{n>N} \left\{ \exp(W(ne) - 1)^{n - \exp(W(ne)-1)} \frac{R^n}{n!} \right\} = \max\{f(N), f(\hat{n})\}.$$

Substituting this bound into (6) and taking the limit gives

$$\lim_{N\to\infty} \|d_N - c_F^1\|_{\infty,R} \leq \lim_{N\to\infty} \max\{f(N), f(\hat{n})\} = \lim_{N\to\infty} f(N).$$

Now using (7) and the fact that $\lim_{N\to\infty} \ln f(N) = \ln \lim_{N\to\infty} f(N)$, it follows that

$$\lim_{N\to\infty} \ln(f(N)) = \lim_{N\to\infty} N(W(Ne) - \ln(N) + \ln(R) - 1)) + \frac{N}{W(Ne)} - \frac{1}{2}\ln(2\pi N) = -\infty.$$

The identity $\lim_{N\to\infty} W(Ne) - \ln(N) = -\infty$ has also been used above. Thus, for any $R > 0$

$$\lim_{N\to\infty} \|d_N - c_F^1\|_{\infty,R} \leq \lim_{N\to\infty} f(N) = 0.$$

Hence, the sequence $\{d_N\}_{N\geq 0}$ in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ converges to $c_F^1 \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ in the semi-norm topology, and, consequently, $c_F^1 \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \setminus \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. ■

The next two classical theorems from complex analysis are needed to prove the main results of this section, namely, the relationship between strongly and weakly locally convergent generating series and the convergence characteristics of their corresponding Fliess operators.

Theorem 6: [21] Consider a power series $f(z) = \sum_{n\geq 0} a_n z^n$ defined on $\mathbb{C}$. There exists a real number $0 \leq R \leq \infty$, called the radius of convergence of the series f, such that the series converges for all values of z with $|z| < R$ and diverges for all z such that $|z| > R$ with $R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$ ($1/0 := \infty$, $1/\infty := 0$).

Theorem 7: [21] Let $f(z) = \sum_{n\geq 0} a_n z^n/n!$ be a function which is analytic at $z = 0$. Suppose z_0 is a singularity of f having smallest modulus. Then for any $\epsilon > 0$ there exists an integer $N \geq 0$ such that for all $n > N$, $|a_n| < (1/|z_0| + \epsilon)^n n!$. Furthermore, for infinitely many n, $|a_n| > (1/|z_0| - \epsilon)^n n!$.

The next two theorems are the main results of the paper.

Theorem 8: If $c \in \mathbb{R}_1\langle\langle X\rangle\rangle$ is strongly locally convergent, then the radius of convergence of series (1) is finite.

Proof: Since c is locally convergent, there exists $R_u, T > 0$ such that for any $u \in B_1^m(R_u)[0,T]$ the series converges absolutely and uniformly on $[0,T]$. Define the truncation $c_N = \sum_{n=0}^N \sum_{\eta\in X^n} (c,\eta)\eta$. Clearly, $c_N \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, and thus, the series defining $F_{c_N}[u](t)$ converges absolutely and uniformly on $[0,T]$ for any $T > 0$ and $u \in L_{1,e}(0)$. Furthermore, observe that for any fixed $N > 0$, the radius of convergence of the series

$$F_c[u](t) = F_{c_N}[u](t) + F_{c-c_N}[u](t),$$

is finite if and only if

$$F_{c-c_N}[u](t) = \sum_{k=N+1}^{\infty} \sum_{\eta\in X^k} (c - c_N, \eta) E_\eta[u](t)$$

has a finite radius of convergence. The key observation is that the sequence $\{c_N\}_{N\geq 0}$ can not converge to c in the semi-norm topology, otherwise $c \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$, which contradicts the assumption that c is strongly locally convergent. Using this fact, a finite singularity of $F_{c-c_N}[u](t)$ can be constructed. This implies that $F_c[u](t)$ also has a finite singularity, and therefore, a finite radius of convergence. Following [5, Example 1], it is immediate that

$$|F_{c-c_N}[u](t)| \leq \sum_{n=N+1}^{\infty} \sum_{\eta\in X^n} |(c - c_N, \eta)| \frac{\hat{R}^n}{n!} \leq \sum_{n=0}^{\infty} a_n \frac{\hat{R}^n}{n!}, \tag{8}$$

where $\hat{R} := 2\max\{R_u, T\} > 0$ and

$$a_n := \begin{cases} \max_{\eta\in X^n} |(c - c_N, \eta)| & : \ n > N \\ 0 & : \ n \leq N. \end{cases}$$

Define

$$L = \lim_{N\to\infty} \|c_N - c\|_{\infty,R} = \lim_{N\to\infty} \sup_{\eta\in X^*} \left\{ |(c - c_N, \eta)| \frac{R^{|\eta|}}{|\eta|!} \right\}.$$

Note that $L > 0$ for some $R > 0$ since $\{c_N\}_{N\geq 0}$ does not converge to c in the semi-norm topology. In particular, choosing $R = \hat{R}$ gives

$$L = \lim_{n\to\infty} \sup_{n\geq 0} \left\{ |a_n| \frac{\hat{R}^n}{n!} \right\}.$$

The definition of the limit superior implies that for any $0 < \epsilon < 1$, there exists an integer $N \geq 0$ such that for all $n > N$, $|a_n| \hat{R}^n/n! < L + \epsilon$. Furthermore, for infinitely many n, $|a_n| \hat{R}^n/n! > L - \epsilon$ [21, p. 46]. From the first inequality

$$|a_n| < \frac{(L+\epsilon)n!}{\hat{R}^n} \leq \left(\frac{L^{1/n}}{\hat{R}} + \epsilon' \right)^n n!,$$

and for infinitely many n,

$$|a_n| > \frac{(L-\epsilon)n!}{\hat{R}^n} \geq \left(\frac{L^{1/n}}{\hat{R}} - \epsilon' \right)^n n!,$$

where $\epsilon' := \epsilon^{1/N}/\hat{R}$. Thus, from Theorems 6 and 7 it follows that

$$z_0 := \lim_{n\to\infty} \frac{\hat{R}}{L^{1/n}} = \frac{1}{\limsup_{n\to\infty} (|a_n|/n!)^{1/n}}.$$

Since $L > 0$, the real number $z_0 \neq 0$ is a finite singularity of $f(z) := \sum_{n=0}^{\infty} a_n z^n/n!$. In light of (8) then $F_{c-c_N}[u](t)$ must also have a finite singularity, and the theorem is proved. ■

Another characterization of a strongly locally convergent series is below.

Corollary 1: Let $c \in \mathbb{R}_1\langle\langle X\rangle\rangle$ be strongly locally convergent with growth constants $K, M > 0$. Then there exists a subseries $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ whose coefficients are each growing exactly at the rate $KM^{|\eta|}|\eta|!$.

Proof: Following the proof of Theorem 8, for any $\epsilon > 0$ and $L > 0$, there must exist an integer $N > 0$ such that

$$\left(\frac{L^{1/n}}{\hat{R}} - \epsilon' \right)^n n! < |a_n| < \left(\frac{L^{1/n}}{\hat{R}} + \epsilon' \right)^n n! \tag{9}$$

for all $n > N$. Let

$$a_n := \begin{cases} \max_{\eta\in X^n} |(c - c_N, \eta)| & : \ n > N \\ 0 & : \ n \leq N, \end{cases}$$

and for each $n > N$ define

$$\eta_n^* := \arg\max_{\nu \in X^n} |(c,\nu)|.$$

Construct $\hat{c} \in \mathbb{R}\langle\langle X \rangle\rangle$ such that for all $\eta \in X^n$, $n \geq 0$

$$(\hat{c},\eta) := \begin{cases} (c,\eta_n^*) & : \ \eta = \eta_n^*, \ n > N \\ 0 & : \ \text{otherwise.} \end{cases}$$

Clearly $\hat{c}$ is a subseries of c, and by design $|a_n| = |(\hat{c},\eta)|$ for all $\eta \in X^n$ since $\operatorname{supp}(\hat{c}) \subset X^* \setminus X^N$. Thus, a direct application of (9) gives for some $K > 0$

$$|(\hat{c},\eta)| = KM^{|\eta|}\,|\eta|!, \ \forall \eta \in \operatorname{supp}(\hat{c}),$$

where

$$M := \lim_{n\to\infty} \frac{L^{1/n}}{\hat{R}} = \lim_{N\to\infty} \left(\sup_{\eta \in X^*} \left\{ |(c - c_N,\eta)| \frac{1}{|\eta|!} \right\} \right)^{1/N}.$$

■

Theorem 9: If $c \in \mathbb{R}_1\langle\langle X \rangle\rangle$ is weakly locally convergent, then the radius of convergence of series (1) is infinite.

Two lemmas are needed to prove the Theorem 9. The first lemma is a generalization of Example 2.

Lemma 1: Let $c \in S_{\infty,e}$ and define $c_N = \sum_{n=0}^{N}\sum_{\eta\in X^n} (c,\eta)\eta$, $N \geq 0$. Then there exists an $R > 0$ such that $c_N \to c$ as a sequence in the normed linear space $S_\infty(R)$.

Proof: If $c \in S_{\infty,e} = \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ then $|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^s$, $\forall \eta \in X^*$ for some $K, M > 0$ and $0 \leq s \leq 1$. Therefore,

$$\|c_N - c\|_{\infty,R} \leq \sup_{n>N} \ K\frac{(MR)^n}{(n!)^{1-s}}.$$

When $0 \leq s < 1$ it follows that

$$\begin{aligned} \lim_{N\to\infty} \|c_N - c\|_{\infty,R} &\leq \lim_{N\to\infty} \sup_{n>N} \ K\frac{(MR)^n}{(n!)^{1-s}} \\ &= \lim_{N\to\infty} K\frac{(MR)^{N+1}}{((N+1)!)^{1-s}} = 0 \quad (10) \end{aligned}$$

for all $R > 0$. On the other hand, when $s = 1$

$$\lim_{N\to\infty} \|c_N - c\|_{\infty,R} \leq \lim_{N\to\infty} K(MR)^{N+1} = 0,$$

when $R < 1/M$ and infinity otherwise. This implies in both cases that there exists an $R > 0$ such that $c_N \to c$ as a sequence in the normed linear space $S_\infty(R)$. ■

Corollary 2: If $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ then $c_N \to c$ in the semi-norm topology.

Proof: The claim follows directly from (10). ■

Lemma 2: Let $c \in \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$ and define $c_N = \sum_{n=0}^{N}\sum_{\eta\in X^n} (c,\eta)\eta$, $N \geq 0$. Then $c_N \to c$ in the semi-norm topology.

Proof: In light of Corollary 2, the claim only needs to be shown for weakly locally convergence $c \in \mathbb{R}_1\langle\langle X \rangle\rangle$. The proof is by contradiction. If $c \in \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} \subseteq S_\infty$ then

$$\|c\|_{\infty,R} < \infty, \ \forall R > 0. \quad (11)$$

Now suppose $\{c_N\}_{N\geq 0}$ does not converges to c in the semi-norm topology. In which case,

$$L = \lim_{N\to\infty} \|c_N - c\|_{\infty,R} > 0 \quad (12)$$

for some $R > 0$. Note that the proof of Theorem 8 uses only the fact that (12) holds since $\{c_N\}_{N\geq 0}$ does not converge to c in the semi-norm topology. Therefore, from Corollary 1, if $c \in \mathbb{R}_1\langle\langle X \rangle\rangle$ then there exists a subseries $\hat{c} \in \mathbb{R}_1\langle\langle X \rangle\rangle$ of c whose coefficients are each growing exactly at the rate $KM^{|\eta|}\,|\eta|!$ for some $K, M > 0$. This implies that $\|c\|_{\infty,R} = \infty$ when $R > 1/M$. This fact contradicts (11), which completes the proof. ■

Proof of Theorem 9: Following the same approach as in the proof of Theorem 8, one is led to the conclusion in this case that for any $R > 0$

$$L = \lim_{N\to\infty} \|c_N - c\|_{\infty,R} = 0$$

precisely because the sequence $\{c_N\}_{N\geq 0}$ converges to c in the semi-norm topology via Lemma 2. Applying Theorems 6 and 7 as before now gives

$$z_0 := \lim_{n\to\infty} \frac{\hat{R}}{L^{1/n}} = \frac{1}{\limsup_{n\to\infty} \left(|a_n|/n!\right)^{1/n}} = \infty.$$

Thus, f can not have a finite singularity, implying that $F_c[u](t)$ has a infinite radius of convergence. ■

Corollary 3: The series c_F is weakly locally convergent.

Proof: Using Theorem 2, it follows that $c_F \in \mathbb{R}_1\langle\langle X \rangle\rangle$. Applying Theorem 9 in [20] ensures that $F_{c_F}[u](t)$ is well defined on $[0,T]$ for any $T > 0$ and $u \in L_{1,e}(0)$ because $c_F = x_1^* \circ x_1^*$, and x_1^* has Gevrey order $s = 0$. In which case, $F_{c_F}[u](t)$ can not have a finite singularity. Hence, c_F is weakly locally convergent. ■

The next theorem gives the relationship between the space S_∞ and the set $\overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$.

Theorem 10: $S_\infty = \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$.

Proof: From Theorem 4 it is known that $\overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} \subseteq S_\infty$. Thus, it only needs to be shown that $S_\infty \subseteq \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$. The proof is by contradiction. Suppose $c \in S_\infty$ with $c \notin \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$. Then c is strongly locally convergent, and by Corollary 1 there exists a subseries $\hat{c}$ of c such that

$$|(\hat{c},\eta)| = KM^{|\eta|}\,|\eta|!, \ \ \forall \eta \in \operatorname{supp}(\hat{c}).$$

Then for $R > 1/M$

$$\|\hat{c}\|_{\infty,R} = \sup_{\eta\in\operatorname{supp}(\hat{c})} K(MR)^{|\eta|} = \infty.$$

Therefore, $\|c\|_{\infty,R} = \infty$ when $R > 1/M$ since $\|\hat{c}\|_{\infty,R} \leq \|c\|_{\infty,R}$. This is a contradiction since $c \in S_\infty$. ■

Figure 5 summarizes the relationship between $S_{\infty,e}$, S_∞, $\mathbb{R}_1\langle\langle X \rangle\rangle$ and various notions of convergence.

In light of Theorem 9, it now makes more sense to call $S_\infty = \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} \supset \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ the set of *globally convergent* generating series. This means, of course, that the Gevrey order is no longer the sole indicator of whether a given generating series renders a globally convergent Fliess operator.

Example 3: Consider the locally convergent series $c = \sum_{k=0}^{\infty} k!\, x_1^k$. Observe that

$$\|c\|_{\infty,R} = \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} = \sup_{n\geq 0} R^n.$$

Clearly, $\|c\|_{\infty,R} < \infty$ if and only if $R < 1$. This indicates that the radius of convergence of $F_c[u](t)$ is unity. To confirm this, apply the identity $k!\, x_1^k = x_1^{\sqcup\!\sqcup\, k}$ so that

$$F_c[u](t) = \sum_{k=0}^{\infty} k! E_{x_1^k}[u](t) = \sum_{k=0}^{\infty} E_{x_1}^k[u](t) = \frac{1}{1 - E_{x_1}[u](t)}.$$

Setting $u = 1$ gives $F_c[1](t) = 1/(1-t)$, which has a finite

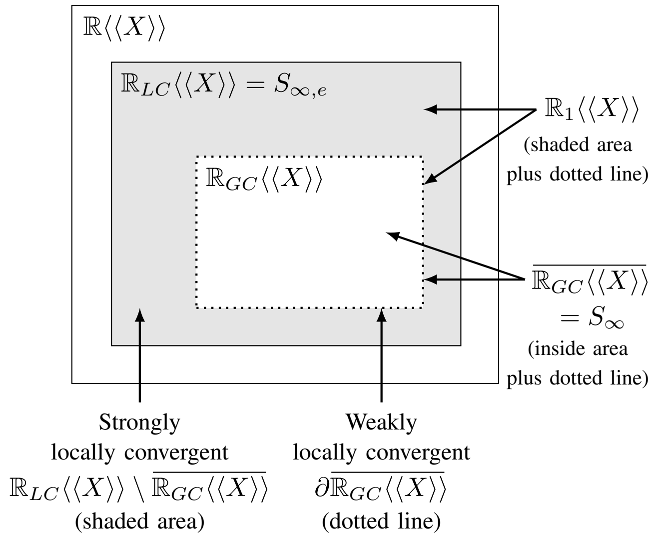

Fig. 5. Relationship between $S_{\infty,e}$, S_∞, Gevrey order and various notions of convergence.

escape time at $t = 1$ as expected. In which case, c is *strongly* locally convergent. □

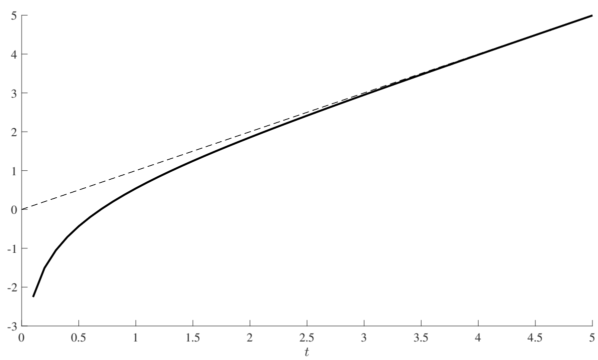

Fig. 6. Zero-input response of the cascade system $c_F = x_1^* \circ x_1^*$ on a double logarithmic scale (solid line) and the function $\hat{y}(t) = t$ (dotted line).

Example 4: Reconsider the series $c = x_1^*$. It is easy to verify that $y = F_c[u]$ has a state space realization $\dot{z} = zu$, $y = z$, $z(0) = 1$ [4]. Cascading two such realizations and simulating in MatLab gives the natural response shown on a double logarithmic scale in Figure 6. From this one can conclude that $F_{c_F}[0]$ is bounded by a double exponential function, which is entire. The response is similar for different inputs, and, in fact, it is shown in [20, Theorem 8] that the cascade of any two systems having generating series with Gevrey order $s = 0$ always has a double exponential bounding function. This further confirms that $c_F = x_1^* \circ x_1^*$ is weakly locally convergent. □

Example 5: Reconsider the series c_F^1 and the truncated version d_N as defined in the proof of Theorem 5. From (8) it follows that

$$\left|F_{c_F^1 - d_N}[u](t)\right| \leq \sum_{n>N} \sum_{k_0=0}^{n} k_0^{n-k_0} \frac{R^n}{n!}.$$

Therefore, $F_{c_F^1 - d_N}[u](t)$ converges for all $R, T > 0$ using the ratio test on the upper bound above. In addition, $F_{c_F^1}[u](t) = F_{d_N}[u](t) + F_{c_F^1 - d_N}[u](t)$ is also bounded, and thus, c_F^1 must be weakly locally convergent. □

V. Conclusions

It was shown that the set of generating series with Gevrey order $s = 1$ can exhibit two fundamentally different types of convergence behavior with respect to its corresponding Fliess operator, either strong or weak local convergence. The former has a finite radius of convergence due to the presence of a singularity in the operator summation, while the latter does not. The Ferfera series was presented as a specific example of a weakly locally convergent series. The main consequence of these results is that the set of generating series known to render global convergence has been expanded, and now the Gevrey order of a generating series is no longer the sole indicator of this behavior.

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