PHYSICAL REALIZABILITY OF AN OPEN SPIN SYSTEM*

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Abstract. Coherent feedback control considers purely quantum controllers in order to overcome disadvantages such as the acquisition of suitable quantum information, quantum error correction, etc. These approaches lack a systematic characterization of quantum realizability. Recently, a condition characterizing when a system described as a linear stochastic differential equation is quantum was developed. Such condition was named physical realizability, and it was developed for linear quantum systems satisfying the quantum harmonic oscillator canonical commutation relations. In this context, open spin systems escape the realm of the current known condition. The goal of this paper is to provide a characterization for physical realizability of a single particle open spin system.

Key words. quantum systems, non-commutative stochastic processes, evolutions on $SU(2)$

AMS subject classifications. 81R25, 81R50, 93E03, 93E99

1. Introduction.

In the last twenty years, the use of quantum feedback control systems have became critical for the development of quantum and nano technologies [1, 4, 5]. However, the majority of approaches consider a classical controller in the feedback loop. In this context, coherent feedback control considers purely quantum controllers in order to overcome disadvantages such as the acquisition of suitable quantum information, quantum error correction, etc [3, 8, 12]. Unfortunately, these approaches lack a systematic characterization of quantum realizability. In [7], a condition characterizing when a system described as a linear stochastic differential equation is quantum was developed. Such condition was named *physical realizability*, and it was developed specifically for linear systems satisfying the quantum harmonic oscillator canonical commutation relations. The class of systems for which this condition is known to be satisfied is still too limited for applications. For example, spin systems escape the realm of the current known condition. In this paper, the focus is on a system describing the spin dynamics of a single particle. Compared to a linear quantum system, the problem is more complicated and requires extra machinery for two basic reasons. The first is that the system being analyzed is bilinear, and the second is that the commutation relations that the system has to obey are now dependent on the system variables, which was not the case for linear quantum systems related to the quantum harmonic oscillator [7, 10]. Thus, the goal of the paper is to provide a characterization for physical realizability of a bilinear quantum stochastic differential equation (QSDE) representing a single particle spin system.

The paper is organized as follows. Section 2 presents the basic preliminaries on open quantum spin systems. In Section 3, the necessary algebraic machinery to study a single particle open quantum spin systems is given. This is followed by Section 4, in which the definition of physical realizability is provided as well as a condition for

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a bilinear QSDE to be physically realizable. Finally, Section 5 gives the conclusions.

2. Open Quantum Spin Systems. Systems governed by the laws of quantum mechanics that interact with an external environment (e.g., electromagnetic field) are known as *open quantum systems*. In order to study such systems, one has to give a quantum description of both the system and the interacting environment. The quantum mechanical behavior of the system is based on the notions of *observables* and *states*. Observables represent physical quantities that can be measured, as self-adjoint operators on a complex separable Hilbert space $\mathfrak{H}$, while states give the current status of the system, as elements of $\mathfrak{H}$, allowing the computation of expected values of observables. Here open quantum systems are treated in the context of quantum stochastic processes (see [2, 11] for more information). For this purpose, observables may be thought as quantum random variables that do not in general commute. A measure of the non commutativity between observables is usually given by the *commutator* between operators. The commutator of two scalar operators x and y in a $\mathfrak{H}$ is an antisymmetric bilinear operation defined as $[x, y] = xy - yx$. Also, for an n-dimensional vector of operators x in $\mathfrak{H}$, the commutator of x and a scalar operator y in $\mathfrak{H}$ is the n-dimensional vector of operators $[x, y] = xy - yx$, and the commutator of x and its adjoint $x^\dagger$ is the $n \times n$ matrix of operators

$$[x, x^\dagger] \triangleq xx^\dagger - (x^\# x^T)^T,$$

where $x^\# \triangleq (x_1^* \; x_2^* \; \cdots \; x_n^*)^T$ and * denotes the operator adjoint. In the case of complex vectors (matrices) * denotes the complex conjugate while † denotes the conjugate transpose. The non-commutativity of observables is a fundamental difference between quantum systems and classical systems in which the former must satisfy certain commutation relations originating from *Heisenberg uncertainty principle*. The environment consists of a collection of oscillator systems each with annihilation field operator $w(t)$ and creation field operator $w^*(t)$ used for the annihilation and creation of quanta at point t, and commonly known as the *boson quantum field* (with parameter t). Here it is assumed that t is a real time parameter. The field operators $w(t)$ and $w^*(t)$ satisfy commutation relations as well. That is,

$$[w(t), w^*(t')] = \delta(t - t'),$$

for all $t, t' \in \mathbb{R}$, where $\delta(t)$ denotes the Dirac delta. Its mathematical description is given in terms of a Hilbert space called a *Fock space*. When the boson quantum field is in the vacuum state, i.e., no physical particles are present, it then represents a natural quantum extension of white noise, and may be described using the quantum Itô calculus [2, 11]. This amounts to have three interacting signals (inputs) in the evolution of the system: the annihilation processes $W(t)$, the creation process $W^*(t)$, and the counting process $\Lambda(t)$. The evolution of an open quantum system (i.e., the system together with the environment) is unitary. That is, if ψ is an state then $\psi(t) = U(t)\psi$, where $U(t)$ is unitary for all t, and is the solution of

$$dU(t) = \left((S - I)\, d\Lambda(t) + L\, dW^*(t) - L^* S\, dW(t) - \frac{1}{2}(L^* L + \boldsymbol{i}\mathcal{H})\, dt \right) U(t),$$

with initial condition $U(0) = I$, I denoting the identity operator and $\boldsymbol{i}$ being the imaginary unit. Here, $\mathcal{H}$ is a fixed self-adjoint operator representing the *Hamiltonian* of the system, and L and S are operators determining the *coupling* of the system to the field, with S unitary. The evolution of ψ is equivalent to the evolution of the observable X given by

$$X(t) = U^*(t)(X \otimes I)\,U(t),$$

whose evolution is referred as the *Heisenberg picture* while the one for ψ is known as the *Schrödinger picture*. This paper exclusively takes the point of view of the Heisenberg picture. Quantum stochastic calculus allows then to express the Heisenberg picture evolution of X as

$$dX = (S^*XS - X)\,d\Lambda + \mathcal{L}(X)\,dt + S^*[X, L]\,dW^* + [L^*, X]S\,dW, \tag{2.1}$$

where $\mathcal{L}(X)$ is the Lindblad operator defined as

$$\mathcal{L}(X) = -\boldsymbol{i}[X, \mathcal{H}] + \frac{1}{2}\left(L^*[X, L] + [L^*, X]L\right). \tag{2.2}$$

The output field is given by $Y(t) = U(t)^*W(t)U(t)$, which amount to the QSDE $dY = Ldt + SdW$. In summary, one can say from the discussion above that the dynamics of an open quantum systems is uniquely determined by the triple of operators $(\mathcal{H}, L, S)$. Hereafter, the operator S is assumed to be the identity operator ($S = I$).

The main focus of this paper is on the dynamics of a single particle open spin system interacting with one boson quantum field. The vector of system variables is

$$x = (x_1, x_2, x_3)^T \triangleq (\sigma_1, \sigma_2, \sigma_3),$$

where σ_1, σ_2 and σ_3 are spin operators. The nature of the spin operators makes x a self-adjoint vector of operators, i.e., $x = x^\#$. In particular $x(0)$ is represented by the well-known Pauli matrices, i.e.,

$$\sigma_1(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2(0) = \begin{pmatrix} 0 & -\boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad \sigma_3(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The product of spin operators satisfy

$$\sigma_i\sigma_j = \delta_{ij} + \boldsymbol{i}\sum_k \epsilon_{ijk}\sigma_k.$$

It is then clear that the commutation relations for Pauli matrices are

$$[\sigma_i, \sigma_j] = 2\boldsymbol{i}\sum_k \epsilon_{ijk}\sigma_k, \tag{2.3}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} denotes the Levi-Civita tensor. For simplicity, the Hamiltonian and coupling operator for this system are chosen to be linear on x, i.e.,

$$\mathcal{H} = \alpha x \;\text{ and }\; L = \Lambda x,$$

where $\alpha^T \in \mathbb{R}^3$ and $\Lambda^T \in \mathbb{C}^3$. Note that $L^\dagger = L^*$. As mentioned before, the coupling operator specifies how the interacting field acts on x. In general, the dimensionality of the coupling matrix Λ depends proportionally on the number of interacting fields.

Observe that, in general, the evolution of x (*standard form*) falls into a class of bilinear QSDEs expressed as

$$dx = F_0\,dt + Fx\,dt + G_1x\,dW^* + G_2x\,dW, \tag{2.4}$$

where $F_0 \in \mathbb{C}^3$, $F \in \mathbb{R}^{3\times 3}$ and $G_1, G_2 \in \mathbb{C}^{3\times 3}$. The output field is

$$dY = Hx\,dt + dW \tag{2.5}$$

with $H \in \mathbb{C}^3$. Thus, a more specific formulation of the paper's goal is that, given a bilinear QSDE as in (2.4), under what conditions there exist operators $\mathcal{H}$ and L such that (2.4) can be written as in (2.1). Such condition is given later in Section 4.

3. Notation and Algebraic Relations. In order to continue the description of the spin quantum system, some linear algebra is needed. Let $\beta \in \mathbb{C}^3$ and define the linear mapping $\Theta : \mathbb{C}^3 \to \mathbb{C}^{3\times 3}$ as

$$\Theta(\beta) = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}. \tag{3.1}$$

Note here that this definition allows β to be either a column or a row vector. The fact that β is either a column or a row vector will be clear from the context. It will also be convenient to rewrite $\Theta(\beta)$ in terms of its columns. That is,

$$\Theta(\beta) = (\Theta_1(\beta), \Theta_2(\beta), \Theta_3(\beta)) = -\begin{pmatrix} \Theta_1(\beta)^T \\ \Theta_2(\beta)^T \\ \Theta_3(\beta)^T \end{pmatrix}. \tag{3.2}$$

The product of Pauli operators can be expressed in a compact matrix form thanks to the mapping Θ. That is,

$$xx^T = I + \boldsymbol{i}\Theta(x).$$

Similarly, the commutation relations for Pauli operators are written as

$$[x, x^T] = 2\boldsymbol{i}\Theta(x).$$

Consider now the *stacking operator* vec : $\mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ whose action on a matrix creates a column vector by stacking its columns below one another. With the help of vec, the matrix $\Theta(\beta)$ can be reorganized so that it gives

$$\text{vec}(\Theta(\beta)) = \begin{pmatrix} \Theta_1(\beta) \\ \Theta_2(\beta) \\ \Theta_3(\beta) \end{pmatrix} = E\beta,$$

where β is a column vector, $\Theta_i(\beta) = \bar{e}_i^T \beta$, $E = (\bar{e}_1, \bar{e}_2, \bar{e}_3)^T$, and

$$\bar{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{e}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $\{-\boldsymbol{i}\bar{e}_1, -\boldsymbol{i}\bar{e}_2, -\boldsymbol{i}\bar{e}_3\}$ can be identified to be the 3-dimensional adjoint representation of $SU(2)$, which has as generators the Pauli matrices. The matrix E satisfies

$$E^T E = 2I. \tag{3.3}$$

If in addition one defines the block matrix $\mathbb{1}_E = \{\mathbb{1}_{ij}\}_{i,j=1}^3 \in \mathbb{R}^{9\times 9}$ whose elements are matrices $\mathbb{1}_{ij} \in \mathbb{R}^{3\times 3}$ consisting of 1 in the (j,i) position and 0 everywhere else, then E also satisfies

$$EE^T = I - \mathbb{1}_E \text{ and } \mathbb{1}_E E = -E.$$

The matrix $\mathbb{1}_E$ can be identified as a tensor permutation matrix, which comes from the fact that the Levi-Civita tensor satisfies the contraction epsilon identity

$$\sum_k \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

The properties of $\Theta(\beta)$ are summarized in the next lemma.

Lemma 3.1. *Let $\beta, \gamma \in \mathbb{C}^3$ be column vectors. The mapping Θ satisfies*

i. $\Theta(\beta)\gamma = -\Theta(\gamma)\beta$,
ii. $\Theta(\beta)\beta = 0$,
iii. $\bar{e}_i\Theta(\beta) = \beta e_i^T - \beta_i I$,
iv. $\Theta(\beta)\Theta(\gamma) = \gamma\beta^T - \beta^T\gamma I$,
v. $\Theta\left(\Theta(\beta)\gamma\right) = \Theta(\beta)\Theta(\gamma) - \Theta(\gamma)\Theta(\beta)$,

where I denotes the identity matrix, and e_i is an element of the canonical basis of $\mathbb{R}^3$ with i indicating the position of the nonzero element, i.e., a unitary vector.

The explicit computation of the vector fields in (2.1) is given by the next lemma.

Lemma 3.2. *The component coefficients of equations (2.1) and (2.2) are*

$$[x,\mathcal{H}] = -2\boldsymbol{i}\Theta(\alpha)x, \tag{3.4a}$$
$$[x,L] = -2\boldsymbol{i}\Theta(\Lambda)x, \tag{3.4b}$$
$$[x,L^*] = -2\boldsymbol{i}\Theta(\Lambda^\#)x, \tag{3.4c}$$
$$L^*[x,L] = -2\boldsymbol{i}\Theta(\Lambda)\Lambda^\dagger - 2(\Lambda\Lambda^\dagger I - \Lambda^\dagger\Lambda)x, \tag{3.4d}$$
$$[x,L^*]L = 2\boldsymbol{i}\Theta(\Lambda)\Lambda^\dagger + 2(\Lambda\Lambda^\dagger I - \Lambda^T\Lambda^\#)x. \tag{3.4e}$$

Proof: For (3.4a), one has by the definition of the commutator that

$$\begin{aligned}[x,\mathcal{H}] &= [x,\alpha x] \\ &= x\,(\alpha x) - ((\alpha x)\,x^T)^T \\ &= (xx^T)\,\alpha^T - (xx^T)^T\alpha^T \\ &= 2i\Theta(x)\alpha^T.\end{aligned}$$

Given that the components of α and x commute, the commutator $[x,\mathcal{H}]$ is rewritten in standard form by applying property (i) of Lemma 3.1. Thus,

$$[x,\mathcal{H}] = -2i\Theta(\alpha)x.$$

The procedure to compute (3.4b) and (3.4c) is identical to the one above. Hence,

$$[x,L] = -2i\Theta(\Lambda)\,x \text{ and } [x,L^*] = -2i\Theta(\Lambda^\#)\,x.$$

The computation of (3.4d) is done by directly multiplying the scalar operator L^* and the vector operator $[x,L]$. Then,

$$L^*[x,L] = -2ix^T\Lambda^\dagger\Theta(\Lambda)\,x$$

$$
\begin{aligned}
&= -2i\Theta(\Lambda)\begin{pmatrix}\Lambda^{\#}xx_1\\ \Lambda^{\#}xx_2\\ \Lambda^{\#}xx_3\end{pmatrix}\\
&= -2i\Theta(\Lambda)\,(xx^T)^T\Lambda^\dagger\\
&= -2i\Theta(\Lambda)\,(I - i\Theta(x))\Lambda^\dagger\\
&= -2i\Theta(\Lambda)\Lambda^\dagger - 2\Theta(\Lambda)\Theta(x)\Lambda^\dagger.
\end{aligned}
$$

Therefore, $L^*[x,L] = -2i\Theta(\Lambda)\Lambda^\dagger - 2(\Lambda\Lambda^\dagger I - \Lambda^\dagger\Lambda)\,x$. Finally, (3.4e) is computed similarly. That is,

$$
\begin{aligned}
[x,L^*]L &= -2i\Theta(\Lambda^{\#})x\Lambda x\\
&= -2i\Theta(\Lambda^{\#})xx^T\Lambda^T\\
&= -2i\Theta(\Lambda^{\#})(I + i\Theta(x))\Lambda^T\\
&= -2i\Theta(\Lambda^{\#})\Lambda^T + 2\Theta(\Lambda^{\#})\Theta(x)\Lambda^T\\
&= 2i\Theta(\Lambda)\Lambda^\dagger + 2(\Lambda^{\#}\Lambda^T I - \Lambda^T\Lambda^{\#})x\\
&= 2i\Theta(\Lambda)\Lambda^\dagger + 2(\Lambda\Lambda^\dagger I - \Lambda^T\Lambda^{\#})x.
\end{aligned}
$$

■

From (3.4a)-(3.4e), one can now write equation (2.1) as the following bilinear QSDE

$$
\begin{aligned}
dx = &-2\boldsymbol{i}\Theta(\Lambda)\Lambda^\dagger\,dt - 2\Theta(\alpha)x\,dt + \left(-2\Lambda\Lambda^\dagger I + \Lambda^\dagger\Lambda + \Lambda^T\Lambda^{\#}\right)x\,dt\\
&-2\boldsymbol{i}\Theta(\Lambda)x\,dW^* + 2\boldsymbol{i}\Theta(\Lambda^{\#})x\,dW.
\end{aligned}
\tag{3.5}
$$

As mentioned in Section 2, the output field Y depends linearly on the coupling operator and the input field W, i.e.,

$$dY = L\,dt + dW.$$

4. Physical Realizability. In an environment where the classical laws of physics apply, standard control techniques such as optimization or a Lyapunov procedures do not worry in general of the nature of the controller they synthesized. In other words, their implementation is always possible since the physics behind them still hold. However, if one desires to implement a controller that obeys the laws imposed by quantum mechanics (quantum coherent control), then such a task is not so easily achieved unless an explicit characterization of that laws is given in terms of the control system vector fields. This is exactly the purpose for introducing the concept of a *physically realizable* system in the next definition.

DEFINITION 4.1. *System (2.4) is said to be physically realizable if there exist* $\mathcal{H} = \alpha x$, *with* $\alpha^T \in \mathbb{R}^3$, *and* $L = \Lambda x$, *with* $\Lambda^T \in \mathbb{C}^3$ *such that*

$$F_0 = -2\boldsymbol{i}\Theta(\Lambda)\Lambda^\dagger, \tag{4.1a}$$

$$F = -2\Theta(\alpha) - 2\Lambda\Lambda^\dagger I + \Lambda^\dagger\Lambda + \Lambda^T\Lambda^{\#}, \tag{4.1b}$$

$$G_1 = -2\boldsymbol{i}\Theta(\Lambda), \tag{4.1c}$$

$$G_2 = 2\boldsymbol{i}\Theta(\Lambda^{\#}), \tag{4.1d}$$

$$H = \Lambda. \tag{4.1e}$$

Note by direct inspection that for a physically realizable system $G_1 = -G_2^\dagger$.

From a control perspective, it is necessary to characterize when a bilinear QSDE posses underlying Hamiltonian and coupling operators which allows to express the matrices comprising (2.4) as in Definition 4.1. Thus, the main result of the paper is given in the next theorem, which establishes necessary and sufficient conditions for the physical realizability of a bilinear QSDE.

THEOREM 4.2. *System (2.4) with* $G \triangleq G_1 = -G_2^\dagger$ *is physically realizable if and only if*

i. $F_0 = GH^\dagger$,

ii. $G = -2\boldsymbol{i}\Theta(H)$,

iii. $F + F^T + \frac{1}{2}\left(GG^\dagger + G^\dagger G\right) = 0$.

In which case, one can identify the matrix α *defining the system Hamiltonian as*

$$\alpha = \frac{1}{8}\mathrm{vec}(F^T - F)^T E,$$

and the corresponding coupling matrix can be identified to be $\Lambda = H$.

Proof: Assuming that (2.4) and (2.5) are physically realizable implies that (4.1a)-(4.1e) are satisfied. By comparison, conditions (*i*)-(*iii*) hold. By property (iv) of Lemma 3.1, it follows that

$$GG^\dagger = -4\Theta(\Lambda)\Theta(\Lambda^\#) = 4\left(\Lambda\Lambda^\dagger I - \Lambda^\dagger\Lambda\right).$$

Similarly,

$$G^\dagger G = -4\Theta(\Lambda^\#)\Theta(\Lambda) = 4\left(\Lambda^\#\Lambda^T I - \Lambda^T\Lambda^\#\right).$$

Thus,

$$GG^\dagger + G^\dagger G = 4(2\Lambda\Lambda^\dagger I - \Lambda^\dagger\Lambda - \Lambda^T\Lambda^\#).$$

One can now rewrite F in terms of α, G_1 and G_2 as

$$F = -2\Theta(\alpha) - \frac{1}{4}(GG^\dagger + G^\dagger G). \tag{4.2}$$

Similarly, $F^T = 2\Theta(\alpha) - \frac{1}{4}(GG^\dagger + G^\dagger G)$. Hence, $F + F^T + \frac{1}{2}(GG^\dagger + G^\dagger G) = 0$. Conversely, one needs to show that if conditions (*i*), (*ii*) and (*iii*) of Theorem 4.2 are satisfied, then there exist matrices α and Λ such that system (2.4) is physically realizable. Let

$$\Theta(\alpha) = \frac{1}{4}(F^T - F). \tag{4.3}$$

It is trivial to check that the right-hand-side of (4.3) is antisymmetric with zero diagonal and hence this equation uniquely defines α via (3.1). Also, let $\Lambda = H$. It follows directly that $G = -2\boldsymbol{i}\Theta(\Lambda)$, so that

$$GG^\dagger = 4(\Lambda\Lambda^\dagger I - \Lambda^\dagger\Lambda).$$

From condition (*iii*), one obtains

$$F = -F^T - \frac{1}{2}(GG^\dagger + G^\dagger G).$$

Then,

$$\Theta(\alpha) = \frac{1}{4}\left(2F^T - \frac{1}{2}(GG^\dagger + G^\dagger G)\right).$$

A little algebra shows, as required, that

$$F = -2\Theta(\alpha) - 2\Lambda\Lambda^\dagger I + \Lambda^\dagger\Lambda + \Lambda^T\Lambda^\#.$$

Moreover, from (3.2), (3.3), (4.3) and by applying the stacking operator to $\Theta(\alpha)$, α is explicitly obtained as $\mathrm{vec}(\Theta(\alpha)) = E\alpha^T = \frac{1}{4}\mathrm{vec}\left(F^T - F\right)$. Multiplying both sides by E^T leaves

$$\alpha = \frac{1}{8}\mathrm{vec}\left(F^T - F\right)^T E,$$

which completes the proof. ∎

5. Conclusions. A condition for physical realizability was obtained for a single particle quantum spin system. Under this conditions it was shown that there exist operators $\mathcal{H}$ and L such that QSDE (2.4) can be written as (2.1). Future work includes showing that the conditions for physical realizability imply the preservation of the Pauli commutation relations, and extending the formalism for the case of multilevel spin systems, i.e., for systems evolving in $SU(N)$ for arbitrary $N \in \mathbb{N}$.

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