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Functional Series Expansions for Continuous-Time Switched Systems

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Abstract The main objective of this paper is to describe a class of functional series expansions, known as Fliess operators, which admit inputs from a ball in an L_p space as well as Poisson random processes. Conditions are given under which these functional series expansions converge absolutely in the mean. Then, it is shown that a continuous-time switched input-affine nonlinear system with a Poisson switching signal can be represented as a Fliess operator and that for certain cases a closed form solution can be obtained in terms of Poisson integrals.

Keywords Formal power series · Chen-Fliess series · Switched nonlinear systems · Poisson processes

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1 Introduction

Fliess operators provide a general framework under which analytic nonlinear input-output systems can be studied [6, 7, 9, 10, 19]. In the classical setting, they are described by an

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infinite summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. A series whose support is finite is called a polynomial, and the set of all polynomial is denoted by $\mathbb{R}^\ell\langle X\rangle$. Let $\mathcal{V}^m[a,b]$ denote the set of m-dimensional measurable functions on $[a,b]$. For $u \in \mathcal{V}^m[a,b]$, define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \le i \le m\}$, where $\|u_i\|_{L_p} = \left(\int_a^b |u_i(s)|^p\, ds\right)^{1/p}$ is the usual L_p-norm. Let $L_p^m[a,b] \triangleq \{u \in \mathcal{V}^m[a,b], \|u\|_{L_p} < \infty\}$ and define iteratively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0+T] \to \mathcal{C}[t_0, t_0+T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t,t_0) = \int_{t_0}^t u_i(\tau) E_{\eta'}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\eta' \in X^*$, and $u_0 = 1$. For convenience, assume $t_0 = 0$ and let $E_\eta[u](t,0) = E_\eta[u](t)$. The input-output operator corresponding to c is then

$$F_c[u](t) \triangleq \sum_{\eta \in X^*} (c,\eta)\, E_\eta[u](t),$$

which is called a *Fliess operator*. The most general results regarding the convergence of Fliess operators were presented in [10]. There it was shown that if the generating series c is *globally convergent*, i.e., satisfies the growth condition,

$$|(c,\eta)| \le K M^{|\eta|},\ \ \forall \eta \in X^*,$$

where $|z| \triangleq \max_{1\le i\le \ell}\{|z_i|\}$ when $z \in \mathbb{R}^\ell$, $|\eta|$ denotes the number of symbols in η and $K, M > 0$, then $F_c[u]$ converges absolutely on the L_p-extended space $L_{p,e}^m(0) = \{u : u \in L_p^m[0,t],\ 0 < t < \infty\}$. On the other hand, if the generating series c is *locally convergent*, that is,

$$|(c,\eta)| \le\ K M^{|\eta|}|\eta|!, \forall \eta \in X^*, \tag{1}$$

then $F_c[u]$ converges absolutely on $[0,T]$ for

$$u \in B_p^m(R)[0,T] \triangleq \{u \in L_p^m[0,T] : \|u\|_{L_p} \le R\} \tag{2}$$

provided that T and R are sufficiently small. More recently, in [4], it was shown that the notion of a Fliess operator can be generalized to admit a class of L_2-Itô stochastic processes. Specifically, such operators were defined as infinite summations of iterated Lebesgue and Stratonovich integrals, and conditions for their absolute mean square convergence were given. This class of input-output systems, however, is still too limited for many engineering applications.

A number of systems encountered in engineering involve the stochastic coupling of several subsystems. It is well known, for example, that flight control computers on board a fly-by-wire aircraft are subject to faults induced by electromagnetic disturbances and cosmic rays [18, 20]. In turn, these faults can induce system-level errors by corrupting the control law computations. Once the system detects a fault, it switches from a nominal mode, which models the aircraft under ideal conditions, to a recovery mode, which models the effect of the fault and the recovery mechanism used to restore the system back to the nominal mode. Such dynamics can be modeled as a *switched input-affine nonlinear system* as defined below.

Definition 1 A *switched input-affine system with* k+1 *modes* is a state system

$$\begin{aligned} \dot{z} &= f_v(z) + \sum_{i=1}^{m} g_{vi}(z)\, u_i,\ z(0) = z_0 \\ y &= h_v(z), \end{aligned} \tag{3}$$

where z is an n-dimensional state vector defined on some neighborhood $W \in \mathbb{R}^n$ containing z_0, $u \in L_p^m[0, T]$ is the external input, $v : [0, T] \to \{0, \ldots, k\}$ is an arbitrary switching signal, f_j, g_{ji} are analytic vector fields for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, k$, and each h_j is an analytic output function for $j = 0, 1, \ldots, k$.

Here, (f_0, g_{0i}, h_0) is assumed to represent the nominal mode. The remaining systems represent the k possible recovery modes. It is common in many applications to set the external inputs u_i, $i = 1, 2, \ldots, m$ during recovery to some fixed value since they are usually ignored by the system in such circumstances. Without loss of generality, it is assumed here that $u_i = 1$, $i = 1, 2, \ldots, m$ during any recovery mode. In the case of having only two modes of operation ($k = 1$) and one external input ($m = 1$), a switched input-affine system can be rewritten as

$$\begin{aligned} \dot{z} &= (f_0(z) + g_0(z)\, u)\,(1 - v) + (f_1(z) + g_1(z)u)v \\ &= f_0(z) + g_0(z)\, u + (f_1(z) - f_0(z))\, v + (g_1(z) - g_0(z))\, uv \\ y &= h_0(z) + (h_1(z) - h_0(z))\, v. \end{aligned}$$

When the integral process induced by the switching signal v is a Poisson process, say N, then

$$\begin{aligned} z(t) = z_0 \ &+ \int_0^t f_0(z(s)) + g_0(z(s))\, u(s)\, ds + \int_0^t f_1(z(s)) - f_0(z(s))\, dN(s) \\ &+ \int_0^t (g_1(z(s)) - g_0(z(s)))\, u(s)\, dN(s), \end{aligned}$$

where $\int \cdot\, dN$ denotes a stochastic integral with respect to N. Observe that for each $t \in [0, \infty)$, $v(t)$ is actually representing $\Delta N(t) \triangleq N(t) - N(t-)$, where $N(t-) = \lim_{s \to t, s<t} N(s)$ is the left continuous version of N.

Poisson processes fall into the class of *jump processes* or *Lévy processes* [14], which are distinct from the class of processes being considered in [1, 4]. It is possible, however, to describe (3) with a Poisson switching process in terms of a Fliess operator if a more general type of stochastic integral is used, namely an integral with respect to a semimartingale. The set of semimartingales includes, for example, Wiener processes, Poisson processes, and L_2-Itô processes. One challenge of allowing jumps in the integral is the loss of the chain rule, which cannot be recovered as is done for the Itô integral by using the Stratonovich integral [13, 14]. In addition, the underlying algebraic structure is no longer the shuffle algebra since the integration by parts formula admits extra terms [12], so in this paper, the necessary extension of the existing theory is fully developed. As a result, it will be possible to give a series solution for (3) with a Poisson switching process v and to express the map $(u, v) \mapsto y$ as a Fliess operator whose coefficients satisfy inequality (1). This will ensure that the Fliess operator representation of (3) converges absolutely in the mean.

The paper is organized as follows. Section 2 presents the main results of the paper. In Section 3, some analysis tools from stochastic integration of semimartingales are introduced. In particular, the Poisson integral and its properties are summarized. Then, in Section 4, the proofs of the main results are given. Finally, Section 5 provides the conclusions and suggestions for future work.

2 Main Results

It is assumed throughout that there is an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is a filtration of Ω, $\mathcal{F}_0$ contains all the P-null subsets of $\mathcal{F}$, and $\mathbf{F}$ is right continuous. Unless otherwise stated, every adapted process is with respect to the filtration $\mathbf{F}$. In this context, a Poisson process is defined.

Definition 2 Let $\{\tau_i\}_{i\geq 0}$ be an increasing sequence of $\mathcal{F}_t$-stopping times, and $1_{\{\cdot\}}$ denotes the indicator function. A stochastic process N such that $N(t) \triangleq \sum_{i\geq 1} 1_{\{t\geq\tau_i\}}$, for $t \geq 0$, taking values in $\mathbb{N}$ is called a *Poisson process* with intensity λ if it satisfies:

(i) $N(t) - N(s)$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t < \infty$.
(ii) N has stationary increments with Poisson distribution, i.e.,

$$N(t) - N(s) \sim \frac{e^{-(t-s)\lambda}(\lambda(t-s))^k}{k!}$$

for any $0 \leq s < t < \infty$.

From this definition, it can be inferred that N is adapted and $N(t) = \sum_{0<s\leq t} \Delta N(s)$.

To model switched systems with more than two modes, the idea of "thinning" a Poisson process is useful [16]. Suppose a Poisson process N with intensity λ models the arrival of fault events. The fault events are classified into k disjoint types: type 1, type 2, ..., type k. Let p_j denote the probability that a given event is of type j, and let N_j denote the process counting the events of type j. Then, N_j is a Poisson process with intensity $\lambda_j = p_j\lambda$. It is not difficult to see that the random variables $N_1(t_1), \ldots, N_k(t_k)$ are independent for any set of distinct positive numbers $t_1, t_2, \ldots, t_k$. The original Poisson process N with k-types of events corresponds to a *compound Poisson process* $\bar{N}$. That is,

$$\bar{N}(t) \triangleq \sum_{k=1}^{N(t)} Z_k = \sum_{0<s\leq t} \Delta\bar{N}(s) = \sum_{0<s\leq t}\sum_{j=0}^{k} j\,\Delta N_j(s),$$

where $\{Z_k\}_{k\geq 0}$ is an i.i.d sequence of random variables taking values in $\{1, \ldots, k\}$ with probability distribution given by $\{p_1, \ldots, p_k\}$, and $\{N_1, \ldots, N_k\}$ is the set of Poisson processes obtained after the thinning of the original Poisson process $N(t) = \sum_{0<s\leq t} 1_{\{\Delta\bar{N}(s)\neq 0\}}$.

Definition 3 Let N be a Poisson process with events classified into k types. A *Poisson switching signal of k-types* with probabilities p_j, $j = 1, \ldots, k$ is defined as

$$v : [0, T] \to \{0, \ldots, k\} : t \mapsto v(t) = \sum_{j=1}^{k} j\, v_j(t), \tag{4}$$

where $v_j(t) \triangleq \Delta N_j(t)$, and N_j comes from the thinning of N. In addition, the process $\bar{v} = (v_1, \ldots, v_k)$ is called the *decomposition* of a Poisson process N of k-types.

The relationship between $\bar{N}$ and v is given directly by $v(t) = \Delta\bar{N}(t)$.

To introduce Poisson processes into the Fliess operator formalism, consider stochastic processes defined on the probability space $([0, T]\times\Omega, \mathcal{P}, \mu\otimes P)$ under the measure $\mu\otimes P$ with μ being the Lebesgue measure. $\mathcal{P}$ is known as the *predictable σ-algebra*, i.e., the σ-algebra generated by sets of the form $(s, t]\times F \subset [0, T]\times\Omega$ with $F \in \mathcal{F}_s$ for $0 \leq s < t \leq T$

and $\{0\} \times F'$ with $F' \in \mathcal{F}_0$. Processes that are $\mathcal{P}$-measurable are called *predictable*. For a predictable process u, define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \le i \le m\}$, where $\|u_i\|_{L_p} \triangleq \left(\mathbb{E}[\int_a^b |u_i(s)|^p ds]\right)^{1/p}$. The set of m-dimensional predictable processes having finite L_p-norm over $[a, b]$ is denoted as $L_p^m([a, b] \times \Omega, \mathcal{P}, \mu \otimes P)$. Since this space will be used exclusively henceforth, it will be also abbreviated as $L_p^m[a, b]$. Exactly, as in (2), a ball in this space is denoted by $B_p^m(R)[0, T] \triangleq \{u \in L_p^m[a, b] : \|u\|_{L_p} \le R\}$. The L_p-norm for a random variable Z is taken to be $\|Z\|_p \triangleq (\mathbb{E}[Z^p])^{1/p}$. Consider the following alphabets: $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_k\}$, and $XY = X \cup Y$. For each $\eta \in XY^*$, define a Poisson-Lebesgue iterated integral E_η by first setting $E_\emptyset = 1$ and then letting

$$E_{x_i\eta'}[w](t) \triangleq \int_0^t u_i(s) E_{\eta'}[w](s)\, ds,\ x_i \in X \tag{5}$$

$$E_{y_j\eta'}[w](t) \triangleq \int_0^t E_{\eta'}[w](s-)\, dN_j(s),\ y_j \in Y, \tag{6}$$

where $\eta' \in XY^*$, $w = (u, \bar{v})$, $u \in B_p^m(R)[0, T]$, $u_0 = 1$, and $\bar{v} = (v_1, \ldots, v_k)$ is the decomposition of a Poisson process N of k-types. The definition of the Poisson integral in (6) will be given precisely in Definition 8. Note that u does not appear in (6) as a consequence of the assumption concerning u during a recovery mode. The iterated integral defined in (5) and (6) can be extended linearly to a polynomial $q \in \mathbb{R}\langle XY\rangle$ as

$$E_q[w](t) = \sum_{\eta \in \text{supp}(q)} (q, \eta) E_\eta[w](t),$$

where $\text{supp}(q) = \{\eta \in XY^* : (q, \eta) \neq 0\}$. A Fliess operator over $B_p^m(R)[0, T]$ with Poisson jumps is defined as follows.

Definition 4 A causal m-input, ℓ-output *Fliess operator* F_c, $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ driven by $u \in B_p^m(R)[0, T]$ and a Poisson process N of k-types is formally defined as

$$F_c[w](t) = \sum_{\eta \in XY^*} (c, \eta)\, E_\eta[w](t), \tag{7}$$

where each E_η is given in (5) and (6) and $w = (u, \bar{v})$.

The first main theorem of the paper is given below.

Theorem 5 *Suppose $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ satisfies the locally convergent growth condition* (1)*, where XY now replaces X. Then, there exist $R, T > 0$ such that for each predictable process $u \in B_p^m(R)[0, T]$ and Poisson process of k-types with probabilities p_j, $j = 1, \ldots, k$, the series* (7) *converges uniformly and absolutely in the mean on* $[0, T]$.

Using the Fliess operator formalism, one can describe the input-output behavior of the switched system (3) driven by the Poisson process (4). The original Poisson process can be rewritten as $N(t) = \sum_{0<s\le t} 1_{\{v(s)\neq 0\}}$. It is important to observe that the processes $N_1, \ldots, N_k$ never jump at the same time instant. That is, for any $j_1 \neq j_2$, one has $P(\Delta_\delta(N_{j_1}(t) + N_{j_2}(t)) \ge 2) = P(\Delta_\delta N_{j_1}(t) \ge 2)\ P(\Delta_\delta N_{j_2}(t) \ge 2) \approx 0$ for $\delta > 0$ small, where $\Delta_\delta N_i(t) \triangleq N_i(t + \delta) - N_i(t)$. The second main theorem is given below.

Theorem 6 *A switched input-affine nonlinear system with $k+1$ modes as in (3) driven by a predictable $u \in B_p^m(R)[0,T]$ and a Poisson switching signal of k-types as defined in (4) can always be written uniquely in the form of (7) for some locally convergent $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$.*

The next example shows how to express a bilinear switched system as a Fliess operator. If its vector fields are commutative, then its input-output behavior can be written in closed form.

Example 7 Consider an n-dimensional switched bilinear system

$$\begin{aligned}\dot{z} &= A_v z + B_v z\, u, \quad z(0) = z_0\\ y &= h_v(z) = C_v z,\end{aligned}$$

where v is a Poisson switching signal of k-types, $u \in B_1(R)[0,T]$, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times 1}$, $C_j \in \mathbb{R}^{\ell\times n}$, $j = 0, 1, \ldots, k$, and $z_0 \in \mathbb{R}^{n\times 1}$. Using (4), one can then represent the state equation as

$$\dot{z} = (A_0 z + B_0 z\, u)\left(1 - \sum_{j=1}^{k} v_j\right) + \sum_{j=1}^{k}\left(A_j z + B_j z\, u\right) v_j.$$

Since it has been assumed that $u = 1$ during jumps, it follows that

$$\dot{z} = M_{00} z + M_{01} z\, u + \sum_{j=1}^{k} M_{1j} z\, v_j,$$

where $M_{00} = A_0$, $M_{01} = B_0$ and $M_{1j} = A_j + B_j - A_0 - B_0$. It will be shown at the end of Section 4 that $y = h_0(z) + \sum_{j=1}^{k}(h_j(z) - h_0(z))\, v_j$, where $h_j(z) = F_{c_j}[w]$, $w = (u, \bar{v})$ and $(c_j, \eta) = C_j M_\eta z_0$ with $M_{x_i\eta} = M_{0i} M_\eta$ and $M_{y_j\eta} = M_{1j} M_\eta$ for $\eta \in XY^*$. In which case, each c_j is globally convergent. If the M_{ij}'s commute then

$$h_i(z(t)) = C_i \exp\left(M_{00}\, t + M_{01}\int_0^t u(s)\, ds\right)\prod_{j=1}^{k}\left(I + M_{1j}\right)^{N_j(t)} z_0$$

for all $i = 0, \ldots, k$. □

3 The Poisson Integral

This section gives a brief summary of the main concepts needed from the theory of stochastic integration to prove Theorems 5 and 6. The treatment is based on [14] and the references therein. There exists a well-established theory of integration for semimartingales, i.e., for processes that can be decomposed into a Martingale and a bounded variation process. The main requirement to integrate a locally bounded process H with respect to such a process is that the function H be predictable. In particular, all left-continuous adapted processes are predictable. Hereafter, only the left-continuous version of a process $\{H(s)\}_{s\geq 0}$ will be considered as integrands, namely $\{H(s-)\}_{s\geq 0}$. In particular, an integral with respect to a Poisson process N is well-defined for locally bounded predictable processes since N is a semimartingale.

Definition 8 Let H be a predictable locally bounded stochastic process and N a Poisson process. The *Poisson integral of H* is defined as

$$J_N(H)_t = \int_0^t H(s-)\,dN(s) = \sum_{k=1}^{N(t)} H(t-)[N(\tau_k \wedge t) - N(\tau_{k-1} \wedge t)],$$

where $\tau \wedge t \triangleq \min\{\tau, t\}$.

It is not difficult to see that $J_N(H)$ is also a semimartingale. It coincides with the Itô integral because H is evaluated at the left end of (τ_i, τ_{i+1}), and the jumps of a Poisson integral occur at the jump points of N, i.e., $\Delta\left(\int_0^t H(s-)\,dN(s)\right) = H(t)\,\Delta N(t)$. Thus, the Poisson integral is only active at the instant of a jump, which for system (3) is the dwell time for the modes $\{1, 2, \ldots, k\}$. Nevertheless, it is important to point out that if, for example, $\int_0^t N(s)\,dN(s)$ were defined as $\sum_{\tau_i \le t} N(\tau_i)$, then it is a well-defined Stieltjes integral since $N(t)$ is an increasing process of finite first variation, but it is not a stochastic integral because $N(t)$ is not predictable. On the other hand, $\int_0^t N(s-)\,dN(s)$ is a stochastic integral which is indistinguishable from the re-defined Stieltjes integral $\int_0^t N(s)\,dN(s) = \sum_{\tau_i \le t} N(\tau_{i-1})$.

The advantages of the Stratonovich integral over the Itô integral with respect to a Wiener process are well known. In particular, by using the Stratonovich formulation, one can suppress the extra terms generated by the Itô integral and thus recover the standard rules of calculus such as the integration by parts formula and the chain rule. Convenient algebraic structures also arise when considering iterated Stratonovich integrals rather than its Itô counterpart [1, 8]. Such advantages, however, do not appear when considering stochastic integrals with respect to Poisson processes since their Stratonovich and Itô formulation coincide. This implies that it is not possible to recover, for example, the standard integration by parts formula. For this reason, one expects a fundamentally different underlying algebraic structure for iterated Poisson integrals. In order to see this fact more clearly, consider the following chain rule for stochastic integrals with respect to semimartingales [3, 14].

Theorem 9 *Let $F \in \mathcal{C}^2[0, \infty]$. The change of variables formula for semimartingales is*

$$\begin{aligned} F(X(t)) = F(X(0)) &+ \int_0^t F'(X(s-))dX(s) + \frac{1}{2}\int_0^t F''(X(s-))d[X, X]_s^c \\ &+ \sum_{0<s\le t} (F(X(s)) - F(X(s-))) - F'(X(s-))\Delta X(s), \end{aligned} \tag{8}$$

where $[X, X]_t^c$ is the continuous part of the quadratic variation of X, namely

$$[X, X]_t = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2,$$

where $\|\Pi\| = \max_{i=1,\ldots,n} (t_i - t_{i-1})$ is the measure of the partition Π of $[0, t]$.

Observe that all the integrals above are well-defined since $F'(X(s-))$ and $F''(X(s-))$ are predictable. On the other hand, the Stratonovich change of variables rule is

$$F(X(t)) = F(X(0)) + \oint_0^t F'(X(s-))\,dX(s) + \sum_{0<s\leq t} (F(X(s)) - F(X(s-))) - F'(X(s-))\Delta X(s). \tag{9}$$

Using the identity $[N, N]_t^c = 0$ and by direct comparison of (8) and (9), it follows that

$$\oint_0^t F'(N(s-))\,dN(s) = \int_0^t F'(N(s-))\,dN(s),$$

which shows explicitly that the Poisson-Stratonovich formulation of the Poisson integral coincides with the Poisson-Itô formulation. As iterated integrals are a fundamental feature of any Fliess operator, the next example illustrates how to work with iterated stochastic integrals of pure jump processes such as Poisson processes. In particular, it is shown how they behave naturally as exponentials.

Example 10 For a general pure jump process $X(t) = X(0) + \sum_{i\geq 0} Z_i 1_{\{\tau_i \leq t\}}$ generated by the sequence of random variables $\{\tau_i, Z_i\}_{i\geq 0}$, observe that the nth iterated integral of X is

$$X^{\{n\}}(t) \triangleq \int_0^t X^{\{n-1\}}(s-)\,dX(s) = \sum_{0\leq t_1<\cdots<t_n\leq t} \Delta X(t_1)\cdots\Delta X(t_n)$$

with $X^{\{0\}} = 1$. Note that $X^{\{n\}}(t) \neq 0$ only if the times $t_1 < \cdots < t_n$ take values in $\{\tau_i\}_{i\geq 0}$. Therefore, the number of terms in the summation is exactly $\binom{\#X(t)}{n}$, where $\#X(t) \triangleq \max\{k : \tau_k \leq t\}$. The addition of all iterated integrals can be computed as

$$\begin{aligned} Y(t) \triangleq \sum_{n=0}^{\infty} X^{\{n\}}(t) &= 1 + \sum_{i=0}^{\infty} \sum_{0\leq t_1<\cdots<t_n\leq t} \Delta X(t_1)\cdots\Delta X(t_n) \\ &= \prod_{i=1}^{\#X(t)} (1 + \Delta X(\tau_i)). \end{aligned}$$

If $X = N$, then $\#X(t) = N(t)$. Therefore, one has that $Y(t) = 2^{N(t)}$ since $\Delta N(t) = 1$ when there is a jump and 0 otherwise. □

The following integration by parts formula for semimartingales will play a key role in the description of the underlying algebraic structure for stochastic iterated integrals to be presented in the next section (see Lemma 21).

Lemma 11 *Let X and Y be two semimartingales. It follows that*

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s-)\,dY(s) + \int_0^t Y(s-)\,dX(s) + [X, Y]_t, \tag{10}$$

where $[X, Y]_t$ *is the quadratic covariation of X and Y defined as*

$$[X, Y]_t = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)\left(Y_{t_i} - Y_{t_{i-1}}\right).$$

Note that the quadratic variation is a special case of the quadratic covariation. In the case where X and Y are pure jump processes, it follows that $[X, Y]_t = \sum_{0<s\leq t} \Delta X_s \Delta Y_s$. Moreover, if $X = \int u(s)\, dN(s)$ and $Y = \int v(s)\, dN(s)$, then one has that

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s-)v(s)\, dN(s) + \int_0^t Y(s-)u(s)\, dN(s) + \underbrace{\sum_{0<s\leq t} u(s)v(s)\Delta N(s)\Delta N(s)}_{\int_0^t u(s)v(s)\, dN(s)}.$$

Note that the last term can also be written as $\int_0^t u(s)v(s)\, d[N, N]_s$.

To conclude the section, some tools needed for computing estimates of iterated Poisson integrals are given.

Definition 12 A stochastic process X is called *increasing* if it is adapted, $X(0) = 0$, and its sample paths are non-decreasing and a.s. right continuous.

Theorem 13 *Let X be an increasing stochastic process such that $\mathbb{E}[X(t)] < \infty$. Then, there exists a unique increasing process $\bar{X}$ such that*

$$\mathbb{E}\left[\int_0^t Y(s-)\, dX(s)\right] = \mathbb{E}\left[\int_0^t Y(s)\, d\bar{X}(s)\right]$$

for all $t \geq 0$ and for each non-negative predictable process Y. The process $\bar{X}$ is called the dual predictable projection or compensator of X. Moreover, $\bar{X}$ is the only process such that $X - \bar{X}$ is a Martingale.

Example 14 The dual predictable projection of the Poisson process N is easily shown to be the process λt, which makes $N(t) - \lambda t$ a Martingale. Therefore,

$$\mathbb{E}\left[\int_0^t Y(s-)\, dN(s)\right] = \lambda\, \mathbb{E}\left[\int_0^t Y(s)\, ds\right].$$

□

The next lemma, which will be used in the proof of Theorem 17, is an application of the compensator of a Poisson process.

Lemma 15 *Let X and Y be predictable. The equality*

$$\int_0^t Y(s-)\, dN_s + \int_0^t X(s)\, ds = 0 \tag{11}$$

holds if and only if $X = Y = 0$ for X non-negative.

Proof The sufficiency claim is immediate. For necessity, assume (11) holds. It then follows that

$$\int_0^t Y(s-)\, d\widetilde{N}_s = \int_0^t (-X(s) - Y(s)\lambda)\, ds,$$

where $\widetilde{N}$ is a Martingale obtained by compensating the Poisson process N. It is well known that the quadratic variation of Lebesgue integrals is zero. Therefore, taking into consideration that the only Martingale having quadratic variation zero is the zero Martingale, one must conclude that $Y = 0$. So one has that

$$\int_0^t X(s)\,ds = 0,$$

which by the properties of the Lebesgue integral and the fact that $X \geq 0$ implies $X = 0$. □

Although the stochastic setting presented in this section has been focused on Poisson integrals, the incorporation of noise components in the formulation (e.g., Wiener processes) is straightforward.

4 Proof of Main Results

In order to prove Theorem 5, an upper bound for the Poisson-Lebesgue iterated integral given in (5) and (6) is needed. Let $|\eta|_A$ denote the number of letters in η that belongs to $A \subset XY$ and define the language $X^{n_1}Y^{n_2} = \{\eta \in XY^* : |\eta|_X = n_1, |\eta|_Y = n_2\}$. In this context, it is desired to obtain, for a fixed time t, an upper bound for $\|E_\eta[w](t)\|_1$ in terms of the size of the input u given by its L_1-norm $\|u\|_{L_1}$ over the time interval $[0, t]$. The next lemma gives such a result.

Lemma 16 *Let $\eta \in X^{n_1}Y^{n_2}$ and $w = (u, \bar{v})$, where $u \in B_p^m(R)[0, T]$, and $\bar{v}$ is a decomposition of a Poisson process N of k-types. Then, for a fixed $t \in [0, T]$,*

$$\|E_\eta[w](t)\|_1 \leq \lambda^{n_2} \left(\prod_{j=1}^{k} p_j^{\beta_j}\right) \prod_{i=0}^{m} \frac{U_i^{\alpha_i}(t)}{\alpha_i!}, \tag{12}$$

where $\beta_j = |\eta|_{y_j}$, $U_i(t) = \mathbb{E}\left[\int_0^t |u_i(s)| ds\right]$ for $i \geq 0$, $\alpha_0 = |\eta|_{x_0} + |\eta|_Y$ and $\alpha_i = |\eta|_{x_i}$ for $i \geq 1$.

Proof The inequality is proved by induction over the total number of $n_1 + n_2$ integrals. For $n_1 + n_2 = 0$, the claim is trivial. If $n_1 + n_2 = 1$, then there are two cases to consider. First, if $\eta = x_i$, then $\|E_{x_i}[w](t)\|_1 = U_i(t)$. If $\eta = y_j$, since a Poisson process is an increasing process, it follows from Theorem 13 that

$$\|E_{y_j}[w](t)\|_1 \leq \mathbb{E}\left[\int_0^t dN_j(s)\right] = \lambda p_j\, U_0(t).$$

Now, assume that (12) holds for every $\eta' \in X^{n_1}Y^{n_2}$ with $n_1 + n_2$ fixed. If $\eta = x_i\eta'$, then

$$\begin{aligned}
\|E_{x_i\eta'}[w](t)\|_1 &= \mathbb{E}\left[\left|\int_0^t u_i(s) E_{\eta'}[w](s)\, ds\right|\right] \\
&\leq \int_0^t \mathbb{E}[|u_i(s)|]\mathbb{E}\left[|E_{\eta'}[w](s)|\right] ds \\
&\leq \int_0^t \mathbb{E}[|u_i(s)|]\lambda^{n_2} \left(\prod_{l=1}^{k} p_l^{\beta_l}\right) \prod_{j=1}^{m} \frac{U_j^{\alpha_j}(s)}{\alpha_j!} ds.
\end{aligned}$$

Since $U_j(s)$ is non-decreasing and non-negative for all j, then $U_j(s) \leq \sup_{s\in[0,t]} U_j(s) = U_j(T)$ for any $s \in [0, t]$. Thus, one can conveniently remove from the integral above all U_j, $j \neq i$, which leaves

$$\begin{aligned}\|E_{x_i\eta'}[w](t)\|_1 &\leq \lambda^{n_2}\left(\prod_{l=1}^{k} p_l^{\beta_l}\right)\prod_{\substack{j=0\\ j\neq i}}^{m}\frac{U_l^{\alpha_j}(t)}{\alpha_j!}\int_0^t \mathbb{E}[|u_i(s)|]\frac{U_i^{\alpha_i}(s)}{\alpha_i!}\,ds\\ &= \lambda^{n_2}\left(\prod_{l=1}^{k} p_l^{\beta_l}\right)\frac{U_0^{\alpha_0}(t)\cdots U_i^{\alpha_i+1}(t)\cdots U_m^{\alpha_m}(t)}{\alpha_0!\cdots(\alpha_i+1)!\cdots\alpha_m!}.\end{aligned}$$

On the other hand, if $\eta = y_i\eta'$, then it follows that

$$\begin{aligned}\|E_{y_i\eta'}[w](t)\|_1 &= \mathbb{E}\left[\left|\int_0^t E_{\eta'}[w](s-)\,dN_i(s)\right|\right]\\ &\leq \mathbb{E}\left[\int_0^t \left|E_{\eta'}[w](s-)\right|dN_i(s)\right]\\ &= \lambda p_i\int_0^t \mathbb{E}\left[\left|E_{\eta'}[w](s)\right|\right]ds\\ &\leq \lambda p_i\int_0^t \lambda^{n_2}\left(\prod_{l=1}^{k} p_l^{\beta_l}\right)\prod_{j=0}^{m}\frac{U_j^{\alpha_j}(s)}{\alpha_j!}ds\\ &\leq \lambda^{n_2+1}p_1^{\beta_1}\cdots p_i^{\beta_i+1}\cdots p_k^{\beta_k}\prod_{j=1}^{m}\frac{U_j^{\alpha_j}(t)}{\alpha_l!}\int_0^t\frac{U_0^{\alpha_0}(s)}{\alpha_0!}\,ds\\ &= \lambda^{n_2+1}p_1^{\beta_1}\cdots p_i^{\beta_i+1}\cdots p_k^{\beta_k}\frac{U_0^{\alpha_0+1}(t)}{(\alpha_0+1)!}\prod_{j=1}^{m}\frac{U_j^{\alpha_j}(t)}{\alpha_j!}.\end{aligned}$$

Note that the increment on α_0 is due to the fact that now, $|\eta|_Y = n_2 + 1$ while $|\eta|_X$ remains n_1. Hence, the proof is complete. □

Proof of Theorem 5 Assume c is locally convergent with growth constants $K, M > 0$. Here, it is assumed that $\ell = 1$ since the computation below is performed component-wise on the components of c in the general case. Fix some $T > 0$ and pick any $u \in L_p^m[0, T]$. Note that if $p > 1$, then

$$\|u\|_1 \leq \|u\|_p T^{1/q}$$

when p and q are conjugate exponents. Therefore, $u \in L_p^m[0, T]$ implies $u \in L_1^m[0, T]$. Let $R = \max\{\|u\|_1, T\}$. For $(a_1, \ldots, a_m) \in \mathbb{N}^m$, define $a! = a_1!\cdots a_m!$. From Lemma 16 and since $p_j \leq 1$, it follows for any $\eta \in X^{n_1}Y^{n_2}$ and $t \in [0, T]$ that

$$\mathbb{E}\left[\left|(c,\eta)E_\eta[w](t)\right|\right] \leq KM^r r!\frac{\lambda^{n_2}R^r}{\alpha_0!\alpha!}, \tag{13}$$

where $r = |\eta| = n_1 + n_2$. Also, recall that $\beta_j = |\eta|_{y_j}$, $\sum_{j=1}^{k}\beta_j = n_2$ and $\alpha_0 = |\eta|_{x_0} + n_2$. Next, define $\bar{\lambda} = \max\{1, \lambda\}$ and $a_r(t) = \sum_{|\eta=r}|(c,\eta)E_\eta[w](t)|$. Then, from (13) and the fact that $\lambda^{n_2} \leq \bar{\lambda}^r$, observe that

$$\begin{aligned}
\mathbb{E}[a_r(t)] &= \sum_{\substack{\eta \in XY^* \\ |\eta| = r}} \mathbb{E}\left[\left|(c,\eta)E_\eta[w](t)\right|\right] \\
&\leq KM^r\bar{\lambda}^r R^r \sum_{\substack{\eta \in XY^* \\ |\eta| = r}} \frac{r!}{(|\eta|_{x_0} + \beta_1 + \cdots + \beta_k)!\alpha!} \\
&= KM^r\bar{\lambda}^r R^r! \sum_{\substack{|\eta|_{x_0} + \beta_1 + \cdots + \beta_m \\ +\alpha_1 \cdots + \alpha_m = r}} \frac{r!}{(|\eta|_{x_0} + \beta_1 + \cdots + \beta_k)!\alpha!} \cdot \frac{r!}{|\eta|_{x_0}!\beta!\alpha!} \\
&\leq KM^r\bar{\lambda}^r \left(\sum_{\substack{|\eta|_{x_0} + \beta_1 + \cdots + \beta_m \\ +\alpha_1 + \cdots + \alpha_m = r}} \frac{r!}{|\eta|_{x_0}!\beta!\alpha!} \right)^2 \\
&= KM^r\bar{\lambda}^r R^r (2m+2)^{2r}.
\end{aligned}$$

It then follows that

$$\sum_{r=0}^{\infty} \mathbb{E}[a_r(t)] \leq \sum_{r=0}^{\infty} K(4M\bar{\lambda}R(m+1)^2)^r.$$

Thus, if $R < 1/(4M\bar{\lambda}(m+1)^2)$, then (7) converges uniformly and absolutely in the mean on $[0, T]$. □

An important operator on formal power series used hereafter is the *left-shift operator*. It is defined as

$$q_k^{-1} : XY^* \to \mathbb{R}\langle\langle XY \rangle\rangle : \eta \mapsto \begin{cases} \eta' \ : \ \eta = q_k\eta', q_k \in XY, \eta' \in XY^* \\ 0 \ \ : \ \text{otherwise.} \end{cases} \tag{14}$$

For any series $c \in \mathbb{R}\langle\langle XY \rangle\rangle$, this definition can be extended linearly as

$$q_k^{-1}(c) = \sum_{\eta \in XY^*} (c,\eta) q_k^{-1}(\eta) = \sum_{\eta \in XY^*} (c, q_k\eta)\eta.$$

The next theorem addresses the uniqueness of generating series of Fliess operators in the present context.

Theorem 17 *Let $c, d \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$ be two locally convergent generating series. Then, $F_c[w] = F_d[w]$ for all admissible w if and only if $c = d$.*

Proof The proof is partially based on that appearing in [19, Section 2.2], which is the standard procedure to prove uniqueness of Fliess operators with deterministic inputs. First, since $F_c[w] = F_d[w]$ implies $F_{c-d}[w] = 0$, it is sufficient to prove that $F_c[w] = 0$ for every input w if and only if $c = 0$. If $c = 0$, then trivially $F_c[w] = 0$. Conversely, an arbitrary operator F_c for a locally convergent $c \in \mathbb{R}\langle\langle XY\rangle\rangle$ can always be decomposed as

$$F_c[w](t) = (c, \emptyset) + \int_0^t \sum_{i=0}^{m} u_i(s) \Bigg(\underbrace{(c, x_i) + \sum_{\eta \in XY^+} (c, x_i\eta) E_\eta[w](s)}_{F_{x_i^{-1}(c)}[w](s)} \Bigg)\, ds$$

$$+ \int_0^t \sum_{i=0}^{k} \Bigg(\underbrace{(c, y_i) + \sum_{\eta \in XY^*} (c, y_i\eta) E_\eta[w](s)}_{F_{y_i^{-1}(c)}[w](s)} \Bigg)\, dN_i(s).$$

It is evident that $F_c[w](0) = (c, \emptyset) = 0$. Also, it is clear that $F_{x_i^{-1}(c)}[w]$ and $F_{y_i^{-1}(c)}[w]$ are convergent because $x_i^{-1}(c)$ and $y_i^{-1}(c)$ are also locally convergent series. Since the sum of independent Martingales is again a Martingale, then applying Lemma 15 gives

$$\int_0^t \sum_{i=0}^{m} u_i(s) F_{x_i^{-1}(c)}[w](s)\, ds = 0 \tag{15}$$

and

$$\int_0^t \sum_{i=0}^{k} F_{y_i^{-1}(c)}[w](s)\, dN_i(s) = 0. \tag{16}$$

Given that equality (15) must be true for every input u, it must follow that $F_{x_i^{-1}(c)}[w] = 0$ for all $i \in \{1, \ldots, m\}$, so in particular, $F_{x_i^{-1}(c)}[0] = (c, x_i) = 0$. It is not so obvious that $F_{y_i^{-1}(c)}[w] = 0$ in (16). Recall that the Poisson processes $\{N_1, \ldots, N_k\}$ are independent. In general, one needs to prove that

$$\int_0^t v_1(s-) dN_1(s) + \cdots + \int_0^t v_k(s-) dN_k(s) = 0 \tag{17}$$

implies $v_1 = \cdots = v_j = 0$. A consequence of the Poisson processes' mutual independence is that simultaneous arrivals never occur. Assume that the first arrival happens at τ_1 for Poisson process N_{i_1} with $i_j \in \{1, \ldots, k\}$. Since (17) holds for all times, assume $t = \tau_1$. Then, (17) is equivalent to $v_{i_1}(\tau_1)\Delta(N_{i_1})_{\tau_1} = 0$, which implies $v_{i_1}(\tau_1) = 0$. Similarly, for the jth arrival at $t = \tau_j$, it follows that $v_{i_j}(\tau_j)\Delta(N_{i_j})_{\tau_j} = 0$, hence, $v_{i_j}(\tau_j) = 0$. This conclusion holds for every realization of the Poisson processes and for every $v_i \in B_1^m(R)[0, T]$ and thus $v_i = 0$ for $i = 1, \ldots, k$. Therefore, $F_{y_i^{-1}(c)}[w] = 0$, and as before $(c, y_i) = 0$. Proceeding inductively gives $(c, \eta) = 0$ for all $\eta \in XY^*$. □

A property of any Fliess operator is that its input-output behavior is completely determined by its generating series, independent of whether a state space realization exists. But when a state space realization does exist, it was shown by Fliess in [6, 7] that the output can be written in terms of a series solution of the state equation using the *shuffle algebra* on $\mathbb{R}\langle\langle X\rangle\rangle$. The existence and uniqueness of a local solution for the state equation in (3) can be established using the techniques in [1, 21]. In a similar vein, a variety of analogous tools are needed in order to prove Theorem 6: two new products on $\mathbb{R}\langle\langle XY\rangle\rangle$, two operators, and five lemmas. Some notation is introduced first. The set of all ℓ-dimensional real-valued analytic functions on some neighborhood W of an arbitrary point $z_0 \in \mathbb{R}^n$ will be denoted by $\mathcal{C}_\omega^\ell(W)$. Since one is often interested in working with arbitrary letters in the alphabet XY, hereafter, q_i will denote any letter of XY. The shuffle product on $\mathbb{R}\langle\langle XY\rangle\rangle$ is the commutative $\mathbb{R}$-bilinear mapping $\mathbb{R}\langle\langle XY\rangle\rangle \times \mathbb{R}\langle\langle XY\rangle\rangle \to \mathbb{R}\langle\langle XY\rangle\rangle$ uniquely specified by the shuffle product of two words $(q_i\eta) \sqcup\!\sqcup (q_j\xi) = q_i(\eta \sqcup\!\sqcup (q_j\xi)) + q_j((q_i\eta) \sqcup\!\sqcup \xi)$ and $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ for all $\eta, \xi \in XY^*$ [5–8, 12].

Definition 18 For $\eta', \xi' \in XY^*$, $q_i, q_j \in XY$, and $\eta = q_i\eta'$, $\xi = q_j\xi'$, define the product

$$\eta \diamondsuit \xi = q_i(\eta' \diamondsuit \xi) + q_j(\eta \diamondsuit \xi') + \delta_{q_i,q_j} q_i \left(\eta' \sqcup\!\sqcup \xi' + \eta' \diamondsuit \xi'\right),$$

where $q_l \diamondsuit \emptyset = \emptyset \diamondsuit q_l = 0$, and $\delta_{q_i,q_j} = 1$ if $i = j$ and $q_i, q_j \in Y$, otherwise $\delta_{q_i,q_j} = 0$.

Example 19 Some computations of this product are:

$$\begin{aligned} x_0 \diamondsuit y_1 &= 0, \\ y_1 \diamondsuit y_1 &= y_1, \\ y_2 x_0 y_1 \diamondsuit y_2 y_1 &= y_2 x_0 y_2 y_1 + 2 y_2^2 x_0 y_1. \end{aligned}$$

Observe this product only produces nonempty words when η and ξ have common letters in Y. Furthermore, if $\mathrm{supp}(\eta \diamondsuit \xi)$ is nonempty and $\nu \in \mathrm{supp}(\eta \diamondsuit \xi)$ then $|\eta| + |\xi| - \sum_{j=1}^{k} \min(|\eta|_{y_j}, |\xi|_{y_j}) \leq |\nu| \leq |\eta| + |\xi| - 1$. □

Let $f_0, \ldots, f_m, g_0, \ldots, g_m \in \mathcal{C}^n_\omega(W)$ and $h \in \mathcal{C}^\ell_\omega(W)$. The *Lie derivative* of h is defined for any $\eta = q_{i_k} \cdots q_{i_1} \in XY^*$ as

$$L_\eta h = L_{q_{i_1}} \cdots L_{q_{i_k}} h, \tag{18}$$

where

$$L_{q_i} h : W \to \mathbb{R}^\ell \quad : \quad z \mapsto \begin{cases} \dfrac{\partial h}{\partial z}(z) f_i(z) & : q_i \in X \\ \Delta_{g_i} h(z) & : q_i \in Y \end{cases}$$

with $\Delta_{g_i} h(z) \triangleq h(z + g_i(z)) - h(z)$, and $L_\emptyset h = h$.

Definition 20 The *mixed shuffle* product $\overline{\sqcup\!\sqcup}$ is defined on XY^* as

$$\overline{\sqcup\!\sqcup} : XY^* \times XY^* \to \mathbb{R}\langle\langle XY\rangle\rangle \quad : \quad (\eta, \xi) \mapsto \eta \sqcup\!\sqcup \xi + \eta \diamondsuit \xi.$$

The next lemma shows how the product of iterated integrals defined in (5) and (6) can be written in terms of the $\overline{\sqcup\!\sqcup}$ product.

Lemma 21 *Let $\eta \in XY^*$ and $w = (u, \bar{v})$, where $u \in B_p^m(R)[0, T]$ and $\bar{v}$ is a decomposition of a Poisson process N of k-types. Then,*

$$E_\eta[w] E_\xi[w] = E_{\eta \overline{\sqcup\!\sqcup} \xi}[w], \quad \eta, \xi \in XY^*. \tag{19}$$

Proof The proof of (19) is done by induction over $|\eta| + |\xi| = n$. The claim is trivial for $n = 0, 1$ since $E_\emptyset[w] = 1$ and $\eta \overline{\sqcup\!\sqcup} \emptyset = \eta \sqcup\!\sqcup \emptyset = \eta$. Assume (19) holds up to some fixed

$|\eta|+|\xi|=n\geq 1$. From the integration by parts formula (10), it follows that if $\xi = q_j\xi'$, then

$$\begin{aligned} E_{q_i\eta}[w](t)E_\xi[w](t) &= \int_0^t u_i(s)E_\eta[w](s)E_\xi[w](s)\,dN_{q_i}(s) \\ &+ \int_0^t u_j(s)E_{q_i\eta}[w](s)E_{\xi'}[w](s)\,dN_{q_j}(s) \\ &+ \delta_{q_i,q_j}\int_0^t E_\eta[w](s)E_{\xi'}[w](s)\,dN_{q_i}(s), \end{aligned}$$

where $dN_{q_i}(t)$ refers to an integral with respect to t if $q_i \in X$; otherwise, $dN_{q_i}(t)$ refers to a Poisson integral with respect to N_i. Also, the factor δ_{q_i,q_j} appears from (10) noting that $[N_{q_i}, N_{q_j}]_s = \delta_{q_i,q_j}N_{q_i}(s)$. From (5), (6) and the induction hypothesis, it follows that

$$\begin{aligned} E_{q_i\eta}[w](t)E_\xi[w](t) &= E_{q_i(\eta\,\bar{\sqcup\!\sqcup}\,\xi)+q_j(q_i\eta\,\bar{\sqcup\!\sqcup}\,\xi')}[w](t) + E_{\delta_{q_i,q_j}q_i(\eta\,\bar{\sqcup\!\sqcup}\,\xi')}[w](t) \\ &= E_{q_i\eta\sqcup\!\sqcup\xi} + E_{q_i(\eta\Diamond\xi)+q_j(q_i\eta\Diamond\xi')}[w](t) + E_{\delta_{q_i,q_j}q_i(\eta\sqcup\!\sqcup\xi'+\eta\Diamond\xi')}[w](t) \\ &= E_{q_i\eta\sqcup\!\sqcup\xi+q_i\eta\Diamond\xi}[w](t) = E_{q_i\eta\,\bar{\sqcup\!\sqcup}\,\xi}[w](t). \end{aligned}$$

Because, from Theorem 17, generating series are unique, the identity holds for all words $\eta,\xi \in XY^*$. □

The product $\bar{\sqcup\!\sqcup}$ can be linearly extended to series $c, d \in \mathbb{R}\langle\langle XY\rangle\rangle$ to give

$$c\,\bar{\sqcup\!\sqcup}\,d = \sum_{\eta,\xi\in XY^*}[(c,\eta)(d,\xi)]\eta\,\bar{\sqcup\!\sqcup}\,\xi = \sum_{\nu\in XY^*}\underbrace{\sum_{\eta,\xi\in XY^*}[(c,\eta)(d,\xi)](\eta\,\bar{\sqcup\!\sqcup}\,\xi,\nu)}_{(c\,\bar{\sqcup\!\sqcup}\,d,\nu)}\,\nu.$$

For a specific word $\nu \in XY^*$, the coefficient $(\eta\sqcup\!\sqcup\xi,\nu) \neq 0$ if $|\nu| = |\eta| + |\xi|$, and $(\eta\Diamond\xi,\nu) \neq 0$ if $|\eta| + |\xi| - \sum_{j=1}^k \min(|\eta|_{y_j}, |\xi|_{y_j}) \leq |\nu| \leq |\eta| + |\xi| - 1$. Therefore, $(c\,\bar{\sqcup\!\sqcup}\,d,\nu)$ is finite since the set $I_{\bar{\sqcup\!\sqcup}}(\nu) \triangleq \{(\eta,\xi) \in XY^* \times XY^* : (\eta\,\bar{\sqcup\!\sqcup}\,\xi,\nu) \neq 0\}$ is finite. Moreover, observe that the product $\bar{\sqcup\!\sqcup}$ inherits commutativity and associativity from $\sqcup\!\sqcup$ and $\Diamond$. As a consequence, $\mathbb{R}\langle\langle XY\rangle\rangle$ is a commutative $\mathbb{R}$-algebra under $\bar{\sqcup\!\sqcup}$ with multiplicative identity element 1.

The action of the left-shift operator on $\bar{\sqcup\!\sqcup}$ is provided in the next lemma.

Lemma 22 *The left-shift operator acts on* $\bar{\sqcup\!\sqcup}$ *as*

$$q_k^{-1}(\eta\,\bar{\sqcup\!\sqcup}\,\xi) = q_k^{-1}(\eta)\,\bar{\sqcup\!\sqcup}\,\xi + \eta\,\bar{\sqcup\!\sqcup}\,q_k^{-1}(\xi) + \bar{\delta}_{q_k}\left(q_k^{-1}(\eta)\,\bar{\sqcup\!\sqcup}\,q_k^{-1}(\xi)\right), \tag{20}$$

where $q_k \in XY$, $\eta,\xi \in XY^*$ *and* $\bar{\delta}_{q_k} = 1$ *if* $q_k \in Y$ *and 0 otherwise.*

Proof First, the left-shift operator is known to act as a derivation on the shuffle product $\sqcup\!\sqcup$ [9], i.e.,

$$q_k^{-1}(\eta\,\bar{\sqcup\!\sqcup}\,\xi) = q_k^{-1}(\eta)\sqcup\!\sqcup\xi + \eta\sqcup\!\sqcup q_k^{-1}(\xi), \quad \eta,\xi \in XY^*. \tag{21}$$

For the mixed shuffle product $\bar{\sqcup\!\sqcup}$, using (14) and (21), it is clear that

$$q_k^{-1}(\eta\,\bar{\sqcup\!\sqcup}\,\xi) = q_k^{-1}(\eta\sqcup\!\sqcup\xi + \eta\Diamond\xi) = q_k^{-1}(\eta)\sqcup\!\sqcup\xi + \eta\sqcup\!\sqcup q_k^{-1}(\xi) + q_k^{-1}(\eta\Diamond\xi).$$

Thus, it only remains to be shown how the left-shift operator acts on $\Diamond$. For $q_i, q_j, q_k \in XY$, $\eta', \xi' \in XY^*$, $\eta = q_i\eta'$ and $\xi = q_j\xi'$, it follows that

$$\begin{aligned} q_k^{-1}(\eta\Diamond\xi) &= q_k^{-1}\left(q_i(\eta'\Diamond\xi) + q_j(\eta\Diamond\xi') + \delta_{q_i,q_j}q_i\left(\eta'\sqcup\!\sqcup\xi' + \eta'\Diamond\xi'\right)\right) \\ &= \delta_{ki}\left(\eta'\Diamond\xi\right) + \delta_{kj}\left(\eta\Diamond\xi'\right) + \delta_{q_i,q_j}\,\delta_{ki}\left(\eta'\sqcup\!\sqcup\xi' + \eta'\Diamond\xi'\right). \end{aligned}$$

Note that the factor $\delta_{q_i,q_j}\ \delta_{ki}$ is nonzero only if $q_k = q_i = q_j \in Y$. Therefore, one can rewrite $\delta_{q_i,q_j}\delta_{ki} = \bar{\delta}_{q_k}\delta_{ki}\delta_{kj}$. Since all the products involved are bilinear, clearly

$$\begin{aligned} q_k^{-1}(\eta\Diamond\xi) &= \delta_{ki}\left(\eta'\Diamond\xi\right) + \delta_{kj}\left(\eta\Diamond\xi'\right) + \bar{\delta}_{q_k}\delta_{ki}\delta_{kj}\left(\eta' \sqcup\!\sqcup \xi' + \eta'\Diamond\xi'\right) \\ &= q_k^{-1}(\eta)\Diamond\xi + \eta\Diamond q_k^{-1}(\xi) + \bar{\delta}_{q_k}(q_k^{-1}(\eta) \sqcup\!\sqcup q_k^{-1}(\xi) + q_k^{-1}(\eta)\Diamond q_k^{-1}(\xi)). \end{aligned}$$

The desired identity is thus

$$q_k^{-1}(\eta\,\overline{\sqcup\!\sqcup}\,\xi) = q_k^{-1}(\eta)\,\overline{\sqcup\!\sqcup}\,\xi + \eta\,\overline{\sqcup\!\sqcup}\,q_k^{-1}(\xi) + \bar{\delta}_{q_k}\left(q_k^{-1}(\eta)\,\overline{\sqcup\!\sqcup}\,q_k^{-1}(\xi)\right).$$

□

An important identity concerning the Lie derivative is presented next.

Lemma 23 *Let $\varphi_1, \ldots, \varphi_r \in \mathcal{C}_\omega(W)$, $\eta \in XY^*$ and $z_0 \in \mathbb{R}^n$. Then,*

$$\begin{aligned} &L_\eta(\varphi_1\cdots\varphi_r)(z_0) \\ &\quad= \sum_{\xi_1,\cdots,\xi_r\in XY^*} L_{\xi_1}\varphi_1(z_0)\cdots L_{\xi_r}\varphi_r(z_0)(\xi_1\,\overline{\sqcup\!\sqcup}\cdots\overline{\sqcup\!\sqcup}\,\xi_r,\eta). \end{aligned} \tag{22}$$

Proof Identity (22) will be proved by a nested induction, specifically an inner induction over the length of η and an outer induction over r. The case for a fixed $r = 1$ is trivial, so for the initial step fix, $r = 2$. If one defines $I_{\sqcup\!\sqcup}(\eta) = \{(\xi,\nu) \in XY^* \times XY^* : (\xi \sqcup\!\sqcup \nu, \eta) \neq 0\}$ and $I_\Diamond(\eta) = \{(\xi,\nu) \in XY^*\times XY^* : (\xi\Diamond\nu,\eta) \neq 0\}$, then $I_{\overline{\sqcup\!\sqcup}}(\eta) = I_\Diamond(\eta) \cup I_{\sqcup\!\sqcup}(\eta)$. Note that if there is a pair $(\xi,\nu) \in I_\Diamond(\eta) \cap I_{\sqcup\!\sqcup}(\eta)$, this would imply that $|\xi\nu| = |\eta|$ and, at the same time, $|\xi\nu| = |\eta| + 1$; therefore, $I_\Diamond(\eta) \cap I_{\sqcup\!\sqcup}(\eta) = \varnothing$. The $\eta = \emptyset$ case is trivial. For $\eta = x_i$, observe that $I_{\sqcup\!\sqcup}(x_i) = \{(x_i,\emptyset), (\emptyset, x_i)\}$ and $I_\Diamond(x_i) = \varnothing$. Thus, writing $\varphi_1 = L_\emptyset(\varphi_1)$ and $\varphi_2 = L_\emptyset(\varphi_2)$, observe that

$$\begin{aligned} L_{x_i}(\varphi_1\varphi_2)(z_0) &= \left(L_{x_i}\varphi_1(z_0)\right)\varphi_2(z_0) + \varphi_1(z_0)L_{x_i}\varphi_2(z_0) \\ &= \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)(\xi_1\,\overline{\sqcup\!\sqcup}\,\xi_2, x_i). \end{aligned}$$

If, on the other hand, $\eta = y_i$, then $I_{\sqcup\!\sqcup}(y_i) = \{(y_i,\emptyset), (\emptyset, y_i)\}$ and $I_\Diamond(y_i) = \{(y_i, y_i)\}$. Thus,

$$\begin{aligned} L_{y_i}(\varphi_1\varphi_2)(z_0) &= \left(L_{y_i}\varphi_1(z_0)\right)\varphi_2(z_0) + \varphi_1(z_0)L_{y_i}\varphi_2(z_0) + L_{y_i}\varphi_1(z_0)L_{y_i}\varphi_2(z_0) \\ &= \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)\left(\xi_1\,\overline{\sqcup\!\sqcup}\,\xi_2, y_i\right). \end{aligned}$$

Hence, identity (22) holds when $r = 2$ and $|\eta| = 1$. Now assume (22) holds for $r = 2$ and $|\eta| = k \geq 0$. It is easy to show that for any $q_i \in XY$,

$$\begin{aligned} &L_{q_i}(\varphi_1\varphi_2)(z_0) \\ &\quad= \left(L_{q_i}\varphi_1(z_0)\right)\varphi_2(z_0) + \varphi_1(z_0)L_{q_i}\varphi_2(z_0) + \bar{\delta}_{q_i}L_{q_i}\varphi_1(z_0)L_{q_i}\varphi_2(z_0). \end{aligned} \tag{23}$$

Then, using (23), it follows that

$$\begin{aligned}
&L_{q_i\eta}(\varphi_1\varphi_2)(z_0)\\
&\quad= L_\eta((L_{q_i}\varphi_1(z_0))\,\varphi_2(z_0) + \varphi_1(z_0)L_{q_i}\varphi_2(z_0) + \bar{\delta}_{q_i}L_{q_i}\varphi_1(z_0)L_{q_i}\varphi_2(z_0))\\
&\quad= \sum_{\xi_1',\xi_2\in XY^*} L_{q_i\xi_1'}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)(\xi_1' ⧢ \xi_2,\eta)\\
&\qquad+ \sum_{\xi_1,\xi_2'\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{q_i\xi_2'}\varphi_2(z_0)(\xi_1 ⧢ \xi_2',\eta)\\
&\qquad+ \sum_{\xi_1',\xi_2'\in XY^*} \bar{\delta}_{q_i}L_{q_i\xi_1'}\varphi_1(z_0)L_{q_i\xi_2'}\varphi_2(z_0)(\xi_1' ⧢ \xi_2',\eta).
\end{aligned}$$

Make the change of variables $\xi_j = q_i\xi_j'$ in each summation above, which implies that $q_i^{-1}(\xi_j) = \xi_j'$ when $\xi_j = q_i\xi_j'$ and 0 otherwise. Then, from identity (20), one has that

$$\begin{aligned}
L_{q_i\eta}(\varphi_1\varphi_2)(z_0) &= \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)\left(q_i^{-1}(\xi_1) ⧢ \xi_2,\eta\right)\\
&\quad+ \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)\left(\xi_1 ⧢ q_i^{-1}(\xi_2),\eta\right)\\
&\quad+ \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)\left(\bar{\delta}_{q_i}q_i^{-1}(\xi_1) ⧢ q_i^{-1}(\xi_2),\eta\right)\\
&= \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)\left(q_i^{-1}(\xi_1 ⧢ \xi_2),\eta\right)\\
&= \sum_{\xi_1,\xi_2\in XY^*} L_{\xi_1}\varphi_1(z_0)L_{\xi_2}\varphi_2(z_0)\left(\xi_1 ⧢ \xi_2, q_i\eta\right).
\end{aligned}$$

Now the outer induction over r. Assume (22) holds up to some fixed $r \geq 2$. Then, for any $\eta,\xi \in XY^*$,

$$\begin{aligned}
&L_\eta(\varphi_1\cdots\varphi_{r+1})(z_0)\\
&\quad= \sum_{\nu,\xi_{r+1}\in XY^*} L_\nu(\varphi_1\cdots\varphi_r)(z_0)L_{\xi_{r+1}}\varphi_{r+1}(z_0)(\nu ⧢ \xi_{r+1},\eta)\\
&\quad= \sum_{\nu,\xi_{r+1}\in XY^*}\left(\sum_{\xi_1,\ldots,\xi_r\in XY^*} L_{\xi_1}\varphi_1(z_0)\cdots L_{\xi_r}\varphi_r(z_0)(\xi_1 ⧢ \cdots ⧢ \xi_r,\nu)\right)\cdot\\
&\qquad\qquad L_{\xi_{r+1}}\varphi_{r+1}(z_0)(\nu ⧢ \xi_{r+1},\eta)\\
&\quad= \sum_{\xi_1,\ldots,\xi_{r+1}\in XY^*} L_{\xi_1}\varphi_1(z_0)\cdots L_{\xi_r}\varphi_r(z_0)L_{\xi_{r+1}}\varphi_{r+1}(z_0)\cdot\\
&\qquad\qquad \sum_{\nu\in XY^*}(\xi_1 ⧢ \cdots ⧢ \xi_r,\nu)(\nu ⧢ \xi_{r+1},\eta).
\end{aligned} \tag{24}$$

Now, for any $p_1, p_2 \in \mathbb{R}\langle XY\rangle$, one has that

$$(p_1 ⧢ p_2,\nu) = \sum_{\eta\in XY^*}(p_1,\eta)(\eta ⧢ p_2,\nu).$$

Thus, letting $p_1 = \xi_1 ⧢ \cdots ⧢ \xi_r$ and $p_2 = \xi_{r+1}$ in (24), one has that the summation over ν gives

$$\begin{aligned} &L_\eta(\varphi_1 \cdots \varphi_{r+1})(z_0) \\ &= \sum_{\xi_1,\cdots,\xi_{r+1} \in XY^*} L_{\xi_1}\varphi_1(z_0) \cdots L_{\xi_{r+1}}\varphi_{r+1}(z_0)(\xi_1 ⧢ \cdots ⧢ \xi_{r+1}, \eta). \end{aligned}$$

This completes the proof □

The next lemma provides an upper bound for the growth rate of the operator L_η as a function of $|\eta|$. The special case where $\eta \in X^*$ appeared in [15, 17].

Lemma 24 *Let $f_0, \ldots, f_m, g_0, \ldots, g_m \in \mathcal{C}^n_\omega(W)$ and $h \in \mathcal{C}^\ell_\omega(W)$. Then, there exists real numbers $K, M > 0$ such that*

$$|L_\eta h(z_0)| \leq K 2^{|\eta|_Y} (2nKM)^{|\eta|_X} |\eta|_X!, \quad \forall \eta \in XY^*. \tag{25}$$

Proof Since $f_0, \ldots, f_m, g_0, \ldots, g_m, h$ are analytic in a neighborhood W of $z_0 \in \mathbb{R}^n$, each has a convergent series representation, say

$$f_i(z) = \sum_{\theta \in \tilde{X}^*} (c_{f_i}, \theta) \frac{(z - z_0)^\theta}{\theta!}, \quad g_i(z) = \sum_{\theta \in \tilde{X}^*} (c_{g_i}, \theta) \frac{(z - z_0)^\theta}{\theta!}, \tag{26}$$

$$h(z) = \sum_{\theta \in \tilde{X}^*} (c_h, \theta) \frac{(z - z_0)^\theta}{\theta!},$$

where $\tilde{X} = \{\tilde{x}_1, \cdots, \tilde{x}_n\}$ is a commutative alphabet, i.e., $\tilde{x}_i \tilde{x}_j = \tilde{x}_j \tilde{x}_i$. Thus, for any $\theta = \tilde{x}_{i_1} \cdots \tilde{x}_{i_k} \in \tilde{X}^*$, it can be assumed that the letters are ordered such that $i_\ell \leq i_{\ell+1}$ for $\ell = 1, \ldots, k-1$. The set of all such series is denoted by $\mathbb{R}[[\tilde{X}]]$. In addition, let $(z - z_0)^\theta \triangleq (z - z_0)_{i_1} \cdots (z - z_0)_{i_k}$ and $\theta! \triangleq |\theta|_{\tilde{x}_1}! \cdots |\theta|_{\tilde{x}_n}!$ (here, $(\cdot)_i$ refers to the i-th component of a vector). Clearly, one can always find some $K, M > 0$ such that

$$|(c_h, \theta)| \leq K M^{|\theta|} \theta!, \quad |(c_{f_i}, \theta)| \leq K M^{|\theta|} \theta!, \quad |(c_{g_i}, \theta)| \leq K M^{|\theta|} \theta!, \quad \forall \theta \in \tilde{X}^*.$$

Substituting (26) into the definition of the operator $L_\eta h(z)$, it follows that

$$L_\eta h(z) = \sum_{\theta \in \tilde{X}^*} (c_\eta, \theta) \frac{(z - z_0)^\theta}{\theta!}$$

for some $c_\eta \in \mathbb{R}^\ell[[\tilde{X}]]$. Without loss of generality, assume $z_0 = 0$. It will be shown next that

$$|(c_\eta, \theta)| \leq \left(D^\eta(b), x^{|\theta|}\right) \theta!, \quad \forall \eta \in XY^*, \quad \theta \in \tilde{X}^*, \tag{27}$$

where $b \triangleq \sum_{\ell=0}^\infty K M^\ell x^\ell$, x is a letter in an arbitrary one letter alphabet, and

$$D^\eta(b) \triangleq \begin{cases} nb \dfrac{\partial D^{\eta'}(b)}{\partial x} & : q_i \in X \\ D^{\eta'}(b) \left(((m+1)(K+1)M)^{|\beta|} + 1\right) & : q_i \in Y, \end{cases}$$

with $\eta = q_i \eta'$, $\eta' \in XY^*$ and $D^\emptyset(b) = b$. In which case, (25) follows directly by noting that $|L_\eta h(z_0)| = |(c_\eta, \emptyset)|$.

Inequality (27) is proven by induction over $|\eta|$. The $\eta = \emptyset$ case is trivial. If $\eta = x_i$, then

$$L_{x_i} h(z) = L_{f_i} h(z) = \sum_{j=1}^n (f_i(z))_j \frac{\partial h}{\partial z_j}(z). \tag{28}$$

Given that $\tilde{X}$ is commutative, observe

$$\frac{\partial}{\partial z_j}\left(\frac{z^\theta}{\theta!}\right) = \frac{\partial}{\partial z_j}\left(\frac{z_1^{|\theta|_{\tilde{x}_1}}\cdots z_n^{|\theta|_{\tilde{x}_n}}}{|\theta|_{\tilde{x}_1}!\cdots|\theta|_{\tilde{x}_n}!}\right) = \frac{z_1^{|\theta|_{\tilde{x}_1}}\cdots z_j^{|\theta|_{\tilde{x}_j}-1}\cdots z_n^{|\theta|_{\tilde{x}_n}}}{|\theta|_{\tilde{x}_1}!\cdots(|\theta|_{\tilde{x}_j}-1)!\cdots|\theta|_{\tilde{x}_n}!}.$$

If θ is in the support of series $\frac{\partial}{\partial z_j}h$, then it can be factored into the form $\theta = \tilde{x}_j\beta$ for $\beta \in \tilde{X}^*$, which allows to write $\frac{\partial}{\partial z_j}\left(\frac{z^\theta}{\theta!}\right) = \frac{z^\beta}{\beta!}$. Substituting the series expansions of $f_i(z)$ and $\frac{\partial}{\partial z_j}h(z)$ into (28) gives

$$\begin{aligned}
L_{x_i}h(z) &= \sum_{j=1}^{n}\left(\sum_{\alpha\in\tilde{X}^*}((c_{f_i},\alpha))_j\frac{z^\alpha}{\alpha!}\right)\left(\sum_{\beta\in\tilde{X}^*}(c_h,\tilde{x}_j\beta)\frac{z^\beta}{\beta!}\right) \\
&= \sum_{\alpha,\beta\in\tilde{X}^*}\sum_{j=1}^{n}((c_{f_i},\alpha))_j(c_h,\tilde{x}_j\beta)\frac{z^{\alpha\beta}}{\alpha!\beta!} \\
&= \sum_{\theta\in\tilde{X}^*}\underbrace{\left(\sum_{\substack{\alpha,\,\beta\,\in\,\tilde{X}^* \\ \theta=\alpha\beta}}\sum_{j=1}^{n}((c_{f_i},\alpha))_j(c_h,\tilde{x}_j\beta)\frac{\theta!}{\alpha!\beta!}\right)}_{(c_{x_i},\theta)}\frac{z^\theta}{\theta!}.
\end{aligned}$$

The coefficient (c_{x_i},θ) can be upper bounded as follows:

$$\begin{aligned}
|(c_{x_i},\theta)| &\le \sum_{\substack{\alpha,\,\beta\,\in\,\tilde{X}^* \\ \theta=\alpha\beta}}\sum_{j=1}^{n}KM^{|\alpha|}\alpha!\,KM^{|\beta|+1}(|\beta|+1)!\frac{\theta!}{\alpha!\beta!} \\
&\le n\sum_{\substack{\alpha,\,\beta\,\in\,\tilde{X}^* \\ \theta=\alpha\beta}}KM^{|\alpha|}\,KM^{|\beta|+1}(|\beta|+1)\theta! \\
&= n\sum_{l=0}^{|\theta|}KM^l\,KM^{|\theta|-l+1}(|\theta|-l+1)\theta! \\
&= n\left(\sum_{l=0}^{\infty}KM^lx^l\sum_{l=0}^{\infty}KM^{l+1}(l+1)x^l,x^{|\theta|}\right)\theta! \\
&= n\left(b\frac{\partial b}{\partial x},x^{|\theta|}\right)\theta!.
\end{aligned}$$

If $\eta = y_i$, then $L_{y_i}h(z) = h(z+g_i(z)) - h(z)$. It is obvious that $\bar{g}_i(z) \triangleq z + g_i(z) \in \mathcal{C}^n_\omega(W)$, which implies that $K, M > 0$ can be chosen to also ensure that $|(c_{\bar{g}_i}, \theta)| \leq KM^{|\theta|}\theta!$. Thus,

$$\begin{aligned}
L_{y_i}h(z) &= \sum_{\theta\in\tilde{X}^*} \frac{(c_h,\theta)}{\theta!}\left(\sum_{\alpha\in\tilde{X}^*}(c_{\bar{g}_i},\alpha)\frac{z^\alpha}{\alpha!}\right)^\theta - \sum_{\theta\in\tilde{X}^*}(c_h,\theta)\frac{z^\theta}{\theta!} \\
&= \sum_{\theta\in\tilde{X}^*} \frac{(c_h,\theta)}{\theta!}\left(\sum_{\alpha_1,\ldots,\alpha_{|\theta|}\in\tilde{X}^*}\prod_{\ell=1}^{|\theta|}\frac{((c_{\bar{g}_i},\alpha_\ell))_{i_\ell} z^{\alpha^\ell}}{\alpha_\ell!} - z^\theta\right) \\
&= \sum_{\theta\in\tilde{X}^*} \underbrace{(c_h,\theta)\left(\sum_{n=1}^{|\theta|}\sum_{\substack{\alpha_1,\ldots,\alpha_{|\theta|}\in\tilde{X}^* \\ \alpha_1\cdots\alpha_{|\theta|}=\theta}}\prod_{\ell=1}^{|\theta|}\frac{((c_{\bar{g}_i},\alpha_\ell))_{i_\ell}}{\alpha_\ell!} - 1\right)}_{\left(c_{y_i},\theta\right)}\frac{z^\theta}{\theta!}.
\end{aligned}$$

Similarly, an upper bound for $\left(c_{y_i},\theta\right)$ can be calculated as

$$\begin{aligned}
\left|\left(c_{y_i},\theta\right)\right| &\leq KM^{|\theta|}\left(\sum_{n=1}^{|\theta|}\sum_{\substack{\alpha_1,\ldots,\alpha_{|\theta|}\in\tilde{X}^* \\ \alpha_1\cdots\alpha_{|\theta|}=\theta}} K^n M^{|\theta|} + 1\right)\theta! \\
&\leq KM^{|\theta|}\left(M^{|\theta|}\sum_{n=1}^{|\theta|}K^n\frac{|\theta|!}{|\theta|_{\tilde{x}_0}!\cdots|\theta|_{\tilde{x}_m}!}\binom{|\theta|-1}{n-1} + 1\right)\theta! \\
&\leq KM^{|\theta|}\left(M^{|\theta|}(m+1)^{|\theta|}\sum_{n=1}^{|\theta|}K^n\binom{|\theta|-1}{n-1} + 1\right)\theta! \\
&\leq KM^{|\theta|}\left(M^{|\theta|}(m+1)^{|\theta|}\sum_{n=1}^{|\theta|}K^n\binom{|\theta|}{n} + 1\right)\theta! \\
&\leq KM^{|\theta|}\left(((m+1)(K+1)M)^{|\theta|} + 1\right)\theta! \\
&= \left(\sum_{l=0}^{\infty}KM^l x^l\left(((m+1)(K+1)M)^{|\theta|}+1\right), x^{|\theta|}\right)\theta! \\
&= \left(b\left(((m+1)(K+1)M)^{|\theta|}+1\right), x^{|\theta|}\right)\theta!.
\end{aligned}$$

Now, assume that (27) holds for words η up to length n. Then, the upper bound for $|(c_{\eta x_i},\theta)|$ is

$$
\begin{aligned}
|(c_{\eta x_i},\theta)| &\leq \sum_{\substack{\alpha,\beta\in\tilde{X}^*\\ \theta=\alpha\beta}} \sum_{j=1}^{n} |((c_{f_i},\alpha))_j||(c_\eta,\tilde{x}_j\beta)|\frac{\theta!}{\alpha!\beta!}\\
&\leq \sum_{\substack{\alpha,\beta\in\tilde{X}^*\\ \theta=\alpha\beta}} \sum_{j=1}^{n} KM^{|\alpha|}\alpha!\left(D^\eta(b),x^{|\beta|+1}\right)(\tilde{x}_j\beta)!\frac{\theta!}{\alpha!\beta!}\\
&\leq n \sum_{\substack{\alpha,\beta\in\tilde{X}^*\\ \theta=\alpha\beta}} KM^{|\alpha|}\left(D^\eta(b),x^{|\beta|+1}\right)(|\beta|+1)\theta!\\
&= \left(nb\frac{\partial D^\eta(b)}{\partial x},x^{|\theta|}\right)\theta!.
\end{aligned}
$$

For $\left|\left(c_{\eta y_i},\theta\right)\right|$ it follows that

$$
\begin{aligned}
|\left(c_{\eta y_i},\theta\right)| &\leq \frac{|(c_\eta,\theta)|}{\theta!}\left(\sum_{n=1}^{|\theta|}\sum_{\substack{\alpha_1,\ldots,\alpha_{|\theta|}\in\tilde{X}^*\\ \alpha_1\cdots\alpha_{|\theta|}=\theta}} K^n M^{|\theta|}+1\right)\theta!\\
&\leq \left(D^\eta(b),x^{|\theta|}\right)\left(((m+1)(K+1)M)^{|\theta|}+1\right)\theta!\\
&= \left(D^{\eta y_i}(b),x^{|\theta|}\right)\theta!.
\end{aligned}
$$

Hence, (27) is valid for all $\eta\in XY^*$. Now, evaluating (27) at $\theta=\emptyset$ gives

$$
|L_\eta h(0)| = |(c_\eta,\emptyset)| \leq \left(D^\eta(b),\emptyset\right).
$$

To finish the proof, it is only necessary to calculate an upper bound for $(D^\eta(b),\emptyset)$. Observe that any $\eta\in XY^*$ can be written as $\eta=\eta_0 x_{i_1}\eta_1\cdots\eta_{k-1}x_{i_k}\eta_k$ with $\eta_i\in Y^{n_i}$, $i=0,1,\ldots,k$. Since $b=(1-Mx)^{-1}$, it follows that

$$
\begin{aligned}
D^\eta(b) &= D^{\eta_1 x_{i_1}\eta_2\cdots\eta_{k-1}x_{i_k}\eta_k}(b)\frac{K(nKM)}{(1-Mx)^3}\left(((m+1)(K+1)M)^{|\theta|}+1\right)^{n_0}\\
&= D^{\eta_2\cdots\eta_{k-1}x_{i_k}\eta_k}(b)\frac{K(nKM)^2 1\cdot 3}{(1-Mx)^5}\left(((m+1)(K+1)M)^{|\theta|}+1\right)^{n_0+n_1}\\
&= \frac{K(nKM)^k 1\cdot 3\cdots(2k+1)}{(1-Mx)^{2k+1}}\left(((m+1)(K+1)M)^{|\theta|}+1\right)^{\sum_{i=0}^k n_i}\\
&= \frac{K(nKM)^k(2k)!}{(1-Mx)^{2k+1}2^k k!}\left(((m+1)(K+1)M)^{|\theta|}+1\right)^{\sum_{i=0}^k n_i}.
\end{aligned}
$$

Observe that $\sum_{i=0}^k n_i=|\eta|_Y$ and $k=|\eta|_X$. Using the fact that $\frac{(2k)!}{k!k!}\leq 2^{2k}$, one has that

$$
(D^\eta(b),\emptyset) = \left.\frac{K(nKM)^k(2k)!}{(1-Mx)^{2k+1}2^k k!}2^{|\eta|_Y}\right|_{x=0} \leq K(2nKM)^{|\eta|_X}|\eta|_X!\,2^{|\eta|_Y}.
$$

It therefore follows immediately that

$$|L_\eta h(0)| \leq (D^\eta(b), \emptyset) \leq K 2^{|\eta|_Y} (2nKM)^{|\eta|_X} |\eta|_X!, \quad \forall \eta \in XY^*.$$

□

The final lemma needed is an extension of the "Fliess pre-lemma" [6, Proposition III.1].

Lemma 25 *Let $\varphi \in \mathcal{C}_\omega^n(W)$ and suppose that*

$$z(t) = \sum_{\eta \in XY^*} L_\eta \varphi(z_0) E_\eta[w](t),$$

where $w = (u, \bar{v})$, $u \in B_p^m(R)[0, T]$, and $\bar{v}$ is a decomposition of a Poisson process N of k-types. Then, for any $h \in \mathcal{C}_\omega^\ell(\varphi(W))$,

$$h(z(t)) = \sum_{\eta \in XY^*} L_\eta (h \circ \varphi)(z_0) E_\eta[w](t).$$

Proof Since, by assumption, h is analytic on a neighborhood of $\varphi(z_0)$, it can be written as

$$h(z) = \sum_{\theta \in \tilde{X}^*} (c_h, \theta) \frac{(z - \varphi(z_0))^\theta}{\theta!} \tag{29}$$

for some $c_h \in \mathbb{R}^\ell[[\tilde{X}]]$. Observe that $z(t) = F_{c_z}[w](t)$, where the generating series $c_z \in \mathbb{R}^n\langle\langle XY \rangle\rangle$ is given by $(c_z, \eta) = L_\eta \varphi(z_0)$. Using Lemmas 21 and 23 and letting $\theta = \tilde{x}_{i_1} \cdots \tilde{x}_{i_{|\theta|}}$, it follows that

$$\begin{aligned}
&(z(t) - \varphi(z_0))^\theta \\
&= \left(F_{(c_z - \varphi(z_0))}[w](t)\right)^\theta \\
&= \sum_{\xi_1, \cdots, \xi_{|\theta|} \in XY^*} ((c_z - \varphi(z_0), \xi_1))_{i_1} \cdots ((c_z - \varphi(z_0), \xi_{|\theta|}))_{i_{|\theta|}} \cdot \\
&\qquad E_{\xi_1}[w](t) \cdots E_{\xi_{|\theta|}}[w](t) \\
&= \sum_{\eta \in XY^*} \sum_{\xi_1, \cdots, \xi_{|\theta|} \in XY^*} L_{\xi_1}(\varphi - \varphi(z_0))_{i_1}(z_0) \cdots L_{\xi_{|\theta|}}(\varphi - \varphi(z_0))_{i_{|\theta|}}(z_0) \cdot \\
&\qquad (\xi_1 ⧢ \cdots ⧢ \xi_{|\theta|}, \eta)\, E_\eta[w](t) \\
&= \sum_{\eta \in XY^*} L_\eta \Big(\underbrace{(\varphi - \varphi(z_0))_{i_1} \cdots (\varphi - \varphi(z_0))_{i_{|\theta|}}}_{(\varphi - \varphi(z_0))^\theta} \Big)(z_0)\, E_\eta[w](t).
\end{aligned} \tag{30}$$

Substituting (30) into (29) gives

$$
\begin{aligned}
h(z(t)) &= \sum_{\eta\in XY^*} L_\eta \left(\sum_{\theta\in\tilde{X}^*} \frac{(c_h,\theta)}{\theta!} (\varphi - \varphi(z_0))^\theta \right)(z_0)\, E_\eta[w](t) \\
&= \sum_{\eta\in XY^*} L_\eta (h\circ\varphi)(z_0)\, E_\eta[w](t).
\end{aligned}
$$

This completes the proof. □

One can now prove Theorem 6 by applying Lemmas 23, 24, and 25.

Proof of Theorem 6 The objective is to write any switched input-affine nonlinear system with a Poisson switching signal as a Fliess operator. Observe that if the system has $k+1$ modes then

$$
\begin{aligned}
\dot{z} &= f_0(z) + \sum_{i=0}^{m} g_{0i}(z)u_i + \sum_{j=1}^{k} (f_j(z) - f_0(z))v_j \\
&\quad + \sum_{i=0,j=1}^{m,k} (g_{ji}(z) - g_{0i}(z))u_i\, v_j, \\
y &= h_0(z) + \sum_{j=1}^{k} (h_j(z) - h_0(z))\, v_j,
\end{aligned}
$$

where the integrals of the v_j's come from a Poisson switching signal, N, of k-types with probabilities p_j for $j = 1, \ldots, k$. That is, for each $t \in [0, T]$, either all v_j's are zero or just a single $v_j = 1$. It is sufficient to show that one can write the following switched system as a Fliess operator

$$
\dot{z} = \sum_{i=0}^{m} f_i(z)\, u_i + \sum_{i=0,j=1}^{m,k} g_{ji}(z)\, u_i\, v_j, \tag{31}
$$

$$
y = h_0(z) + \sum_{j=1}^{k} (h_j(z) - h_0(z))\, v_j, \tag{32}
$$

where $u \in B_p^m(R)[0, T]$ and f_i, g_{ji} are analytic vector fields on some neighborhood of $z_0 \in \mathbb{R}^n$. Assume for brevity that $m = k = 1$ and $f_0 = g_{10} = 0$. Note that $v = \Delta N$, so abusing the notation, let $v = dN$ and $z_t = z(t)$. In integral form, (31) becomes (after dropping the subscripts)

$$
z_t = \int_0^t f(z_s)\, u_s\, ds + \int_0^t g(z_{s-})\, u_{s-}\, dN(s). \tag{33}
$$

From the properties of the Poison integral, $z_t = z_{t-} + g(z_{t-})$ when $\Delta N(t) \neq 0$ and 0 otherwise. Given a differentiable function, F, the change of variables formula (8) for Poisson integrals produces

$$
\begin{aligned}
&F(z_t) - F(z_0) \\
&\quad = \int_0^t F'(z_{s-})\ (f(z_s)\,u_s\,ds + g(z_{s-})\,u_{s-}\,dN(s)) + \frac{1}{2}\int_0^t F''(z_{s-})\,d\,\underbrace{[z,z]_s^c}_{0} \\
&\qquad + \sum_{0<s\le t} (F(z_s) - F(z_{s-})) - \sum_{0<s\le t} F'(z_{s-})g(z_{s-})\,u_{s-}\,\Delta N(s) \\
&\quad = \int_0^t \left(f(z_s)\,\frac{\partial}{\partial z}F(z_s)\right)\,u_s\,ds + \int_0^t (F(z_s) - F(z_{s-}))\ u_{s-}\,dN(s).
\end{aligned} \tag{34}
$$

Recall that $u = 1$ at the time of each jump of N. Using this equation, one can identify the operators $L_f F(z) \triangleq f(z)\frac{\partial F(z)}{\partial z}$ and $\Delta_g F(z) \triangleq F(z+g(z)) - F(z)$. Now, let $F(z)$ in (34) be replaced by either $f(z)$ or $g(z)$ and then substitute for $f(z)$ and $g(z)$ into (33). This yields

$$
z_t = z_0 + f(z_0)\int_0^t u_s\,ds + g(z_0)\int_0^t u_{s-}\,dN(s) + R_1(z_t),
$$

where $R_1(z_t)$ contains all the iterated integrals of order 2 whose integrands do not depend on z_0. In light of (5)-(6), define $X = \{x_1\}$, $Y = \{y_1\}$. Note here that the operator L_η defined in (18) can be used to describe the coefficients of the series expansion of the solution of the state equation, z_t, by identifying $L_{x_1} = L_f$ and $L_{y_1} = \Delta_g$. Repeating this procedure iteratively yields the Peano-Baker formula for equation (31), namely

$$
z(t) = \sum_{\eta\in XY^*} L_\eta \mathrm{id}(z_0)\ E_\eta[w](t), \tag{35}
$$

where id denotes the identity map on W. Thus, (f, g, id, z_0) realizes the operator F_{c_z} driven by u and a Poisson process N of k types, where $(c_z, \eta) = L_\eta \mathrm{id}(z_0)$, $\forall \eta \in XY^*$. Lemma 24 provides that c_z is locally convergent, and therefore, as a result of Theorem 5, series (35) is convergent in the mean for all $t \in [0, T]$. Note now that if (35) is the solution of (31), then

$$
\begin{aligned}
dz(t) &= \sum_{\eta\in XY^*} L_\eta L_f \mathrm{id}(z_0) E_\eta[w](t)u(t)dt \\
&\quad + \sum_{\eta\in XY^*} L_\eta \Delta_g \mathrm{id}(z_0) E_\eta[w](t-)u(t-)dN(t) \\
&= \sum_{\eta\in XY^*} L_\eta f(z_0) E_\eta[w](t)u(t)dt \\
&\quad + \sum_{\eta\in XY^*} L_\eta g(z_0) E_\eta[w](t-)u(t-)\,dN(t).
\end{aligned}
$$

Thus, by Lemma 25, it follows that

$$
\begin{aligned}
z(t) &= \int_0^t f\left(\sum_{\eta\in XY^*} L_\eta \mathrm{id}(z_0) E_\eta[w](s)\right) u(s)ds \\
&\quad + \int_0^t g\left(\sum_{\eta\in XY^*} L_\eta \mathrm{id}(z_0) E_\eta[w](s-)\right) u(s-)dN(s) \\
&= \int_0^t f(z(s))u(s)ds + \int_0^t g(z(s-))u(s-)dN(s).
\end{aligned}
$$

Furthermore, for $h_j \in \mathcal{C}^\ell_\omega(W)$, $j = 0, 1$, Lemma 25 gives

$$h_j(z(t)) = F_{c_j}[w](t) = \sum_{\eta \in XY^*} L_\eta h_j(z_0)\; E_\eta[w](t),$$

where $c_j \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$ and $(c_j, \eta) = L_\eta h_j(z_0)$. Therefore, the output defined in (32) is equivalent to

$$y(t) = F_{c_0}[w](t) + F_{c_1 - c_0}[w](t)\, v_1(t).$$

Finally, Theorem 17 ensures that the generating series c_j is unique, which completes the proof. □

Example 26 Reconsider the switched bilinear system presented in Example 7, namely

$$\begin{aligned} \dot{z} &= M_{00} z + M_{01} z u + \sum_{j=1}^{k} M_{1j} z v_j, \\ y &= C_0 z + \sum_{j=1}^{k} (C_j - C_0) z v_j, \end{aligned} \tag{36}$$

where the initial condition is $z_0 = (1, \ldots, 1)^T \in \mathbb{R}^{n\times 1}$. Let $XY = \{x_0, x_1, y_1, \ldots, y_k\}$. Then, from (35), for each $h_j(z) = C_j z$ and $w = (0, \bar{v})$,

$$h_j(z(t)) = F_{c_j}[w](t) = \sum_{\eta \in XY^*} (c_j, \eta) E_\eta[w](t) = \sum_{\eta \in XY^*} C_j M_\eta z_0\, E_\eta[w](t),$$

where $M_{x_i\eta} = M_{0i} M_\eta$ and $M_{y_j\eta} = M_{1j} M_\eta$ for $\eta \in XY^*$, $i = 0, 1$ and $j = 1, \ldots, k$. Recall that c_j is globally convergent for $j = 0, 1, \ldots, k$, and hence locally convergent. Thus,

$$y = h_0(z) + \sum_{j=1}^{k} (h_j(z) - h_0(z))\, v_j = F_{c_0}[w](t) + \sum_{j=1}^{k} F_{c_j - c_0}[w](t)\, v_j,$$

which, by Theorem 5, is a well-defined random variable for every $t \in [0, T]$ and some $T > 0$. The generating series of (36) is also equivalent to

$$c_j = C_j \left(I - \sum_{i=0}^{1} M_{0i} x_i - \sum_{j=1}^{k} M_{1j} y_j \right)^{-1} z_0,$$

where formally $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$, and I denotes the identity matrix. Observe that

$$E_{q_i^j}[w](t) = (E_{q_i}[w](t))^{\{j\}} = \int_0^t u_i(s) (E_{q_i}[w](s))^{\{j-1\}} dN_{q_i}(s)$$

for any $q_i \in XY$. Recall, in Example 10, $X^{\{n\}}$ was defined as the n-fold iterated integral of X. The same statement is valid if q_i is replaced by a polynomial. From the integration by parts formula (10), the following identity can be obtained

$$(X + Y)^{\{n\}}(t) = \sum_{k=0}^{n} X^{\{i\}}(t) Y^{\{n-i\}}(t), \tag{37}$$

where X and Y are such that $X(0) = 0$, $Y(0) = 0$ and $[X, Y]_t = 0$. From (37) and assuming that the M_{ji}'s commute, then $F_{c_j}[w](t)$ written as an infinite sum of iterated integrals can be expressed as follows

$$
\begin{aligned}
F_{c_j}[w](t) &= \sum_{l=0}^{\infty} C_j E_{\left(\sum_{i=0}^{1} M_{0i} x_i + \sum_{j=1}^{k} M_{1j} y_j\right)^l}[w](t)\; z_0 \\
&= \sum_{l=0}^{\infty} C_j \left(E_{\left(\sum_{i=0}^{1} M_{0i} x_i\right)}[w](t) \; + E_{\left(\sum_{j=1}^{k} M_{1j} y_j\right)}[w](t) \right)^l z_0 \\
&= C_j \sum_{l=0}^{\infty} \sum_{r=0}^{l} \left(E_{\left(\sum_{i=0}^{1} M_{0i} x_i\right)}[w](t) \right)^{\{r\}} \left(\sum_{j=1}^{k} E_{M_{1j} y_j}[w](t) \right)^{\{l-r\}} z_0 \\
&= C_j \sum_{l=0}^{\infty} \left(E_{\left(\sum_{i=0}^{1} M_{0i} x_i\right)}[w](t) \right)^{\{l\}} \sum_{r=0}^{\infty} \left(\sum_{j=1}^{k} E_{M_{1j} y_j}[w](t) \right)^{\{r\}} z_0 \\
&= C_j \prod_{i=0}^{1} \sum_{l=0}^{\infty} E_{(M_{0i} x_i)^l}[w](t) \prod_{j=1}^{k} \sum_{r=0}^{\infty} E_{(M_{1j} y_j)^r}[w](t)\; z_0 \\
&= C_j \prod_{i=0}^{1} \sum_{l=0}^{\infty} \frac{(M_{0i})^l}{l!} (E_{x_i}[w](t))^l \prod_{j=1}^{k} \sum_{r=0}^{N_j(t)} (M_{1j})^r \underbrace{E_{y_j^r}[w](t)}_{\binom{N_j}{r}} z_0 \\
&= C_j \exp\left(M_{00}\, t + M_{01} \int_0^t u(s)\, ds \right) \prod_{j=1}^{k} \left(I + M_{1j} \right)^{N_j(t)} z_0.
\end{aligned}
$$

Note that the same result can be obtained via the Itô formula (8).

It is also worth pointing out that explicit solutions to (36) can be obtained in terms of exponentials when the vector fields are not commutative. The expression for the logarithm of this exponential (known as the Magnus expansion or the Chen-Strichartz formula) can be developed in terms of iterated Lie brackets [2, 11, 12]. □

5 Conclusions and Future Work

This paper described a class of convergent Fliess operators admitting L_p and Poisson process inputs. It was then shown how Poisson switched input-affine nonlinear systems have an input-output map that can be described in terms of such Fliess operators. In the future, the authors plan to introduce a nondifferential dwell time into the formulation and to extend the approach to switching signals having interarrival times that are not necessarily exponentially distributed, nor have independent increments. Moreover, L_p convergence would be required when considering general semimartingales as driving inputs. The L_2 convergence case, which is for instance the case when considering Wiener processes, can be treated by simply employing the isometry property of Martingales. For $p > 2$, the authors believe that either the Burkholder-Davis-Gundy inequality would be useful or the use of T. Lyons' rough path theory.

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