Left Inversion of Analytic SISO Systems via Formal Power Series Methods

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Abstract: Given a single-input, single-output (SISO) system, *F*, and a function *y* in the range of *F*, the left inversion problem is to determine a unique input *u* such that *y* = *F*[*u*]. The goal of this paper is to provide an explicit analytical solution to this problem in the case where *F* is an analytic mapping in the sense that it has a convergent Chen-Fliess functional expansion, and *y* is a real analytic function. In particular, it will be shown that given a certain condition on the generating series *c* of *F*, a corresponding analytic *u* can always be determined via operations on formal power series. The condition on *c* turns out to be exactly equivalent to having a well-defined relative degree when *F* has an input-affine analytic state space realization with finite dimension. But the method is applicable even when *F* does *not* have such a realization.

Keywords: Nonlinear systems, Chen-Fliess series, Feedback linearization, Interconnected systems, Algebraic systems theory.

1. INTRODUCTION

Given a single-input, single-output (SISO) system, F, and a function y in the range of F, the left inversion problem is to determine a unique input u such that $y = F[u]$. An asymptotic version of this problem is implicit in virtually all output tracking and motion planning algorithms. Given a smooth input-affine state space realization of F with finite dimension, it is known that a well defined relative degree is a sufficient (but not necessary) condition for solving the left inversion problem (Hirschorn, 1979; Tanwani & Liberzon, 2010). In such a setting, this is usually accomplished via the method of feedback linearization (Isidori, 1995; Krener, 1999), which is a type of dynamic system inversion (Getz, 1995). The problem is significantly simplified when the system has *full* relative degree, i.e., is differentially flat (Fliess et al., 1995). The goal of this paper is to provide an explicit analytical solution to this problem in the case where F is an analytic mapping in the sense that it has a convergent Chen-Fliess functional expansion (Fliess, 1981, 1983), and where y is a real analytic function. In which case, the Fliess operator F is completely characterized by a noncommutative formal power series c. In particular, it will be shown that given a certain condition on c, a corresponding analytic u can always be determined via operations on formal power series. The condition on c turns out to be exactly equivalent to having a well-defined relative degree when c has an input-affine analytic state space realization. The method, however, is applicable even when c does *not* have such a realization. The derivation relies heavily on recent advances in the analysis of interconnected nonlinear systems, especially Gray & Duffaut Espinosa (2011, 2012). What the present method highlights is that (static) feedback linearization in the SISO analytic case can be characterized entirely in terms of input-output concepts.

The paper is organized as follows. Some preliminaries concerning Fliess operators and their interconnections are briefly reviewed in the next section. Section 2.3, addressing the quotient connection, is entirely new material motivated by the system inversion problem. Section 3 describes the main results of the paper, a system inversion formula. The paper's conclusions are given in the final section.

2. PRELIMINARIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. If $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, that is, the $\mathbb{R}$-bilinear mapping $\mathbb{R}^\ell\langle\langle X\rangle\rangle \times \mathbb{R}^\ell\langle\langle X\rangle\rangle \to \mathbb{R}^\ell\langle\langle X\rangle\rangle$ uniquely specified by the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi)$$

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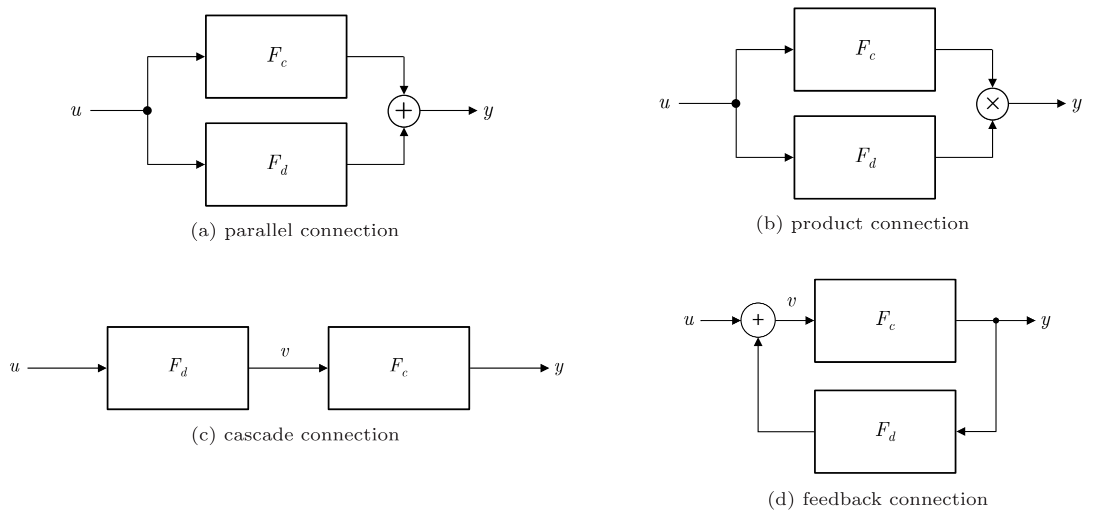

(a) parallel connection

(b) product connection

(c) cascade connection

(d) feedback connection

Fig. 1. Four elementary system interconnections

and $\eta \sqcup\!\sqcup\, \emptyset = \eta$ for all $\eta, \xi \in X^*$ (Fliess, 1981).

2.1 Fliess Operators and Their Convergence

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : \ 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{1}$$

(Fliess, 1981, 1983). If there exists real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \ \ \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0,$ $t_0+T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\ t_0+T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ (Gray & Wang, 2002). (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. When c satisfies the more stringent growth condition

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \ \ \eta \in X^*,$$

the series (1) defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $\mathcal{C}[t_0, \infty)$, where

$$L^m_{\mathfrak{p},e}(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_\mathfrak{p}[t_0, t_1], \ \forall t_1 \in (t_0, \infty)\},$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ (Gray & Wang, 2002). The set of all such *globally convergent* series is designated by $\mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$.

2.2 Elementary System Interconnections

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the parallel and product connections $F_c + F_d$ and F_cF_d as shown in Figs. 1(a) and 1(b) have generating series $c + d$ and $c \sqcup\!\sqcup\, d$, respectively. That is, $F_c + F_d = F_{c+d}$ and $F_cF_d = F_{c \sqcup\!\sqcup d}$. When F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion as shown in Fig. 1(c), the composite system $u \mapsto y$ always has a Fliess operator representation, and the composition product can be used to describe its generating series. Noting that for any $x_i \in X$, $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$

$$E_{x_i\eta}[F_d[u]](t, t_0) = \int_{t_0}^{t} F_{d_i}[u](\tau)\, \underbrace{E_\eta[F_d[u]](\tau, t_0)}_{:=F_{\eta\circ d}[u](\tau)}\, d\tau$$
$$= F_{x_0(d_i \,\sqcup\!\sqcup\, \eta\circ d)}[u](t),$$

it is convenient to first define a family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup\, e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Let $D_\emptyset$ be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_ix_j} := D_{x_i}D_{x_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the catenation product.

Definition 1. (Ferfera, 1979, 1980) The *composition product* of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$(\underbrace{x_{i_k} x_{i_{k-1}} \cdots x_{i_1}}_{\eta}) \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ define

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d.$$

The composition product is associative, distributes to the left over the shuffle product, and has the key property that $F_c \circ F_d = F_{c\circ d}$. While the composition product is $\mathbb{R}$-linear by design in its left argument c, it is linear in its right argument if and only if its left argument is a *linear series*, that is, $\mathrm{supp}(c) \subseteq L$, where

$$L := \{\eta \in X^* : \eta = x_0^{n_1} x_i x_0^{n_0}, \ i \in \{1, 2, \ldots, m\}, \ n_j \geq 0\}$$

is the set of all *linear words*. In addition, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ under the ultrametric dist : $(c, d) \mapsto \sigma^{\mathrm{ord}(c-d)}$ (Ferfera, 1979; Gray & Li, 2005).

Here σ is any real number $0 < \sigma < 1$, and the *order* of a series c, $\mathrm{ord}(c)$, is taken as the length of the smallest word in the support of c.

In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 1(d), the output y must satisfy the feedback equation

$$y = F_c[v] = F_c[u + F_d[y]]$$

for every admissible input u. It was shown by Gray & Li (2005); Gray & Wang (2008) that there always exists a unique generating series e so that $y = F_e[u]$. In which case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d\circ e}[u]] = F_{c\tilde{\circ}(d\circ e)}[u],$$

where $\tilde{\circ}$ denotes the *modified* composition product. That is, the product

$$c\tilde{\circ}d = \sum_{\eta\in X^*} (c,\eta)\, \eta \,\tilde{\circ}\, d,$$

where $\eta \,\tilde{\circ}\, d = \tilde{D}_\eta(1)$ with

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup e)$$

for each letter $x_i \in X$ and $d_0 := 0$. The mapping $d \mapsto c \,\tilde{\circ}\, d$ is also an ultrametric contraction (Gray & Li, 2005; Li, 2004). Therefore, the *feedback product* of c and d, namely $c@d$, is defined as the unique fixed point of the contractive iterated map

$$\tilde{S} : e_i \mapsto e_{i+1} = c\tilde{\circ}(d \circ e_i).$$

Specifically, $c@d = e$, where $e = c\tilde{\circ}(d \circ e)$. Given arbitrary c and d, the feedback product can be computed explicitly using the theory of Hopf algebras (Gray & Duffaut Espinosa, 2011, 2012). Consider, for example, the SISO case where $X = \{x_0, x_1\}$. Define the set of operators

$$\mathscr{F}_\delta = \{I + F_c \;:\; c \in \mathbb{R}\langle\langle X\rangle\rangle\},$$

where I denotes the identity map. It is convenient to introduce the Dirac symbol δ and the definition $F_\delta = I$ such that $I + F_c = F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. In which case,

$$c \,\tilde{\circ}\, d = c \circ (\delta + d).$$

The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. The central idea is that $\mathscr{F}_\delta$ forms a group under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c \,\tilde{\circ}\, d$. In which case, the corresponding group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ is a Faà di Bruno Hopf algebra with antipode, α, satisfying

$$c_\delta^{\circ -1} = \delta + c^{\circ -1} = \delta + \sum_{\eta\in X^*} (\alpha\, a_\eta)(c)\, \eta,$$

where $c^{\circ -1}$ denotes the composition inverse of c,

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta)$$

is the coordinate function for $\eta \in X^*$, and $a_\delta(c_\delta) := 1$. The antipode can be computed directly by either a series expansion or via a matrix representation of the group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ (Gray & Duffaut Espinosa, 2012). The first few antipode terms are:

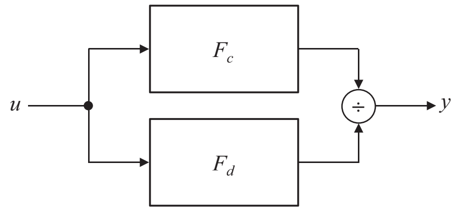

Fig. 2. The quotient connection.

$$\begin{aligned}
\alpha\, 1 &= 1 && \text{(2a)}\\
\alpha\, a_\emptyset &= -a_\emptyset && \text{(2b)}\\
\alpha\, a_{x_0} &= -a_{x_0} + a_\emptyset a_{x_1} && \text{(2c)}\\
\alpha\, a_{x_1} &= -a_{x_1} && \text{(2d)}\\
\alpha\, a_{x_0^2} &= -a_{x_0^2} + a_\emptyset a_{x_0x_1} + a_{x_0}a_{x_1} + a_\emptyset a_{x_1x_0} - && \text{(2e)}\\
&\quad\; a_\emptyset a_{x_1}^2 - a_\emptyset^2 a_{x_1^2} &&\\
\alpha\, a_{x_0x_1} &= -a_{x_0x_1} + a_{x_1}^2 + a_\emptyset a_{x_1^2} && \text{(2f)}\\
\alpha\, a_{x_1x_0} &= -a_{x_1x_0} + a_\emptyset a_{x_1^2} && \text{(2g)}\\
\alpha\, a_{x_1^2} &= -a_{x_1^2} && \text{(2h)}\\
&\;\;\vdots &&
\end{aligned}$$

In this setting, the following theorem holds.

Theorem 2. For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that

$$c@d = c \,\tilde{\circ}\, (-d \circ c)^{\circ -1} = c \circ (\delta - d \circ c)^{\circ -1}.$$

Finally, a few comments about the convergence of interconnected systems. It is known that all four elementary interconnections preserve local convergence (Gray & Duffaut Espinosa, 2011; Gray & Li, 2005; Thitsa, 2011; Thitsa & Gray, 2012). On the other hand, only the parallel and product connections preserve global convergence (Gray, et al., 2009).

2.3 Quotient Connection

In addition to the elementary systems interconnections described above, the quotient interconnection shown in Fig. 2 will be central to solving the system inversion problem. Its corresponding generating series is described in terms of the *shuffle group*. The following theorem appears to be known (Bacher, 2007, 2010), but no explicit proof is readily available to the authors' knowledge.

Theorem 3. The set of non proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ is a group under the shuffle product. In particular, the shuffle inverse of any such series c is

$$c^{\sqcup\!\sqcup -1} = ((c,\emptyset)(1 - c'))^{\sqcup\!\sqcup -1} := (c,\emptyset)^{-1} \sum_{k=0}^{\infty} (c')^{\sqcup\!\sqcup k},$$

where $c' = 1 - c/(c,\emptyset)$ is proper.

Proof. Since the set of non proper series is a subalgebra of the shuffle algebra, it only needs to be shown that a suitable shuffle inverse exists for each non proper series c. Since c' as defined above is proper, it is easy to verify that $\mathrm{ord}\left((c')^{\sqcup\!\sqcup k}\right) \geq k$. In which case, the set of series $\{(c')^{\sqcup\!\sqcup k} : k \geq 1\}$ is locally finite, and therefore, summable (Berstel & Reutenauer, 1988). It is clear that $c^{\sqcup\!\sqcup -1}$ is non proper, and the bilinearity of the shuffle product gives immediately that $c \sqcup\!\sqcup c^{\sqcup\!\sqcup -1} = c^{\sqcup\!\sqcup -1} \sqcup\!\sqcup c = 1$. In fact, since the shuffle algebra is an integral domain, i.e., $c \sqcup\!\sqcup d = 0$ if and only if c and/or d is zero (Wang, 1990), $c^{\sqcup\!\sqcup -1}$ is the only series with this property.

The following theorem states that local convergence is preserved under the shuffle inverse.

Theorem 4. If $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ is non proper then $c^{\sqcup\!\sqcup\, -1} \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$.

Proof. Since $c = (c,\emptyset)(1-c')$ is locally convergent, there exists a real number $M_{c'} > 0$ such that $|(c',\eta)| \leq M_{c'}^{|\eta|}\,|\eta|!$ for all $\eta \in X^*$. In which case,

$$\left|(c^{\sqcup\!\sqcup\, -1},\eta)\right| \leq \frac{1}{(c,\emptyset)}\sum_{k=0}^{\infty}\left|((c')^{\sqcup\!\sqcup\, k},\eta)\right| \leq \frac{1}{(c,\emptyset)}\sum_{k=0}^{\infty}(\bar{c}^{\sqcup\!\sqcup\, k},\eta),$$

where $\bar{c}$ is a series with coefficients $(\bar{c},\eta) = M_{c'}^{|\eta|}\,|\eta|!$. Using the identity

$$(\bar{c}^{\sqcup\!\sqcup\, k},\eta) = M_{c'}^{|\eta|}\binom{(k-1)+|\eta|}{k-1}|\eta|!,\quad \eta \in X^*,$$

it follows that

$$\left|(c^{\sqcup\!\sqcup\, -1},\eta)\right| \leq K_{c'}(8M_{c'})^{|\eta|}\,|\eta|!,\quad \eta \in X^*$$

for some $K_{c'} > 0$. (The fact that the Catalan numbers grow in proportion to 4^n as $n \to \infty$ has been used in the last step.) Hence, the shuffle inverse preserves local convergence.

The following example address the globally convergent case.

Example 5. The polynomial $c = 1 - x_0 - x_1$ is clearly globally convergent. But the series

$$c^{\sqcup\!\sqcup\, -1} = 1 + \sum_{k=1}^{\infty}(x_0+x_1)^{\sqcup\!\sqcup\, k} = \sum_{\eta\in X^*}|\eta|!\,\eta$$

is only locally convergent. So global convergence is *not* preserved under the shuffle inverse.

The main theorem of this subsection is given below.

Theorem 6. For $c,d \in \mathbb{R}\langle\langle X\rangle\rangle$, the quotient connection has a Fliess operator representation if and only if d is non proper. In particular, $F_c/F_d = F_{c/d}$, where $c/d := c \sqcup\!\sqcup\, d^{\sqcup\!\sqcup\, -1}$. In addition, the quotient c/d preserves local convergence.

Proof. Observe that at least formally

$$\begin{aligned}\frac{F_c}{F_d} &= \frac{F_c}{F_{(d,\emptyset)(1-d')}} = F_c\left((d,\emptyset)^{-1}\sum_{k=0}^{\infty}F_{d'}^k\right)\\ &= F_c\left((d,\emptyset)^{-1}\sum_{k=0}^{\infty}F_{(d')^{\sqcup\!\sqcup\, k}}\right) = F_cF_{d^{\sqcup\!\sqcup\, -1}}\\ &= F_{c\,\sqcup\!\sqcup\, d^{\sqcup\!\sqcup\, -1}}.\end{aligned}$$

Clearly from Theorem 3, if d is not proper then F_c/F_d has a well-defined generating series. On the other hand, if F_c/F_d has a well-defined Fliess operator representation then $F_d[u](t) \neq 0$ on some interval $[t_0, t_0+T)$, $T > 0$. So in particular, $F_d[u](t_0) = (d,\emptyset) \neq 0$. Hence, d must be non proper. Finally, the convergence claim follows directly from Theorem 4 and the fact that the shuffle product preserves local convergence.

3. LEFT SYSTEM INVERSION

It was shown by Wang (1990) that F_c will map every input which is analytic at t_0 to an output which is also analytic at t_0 provided $c \in \mathbb{R}_{LC}^{\ell}\langle\langle X\rangle\rangle$. In this section, the problem of computing a left inverse of F_c is considered given an analytic function in its range. The focus is on the SISO case with $t_0 = 0$. Note that every $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be written as $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$. The following definition will provide a sufficient condition under which the left inverse of F_c exists. The subsequent lemma is trivial to prove.

Definition 7. Given $c \in \mathbb{R}\langle\langle X\rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1}X^*$. Then c has *relative degree* r if the linear word $x_0^{r-1}x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

Lemma 8. If $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r then $(x_0^{r-1}x_1)^{-1}(c)$ is non proper.

Observe that this definition coincides with the usual definition of relative degree given in a state space setting (Isidori, 1995). Specifically, in light of the identity

$$\dot{y} = F_{x_0^{-1}(c)}[u] + uF_{x_1^{-1}(c)}[u],$$

where $x_i^{-1}(\cdot)$ denotes the left-shift operator, it follows that

$$y = F_c[u] \tag{3a}$$
$$y^{(1)} = F_{x_0^{-1}(c)}[u] \tag{3b}$$
$$\vdots$$
$$y^{(r-1)} = F_{(x_0^{r-1})^{-1}(c)}[u] \tag{3c}$$
$$y^{(r)} = F_{(x_0^r)^{-1}(c)}[u] + uF_{(x_0^{r-1}x_1)^{-1}(c)}[u]. \tag{3d}$$

From Lemma 8, $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](0) = (c, x_0^{r-1}x_1) \neq 0$ for any admissible u, and furthermore $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](t) \neq 0$ over some interval $[0,T)$, $T > 0$. Hence, any corresponding state space system will have relative degree r. On the other hand, if the word $x_0^{r-1}x_1 \notin \mathrm{supp}(c)$ then $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](0) = 0$ and $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](t) \neq 0$ for some $t > 0$ and input u. This implies that any corresponding state space system does not have a well defined relative degree in the classical sense. Definition 7 makes it clear that a state space realization has relative degree r only if the support of its generating series contains a specific linear word of length r normally associated with a linear time-invariant system. It is also trivial to see in this setting that a Taylor series linearization of any state space realization of F_c and any smooth change of state coordinates will not alter the relative degree when it is well defined.

Example 9. Consider the state space system in Example 4.1.5 of Isidori (1995):

$$\dot{z} = \begin{bmatrix} z_1z_2 - z_1^3 \\ z_1 \\ -z_3 \\ z_1^2 + z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2+2z_3 \\ 1 \\ 0 \end{bmatrix}u,\quad y = z_4.$$

The system is known by the conventional definition to have relative degree $r = 2$ at any point where $z_3(0) \neq -1$, and r is undefined otherwise. It is easy to verify that if $z(0) = 0$ then the corresponding generating series is

$$c = 2x_0x_1 + 2x_0x_1^2 - 2x_0x_1x_0x_1 + 2x_0x_1x_0^2x_1 - \cdots$$

In this case, $\mathrm{supp}(c_F) = \mathrm{supp}(c) \subset x_0X^*$, and the word $x_0x_1 \in \mathrm{supp}(c)$. On the other hand, if $z(0) = [0,0,-1,0]^T$ then

$$c = 2x_0x_1x_0 + 2x_0x_1x_1 - 2x_0x_1x_0^2 - 2x_0x_1x_0x_1 + \cdots$$

Here also $\mathrm{supp}(c_F) = \mathrm{supp}(c) \subset x_0X^*$, but the word $x_0x_1 \notin \mathrm{supp}(c)$. So both cases are consistent with Definition 7.

The next theorem is the main result of the paper.

Theorem 10. Suppose $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r. Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}_{LC}[[X_0]]$ satisfying $(c_y, x_0^k) = (c, x_0^k)$, $k = 0, \dots, r-1$. (Here $X_0 := \{x_0\}$.) Then the input

$$u(t) = \sum_{k=0}^{\infty} (c_u, x_0^k)\frac{t^k}{k!}, \qquad (4)$$

where

$$c_u = ((x_0^r)^{-1}(c - c_y)/(x_0^{r-1}x_1)^{-1}(c))^{\circ -1},$$

is the unique solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$.

Proof. First observe from (3a)-(3c) that $(c_y, x_0^k) = y^{(k)}(0) = (c, x_0^k)$ for $k = 0, 1, \dots, r-1$ due to the assumption that c has relative degree r. In effect, this is describing a set of analytic functions in the range of F_c. The relative degree assumption also ensures that the quotient $1/F_{(x_0^{r-1}x_1)^{-1}(c)}$ is well defined. If the feedback

$$u = \frac{v - F_{(x_0^r)^{-1}(c)}[u]}{F_{(x_0^{r-1}x_1)^{-1}(c)}[u]}$$

is applied, it is clear from (3d) that $y^{(r)} = v$. Noting that $v = F_{(x_0^r)^{-1}(c_y)}[u]$, it follows directly that

$$u = \frac{-F_{(x_0^r)^{-1}(c-c_y)}[u]}{F_{(x_0^{r-1}x_1)^{-1}(c)}[u]} = -F_d[u],$$

where

$$d = \frac{(x_0^r)^{-1}(c - c_y)}{(x_0^{r-1}x_1)^{-1}(c)}. \qquad (5)$$

This implies that

$$u + F_d[u] = 0,$$

and taking the compositional inverse gives

$$u = F_{d^{\circ -1}}[0].$$

This last expression produces exactly (4). It is known in the state space setting that this is the *unique* solution of $y = F_c[u]$ (Isidori, 1995). A general argument, based on analyticity, will be deferred to a later publication.

Note that in the usual state space tracking problem via feedback linearization, *every* analytic output is in the range of some Fliess operator induced by a suitable set of initial conditions for the variables $\xi := [y\ y^{(1)} \cdots y^{(r-1)}]$ (Isidori, 1995). But here c is fixed, so these r degrees of freedom are not available. The following corollary is fundamental in the definition of the zero dynamics of a state space realization.

Corollary 11. Suppose that $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r and $\mathrm{supp}(c_N) \subseteq x_0^r X_0^*$. Then the input

$$u^*(t) = \sum_{k=0}^{\infty} (c^*, x_0^k)\frac{t^k}{k!}, \qquad (6)$$

where

$$c^* = ((x_0^r)^{-1}(c)/(x_0^{r-1}x_1)^{-1}(c))^{\circ -1},$$

has the property that $F_c[u^*] = 0$.

Example 12. Consider the linear time-invariant system

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = [4\ \ 5\ \ 1]z.$$

initialized at $z(0) = [1\ 1\ -9]^T$. The transfer function has two finite zeros at $s = -1$ and $s = -4$, in which case, $r = 1$. Using conventional linear system theory, it can be shown for $t \geq 0$ that

$$\begin{aligned} u^*(t) &= \frac{5}{3}\mathrm{e}^{-t} + \frac{46}{3}\mathrm{e}^{-4t} \\ &= 17 - 63t + 247\frac{t^2}{2!} - 983\frac{t^3}{3!} + 3927\frac{t^4}{4!} - \cdots. \end{aligned} \qquad (7)$$

The corresponding generating series for this system $(A, b, c, z(0))$ is $c = c_N + c_F$, where

$$\begin{aligned} c_N &= s(c(sI - A)^{-1}z(0))|_{s^{-1}\to x_0} \\ &= -17x_0 + 29x_0^2 - 53x_0^3 + 118x_0^4 - 277x_0^5 + \cdots \\ c_F &= s(c(sI - A)^{-1}b)x_1|_{s^{-1}\to x_0} \\ &= (1 + 2x_0 - 4x_0^2 + 7x_0^3 - 15x_0^4 + 35x_0^5 - \cdots)x_1. \end{aligned}$$

Since $\mathrm{supp}(c_N) \subseteq x_0 X_0^*$, the function $y = 0$ is in the range of F_c. To determine u^* via Corollary 11, observe that

$$\begin{aligned} x_0^{-1}(c) &= x_0^{-1}(c_N) + x_0^{-1}(c_F) \\ &= (-17 + 29x_0 - 53x_0^2 + 118x_0^3 - 277x_0^4 + \cdots) + \\ &\quad (2 - 4x_0 + 7x_0^2 - 15x_0^3 + 35x_0^4 - \cdots)x_1 \\ x_1^{-1}(c) &= 1, \end{aligned}$$

and thus, $x_0^{-1}(c)/x_1^{-1}(c) = x_0^{-1}(c)$ and $c^* = (x_0^{-1}(c))^{\circ -1}$. To compute the indicated compositional inverse, the Faà di Bruno Hopf antipode formulas (2) are utilized:

$$\begin{aligned} (c^*, \emptyset) &= -(x_0^{-1}(c), \emptyset) = 17 \\ (c^*, x_0) &= -(x_0^{-1}(c), x_0) + (x_0^{-1}(c), \emptyset)(x_0^{-1}(c), x_1) = -63 \\ (c^*, x_0^2) &= -(x_0^{-1}(c), x_0^2) + (x_0^{-1}(c), \emptyset)(x_0^{-1}(c), x_0x_1) + \\ &\quad (x_0^{-1}(c), x_0)(x_0^{-1}(c), x_1) + (x_0^{-1}(c), \emptyset)(x_0^{-1}(c), x_1)^2 \\ &= 247 \\ &\ \ \vdots \end{aligned}$$

As expected, these coefficients agree with those in (7). Of course, if no closed-form expression is known for the Taylor series of u^*, as will be the case for the nonlinear system in the next example, then some truncation of this series must be used. Therefore, $F_c[\hat{u}^*] \approx 0$ on $[0, T]$ provided T is small relative to the number of terms retailed in the approximation $\hat{u}^*$ of u^*. The MatLab simulated output shown in Fig. 3 gives some idea of the accuracy of the results for approximations of (6) up to third order.

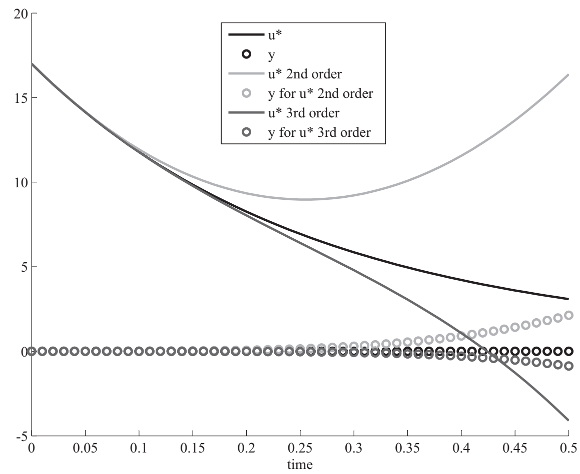

Fig. 3. The output of F_c for u^* and Taylor series approximations of u^* up to third order in Example 12.

Example 13. Reconsider the system in Example 9 when $z(0) = 0$. Since $r = 2$ in this case, and $c_N = 0$, the range of F_c contains all analytic outputs with generating series $c_y = \sum_{k\geq 2}(c_y, x_0^k)x_0^k$. As an illustration, select $y(t) = t^2/2$. Then $c_y = x_0^2$, $(x_0^2)^{-1}(c - c_y) = -1$, and

$$(x_0x_1)^{-1}(c) = 2 + 2x_1 - 2x_0x_1 + 2x_0^2x_1 - 2x_0^3x_1 + \cdots .$$

With the assistance of a Mathematica package for manipulating noncommutative formal power series, NCFPS (which is supported by the package for noncommutative algebra NCAlgebra (2012)), it follows from (5) that

$$d = -\frac{1}{2} + \frac{1}{2}x_1 - \frac{1}{2}x_0x_1 - x_1^2 + \frac{1}{2}x_0^2x_1 + 2x_0x_1^2 + x_1x_0x_1 + 3x_1^3 - \cdots .$$

Applying (2) gives

$$u(t) = \frac{1}{2} - \frac{1}{4}t + \frac{5}{8}\frac{t^2}{2!} - \frac{35}{16}\frac{t^3}{3!} + \frac{307}{32}\frac{t^4}{4!} - \cdots .$$

The corresponding output is compared against y in Fig. 4. If the generating series c_u is truncated to third order to produce the polynomial $\hat{c}_u$, the corresponding output error is characterized by

$$c_y - c\circ\hat{c}_u = \frac{331}{16}x_0^6 - \frac{1999}{32}x_0^7 + \frac{12855}{64}x_0^8 - \frac{83527}{128}x_0^9 + \cdots .$$

Therefore, the output approximation in this case is accurate to fifth order.

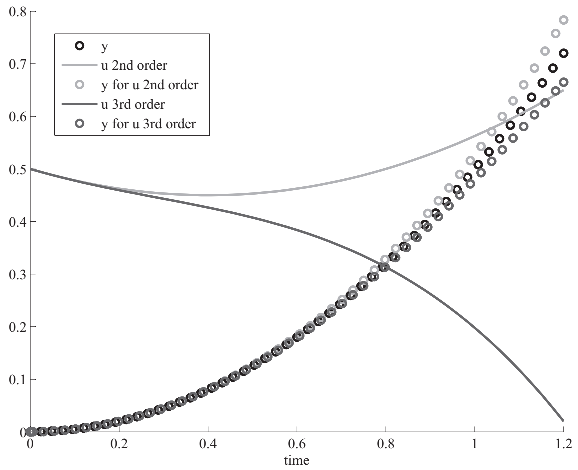

Fig. 4. The output of F_c for Taylor series approximations of u up to third order and $y(t) = t^2/2$ in Example 13.

4. CONCLUSIONS

The left inversion problem was solved for SISO Fliess operators having well defined relative degree, as defined in terms of its generating series, and analytic outputs. An extension of the method to the multivariable case is currently under investigation.

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