Sensitivity of the Left Inverse of a SISO Analytic Nonlinear System

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***Abstract*—The goal of this paper is to describe the parametric sensitivity functions associated with the left inverse of any single-input, single-output system written in terms of a Chen-Fliess functional series whose generating series has a well defined relative degree. A formula is also given to describe how this sensitivity gets propagated to the output. The technique is demonstrated with an example.**

***Keywords*—sensitivity, system inversion, Chen-Fliess series, formal power series**

I. Introduction

Consider an input-output system $F : \mathcal{U} \rightarrow \mathcal{Y}$ defined on suitable function spaces $\mathcal{U}$ and $\mathcal{Y}$. A left inverse of F is a mapping $F^{-1} : \mathcal{Y} \rightarrow \mathcal{U}$ such that $F^{-1}F[u] = u$, $\forall u \in \mathcal{U}$. Therefore, if y is in the range of F then necessarily $F^{-1}[y] = u$, where $y = F[u]$. Some version of this problem is implicit in virtually all output tracking and motion planning problems. In this setting, y is the desired output function to be followed by the plant, and the trajectory generation problem is to identify those inputs which will produce the desired output. In the special case where $F = F_c$ with F_c being a Chen-Fliess functional series/Fliess operator with generating series c, there can be no left inverse in the *same class* since Fliess operators are weighted sums of iterated integrals. Therefore, a left inverse operator would necessarily have to involve differentiation. Nevertheless, it was shown in [9] that when y is analytic and in the range of a single-input, single-output (SISO) operator F_c, it is possible to explicitly compute the unique Taylor series of u that satisfies $F_c^{-1}[y] = u$ provided c has a well defined relative degree. The formula for the left inverse function u is written in terms of the left-shift operator on formal power series (so differentiation is being done implicitly here) and the antipode of the output feedback Hopf algebra that appears naturally when Fliess operators are connected in a feedback configuration [7]. The use of this combinatorial Hopf algebra allows one to identify highly efficient inversion algorithms in the nonlinear setting [1], [3]. This result has been used in a variety of applications including trajectory generation for population dynamics [8], [10] and motion planning for both a crane [6] and a bi-steerable car [2].

Of course in real problems the system to be inverted is usually not known exactly. In which case, any inversion algorithm will ultimately have to address this uncertainty. The goal of this paper is to compute the parametric sensitivity of the left inverse as described in [9]. That is, the left inverse $u = F_c^{-1}[y]$ depends explicitly on the plant parameters that determine the generating series c. In which case, if $\partial u/\partial c$ is *large* in some sense, then the left inverse computed from data may yield an output with a large variability, something that would be useful to know a priori. The classical way to describe the situation is via a sensitivity function of the form

$$\mathcal{S}_c u := \frac{\partial u}{\partial c}\frac{c}{u} = \frac{\partial u/u}{\partial c/c}.$$

While conceptually straightforward to describe, the real difficulty here is explicitly computing $\partial u/\partial c$ since the left inverse u as it appears in [9] involves three nontrivial combinatorial operations: the shuffle product, the shuffle inverse, and the composition inverse written in terms of the output feedback Hopf algebra antipode. So the first result of this paper is to exhibit the actual formula for $\partial u/\partial c$. Once $\mathcal{S}_c u$ is determined, it is also useful to compute $\mathcal{S}_c y$ in order to understand how the sensitivity of u is ultimately propagated forward to the system's output y. So a formula for this sensitivity function is also developed. These tools are then demonstrated on an example from [9], which is ultimately based on an example appearing in [13].

The paper is organized as follows. In the next section some preliminaries are briefly summarized to explicitly describe the problem and set the notation. In Section III, the desired sensitivity formulas are developed. These results are then applied in the subsequent section to a specific example. The conclusions of the paper are given in the final section.

II. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \rightarrow \mathbb{R}^\ell$ is called a *formal power series*.

The value of c at $\eta \in X^*$ is written as (c,η) and called the *coefficient* of η in c. If $(c,\emptyset) = 0$ then c is said to be *proper*. Typically, c is represented as the formal sum $c = \sum_{\eta\in X^*}(c,\eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product defined in terms of the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ [5].

A. Fliess Operators and Their Interconnections

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar\eta}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar\eta}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar\eta \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta\in X^*} (c,\eta)\, E_\eta[u](t, t_0)$$

[5]. If there exists constants $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0, t_0+T]$ into $B^\ell_\mathfrak{q}(S)[t_0, t_0+T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [12]. The set of all such *locally convergent* generating series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$.

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_cF_d = F_{c\,\sqcup\!\sqcup\, d}$, respectively [5]. When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta\in X^*} (c,\eta)\, \psi_d(\eta)(1)$$

[4]. Here 1 denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to the set of vector space endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with $\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := 1$). $\psi_d(\emptyset)$ is defined to be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. If $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, and u is analytic with a Taylor series representation $u = \sum_{n\geq 0}(c_u, x_0^n)(t - t_0)^n/n! = \sum_{n\geq 0}(c_u, x_0^n)E_{x_0^n}[u](t, t_0)$, then $y = F_c[u]$ is also analytic (see [14]) with Taylor series coefficients given by $c_y = c \circ c_u$ so that $c_u \in \mathbb{R}^m_{LC}[[X_0]]$ and $c_y \in \mathbb{R}^\ell_{LC}[[X_0]]$ with $X_0 := \{x_0\}$. But when c is not locally convergent, the mapping $\mathbb{R}^m[[X_0]] \to \mathbb{R}^\ell_{LC}[[X_0]] : c_u \mapsto c_y = c \circ c_u$ is still well defined. It can be viewed as the formal version of $y = F_c[u]$.

When two Fliess operators F_c and F_d are interconnected to form a feedback system with F_c in the forward path and F_d in the feedback path, the generating series for the closed-loop system is denoted by the feedback product $c@d$. It can be computed explicitly using the Hopf algebra of coordinate functions associated with the underlying *output feedback group* [7]. Specifically, in the SISO case where $X = \{x_0, x_1\}$, define the set of operators $\mathscr{F}_\delta = \{I + F_c : c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group of operators under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta\circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c\,\tilde\circ\, d$, and $\tilde\circ$ denotes the modified composition product (a variation of the composition product above, but with a direct feed term on the right-hand side [11]). The coordinate maps for the corresponding Hopf algebra H have the form

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c,\eta),$$

where $\eta \in X^*$. The commutative product is defined as

$$\mu : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi,$$

where the unit $\mathbf{1}$ is defined to map every c to zero. If the *degree* of a_η is defined as $\deg(a_\eta) = 2\,|\eta|_{x_0} + |\eta|_{x_1} + 1$, then H is graded and connected with $H = \bigoplus_{n\geq 0} H_n$, where H_n is the set of all elements of degree n and $H_0 = \mathbb{R}\mathbf{1}$. The coproduct Δ is defined so that the formal power series product $c \circledcirc d := d + c\,\tilde\circ\, d$ for the group $\mathscr{F}_\delta$ satisfies

$$\Delta a_\eta(c, d) = a_\eta(c \circledcirc d) = (c \circledcirc d, \eta).$$

Of primary importance is the following lemma which describes how the group inverse $c_\delta^{\circ -1} := \delta + c^{\circ -1}$ is computed.

Lemma 2.1: [7] The Hopf algebra (H, μ, Δ) has an antipode S satisfying $a_\eta(c^{\circ -1}) = (Sa_\eta)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$.

The first few antipode terms are:

$$\begin{aligned} S\mathbf{1} &= \mathbf{1} \\ Sa_\emptyset &= -a_\emptyset \\ Sa_{x_0} &= -a_{x_0} + a_\emptyset a_{x_1} \end{aligned}$$

$$
\begin{aligned}
Sa_{x_1} &= -a_{x_1}\\
Sa_{x_0^2} &= -a_{x_0^2}+a_\emptyset a_{x_0x_1}+a_{x_0}a_{x_1}+a_\emptyset a_{x_1x_0}-\\
&\quad a_\emptyset a_{x_1}^2-a_\emptyset^2 a_{x_1^2}\\
Sa_{x_0x_1} &= -a_{x_0x_1}+a_{x_1}^2+a_\emptyset a_{x_1^2}\\
Sa_{x_1x_0} &= -a_{x_1x_0}+a_\emptyset a_{x_1^2}\\
Sa_{x_1^2} &= -a_{x_1^2}.
\end{aligned}
$$

Another way to compute S is to use the right augmentation operators $\tilde{\theta}_i(a_\eta) := a_{\eta x_i}$, $i = 0, 1$ with $\tilde{\theta}_{x_i}(\mathbf{1}) := 0$.

Theorem 2.1: [3] For any nonempty word $\eta = x_{i_1}\cdots x_{i_l}$, the antipode $S : H \to H$ for output feedback Hopf algebra can be written as

$$Sa_\eta = (-1)^{|\eta|-1}\tilde{\Theta}'_\eta(a_\emptyset),$$

where $|\eta| = l$ is the length of the word η, and

$$\tilde{\Theta}'_\eta := \tilde{\theta}'_{i_l}\circ\cdots\circ\tilde{\theta}'_{i_1}$$

with

$$\tilde{\theta}'_1(a_\eta) := -a_{\eta x_1}$$

and

$$\tilde{\theta}'_0(a_\eta) := -\tilde{\theta}_0(a_\eta)+a_\emptyset\tilde{\theta}_1(a_\eta) = -a_{\eta x_0}+a_{\eta x_1}a_\emptyset.$$

In addition to the elementary system interconnections described above, the quotient interconnection will be needed to solve the system inversion problem. Its corresponding generating series is described in terms of the *shuffle group*.

Theorem 2.2: [9] The set of non proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ is a group under the shuffle product. In particular, the shuffle inverse of any such series c is

$$c^{\sqcup\!\sqcup -1} = ((c,\emptyset)(1-c'))^{\sqcup\!\sqcup -1} = (c,\emptyset)^{-1}(c')^{\sqcup\!\sqcup *}$$

where $c' = 1 - c/(c,\emptyset)$ is proper and $(c')^{\sqcup\!\sqcup *} := \sum_{k\geq 0}(c')^{\sqcup\!\sqcup k}$.

Theorem 2.3: [9] For $c, d \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, the quotient connection F_c/F_d has a Fliess operator representation if and only if d is non proper. In particular, $F_c/F_d = F_{c/d}$, where $c/d := c \sqcup\!\sqcup d^{\sqcup\!\sqcup -1}$. In addition, the quotient c/d preserves local convergence.

B. Computing the Left Inverse

The following definition provides a sufficient condition under which the left inverse of F_c exists. It uses the fact that every $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$.

Definition 2.1: [9] Given $c \in \mathbb{R}\langle\langle X\rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1}X^*$. Then c has ***relative degree*** r if the linear word $x_0^{r-1}x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

Such a left inverse can be computed by the explicit formula given below. Here $x_i^{-1}(\cdot)$ will denote the $\mathbb{R}$-linear left-shift operator specified by $x_i^{-1}(\eta) = \eta'$ when $\eta = x_i\eta'$ with $\eta' \in X^*$ and zero otherwise. The definition is extended inductively to $\xi^{-1}(\cdot)$ when $\xi \in X^*$.

Theorem 2.4: [9] Suppose $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r. Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}_{LC}[[X_0]]$ satisfying $(c_y, x_0^k) = (c, x_0^k)$, $k = 0, \ldots, r-1$. Then the input

$$u(t) = \sum_{\ell=0}^{\infty}(c_u, x_0^\ell)\frac{t^\ell}{\ell!}, \tag{1}$$

where

$$(c_u, x_0^\ell) = \left(\left(\frac{(x_0^r)^{-1}(c-c_y)}{(x_0^{r-1}x_1)^{-1}(c)}\right)^{\circ -1}, x_0^\ell\right), \tag{2}$$

is the unique analytic solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$.

III. Sensitivity Functions for c_u and $c \circ c_u$

The first goal of this section is to compute the derivatives and sensitivity functions for the left inverse series c_u as defined in (2) with respect to c, that is,

$$\mathcal{D}_\nu^\ell c_u := \frac{\partial(c_u, x_0^\ell)}{\partial(c,\nu)},\;\; \mathcal{S}_\nu^\ell c_u := \mathcal{D}_\nu^\ell c_u\frac{(c,\nu)}{(c_u, x_0^\ell)}, \tag{3}$$

respectively, for any $\ell \geq 0$ and $\nu \in X^*$. This requires determining how to differentiate

$$d^{\circ -1} = \sum_{\eta\in X^*} a_\eta(d^{\circ -1})\eta = \sum_{\eta\in X^*}(Sa_\eta)(d)\eta,$$

for an arbitrary $d \in \mathbb{R}\langle\langle X\rangle\rangle$ and some fixed parameter p. From the chain rule it follows directly that

$$
\begin{aligned}
\frac{\partial d^{\circ -1}}{\partial p} &= \sum_{\eta\in X^*}\frac{\partial}{\partial p}(Sa_\eta)(d)\eta\\
&= \sum_{\eta\in X^*}\sum_{a_{\xi_i}\in N(\eta)}\left.\frac{\partial Sa_\eta}{\partial a_{\xi_i}}\right|_d\frac{\partial(d,\xi_i)}{\partial p}\eta,
\end{aligned}
$$

where $N(\eta)$ denotes the coordinate functions appearing in the polynomial Sa_η. Now, in light of the quotient appearing in (2) (see Theorem 2.3), it is necessary to consider the particular situation above where $d = d_1 \sqcup\!\sqcup d_2$. Clearly, from the bilinearity of the shuffle product

$$\frac{\partial}{\partial p}(d_1 \sqcup\!\sqcup d_2) = \frac{\partial d_1}{\partial p}\sqcup\!\sqcup d_2 + d_1 \sqcup\!\sqcup \frac{\partial d_2}{\partial p}.$$

So setting $d_1 = (x_0^r)^{-1}(c - c_y)$ and $d_2 = ((x_0^{r-1}x_1)^{-1}(c))^{\sqcup\!\sqcup -1}$ as in (2) gives

$$
\begin{aligned}
\frac{\partial}{\partial p}(d_1 \sqcup\!\sqcup d_2) &= \frac{\partial}{\partial p}(x_0^r)^{-1}(c-c_y)\sqcup\!\sqcup((x_0^{r-1}x_1)^{-1}(c))^{\sqcup\!\sqcup -1}+\\
&\quad (x_0^r)^{-1}(c-c_y)\sqcup\!\sqcup\frac{\partial((x_0^{r-1}x_1)^{-1}(c))^{\sqcup\!\sqcup -1}}{\partial p}.
\end{aligned}
$$

Henceforth, the focus will be on the case where $p = (c, \nu)$ for some fixed $\nu \in X^*$. Observe that for any $\eta \in X^*$

$$\frac{\partial\eta^{-1}(c)}{\partial(c,\nu)} = \frac{\partial}{\partial(c,\nu)}\sum_{\xi\in X^*}(\eta^{-1}(c),\xi)\xi$$

$$= \sum_{\xi \in X^*} \frac{\partial}{\partial (c,\nu)} (c, \eta\xi)\xi$$
$$= \begin{cases} \eta^{-1}(\nu) & : \quad (c,\nu) \neq 0 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Therefore, if $(c,\nu) \neq 0$ then

$$\frac{\partial}{\partial (c,\nu)} (x_0^r)^{-1}(c - c_y) = (x_0^r)^{-1}(\nu).$$

The following lemma is needed to compute the other partial derivative above.

Lemma 3.1: For any non proper $c \in \mathbb{R}\langle\langle X \rangle\rangle$ with $(c,\nu) \neq 0$ for some fixed $\nu \in X^*$,

$$\frac{\partial c^{\sqcup\!\sqcup\, -1}}{\partial (c,\nu)} = -(c,\emptyset)^{-2} \sum_{k=0}^{\infty} (k+1)(c')^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, \nu.$$

Proof: The claim follows directly from the known identity $\partial c^{\sqcup\!\sqcup\, k}/\partial p = k c^{\sqcup\!\sqcup\, (k-1)} \sqcup\!\sqcup\, \partial c/\partial p$ and the definition of the shuffle inverse in Theorem 2.2. ■

The development above provides the first main result.

Theorem 3.1: For any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ in Theorem 2.4 with $(c,\nu) \neq 0$, $\nu \in X^*$ and $\ell \geq 0$

$$\mathcal{D}_\nu^\ell c_u = \sum_{a_{\xi_i} \in N(x_0^\ell)} \left.\frac{\partial S a_{x_0^\ell}}{\partial a_{\xi_i}}\right|_{c_u} \Bigg((x_0^r)^{-1}(\nu) \sqcup\!\sqcup\, ((x_0^{r-1}x_1)^{-1}(c))^{\sqcup\!\sqcup\, -1} - (x_0^r)^{-1}(c - c_y) \sqcup\!\sqcup\, (c, x_0^{r-1}x_1)^{-2} \sum_{k=0}^{\ell} (k+1)(c')^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, \nu, \xi_i \Bigg),$$

where $c' = 1 - (x_0^{r-1}x_1)^{-1}(c)/(c, x_0^{r-1}x_1)$.

The corresponding sensitivity function $\mathcal{S}_\nu^\ell c_u$ is then computed directly as defined in (3).

The next goal is to compute $\mathcal{D}_\nu^\ell(c \circ c_u)$ and $\mathcal{S}_\nu^\ell(c \circ c_u)$, which of course will use the results above for c_u. First observe that

$$c \circ c_u = \sum_{\eta \in X^*} (c,\eta)\, \psi_{c_u}(\underbrace{x_{i_1} \cdots x_{i_k}}_{\eta})(1)$$
$$= \sum_{\eta \in X^*} (c,\eta)\, \psi_{c_u}(x_{i_1}) \cdots \psi_{c_u}(x_{i_k})(1).$$

Therefore, if $(c,\nu) \neq 0$ then

$$\frac{\partial (c \circ c_u)}{\partial (c,\nu)} = \nu \circ c_u + \sum_{\eta \in X^*} (c,\eta) \sum_{j=1}^{k} \psi_{c_u}(x_{i_1}) \cdots \frac{\partial}{\partial (c,\nu)} \psi_{c_u}(x_{i_j}) \cdots \psi_{c_u}(x_{i_k})(1)$$
$$= \nu \circ c_u + \sum_{\eta \in X^*} (c,\eta) \sum_{j=1}^{k} \psi_{c_u}(x_{i_1}) \cdots \psi_{\frac{\partial c_u}{\partial (c,\nu)} \delta_{i_j 1}}(x_{i_j}) \cdots \psi_{c_u}(x_{i_k})(1),$$

using the fact that if $x_{i_j} = x_1$ and $e \in \mathbb{R}\langle\langle X \rangle\rangle$ then

$$\left(\frac{\partial}{\partial (c,\nu)} \psi_{c_u}(x_1)\right)(e) = \frac{\partial}{\partial (c,\nu)} x_0(c_u \sqcup\!\sqcup\, e)$$
$$= x_0 \left(\frac{\partial c_u}{\partial (c,\nu)} \sqcup\!\sqcup\, e\right)$$
$$= \left(\psi_{\frac{\partial c_u}{\partial (c,\nu)}}(x_1)\right)(e).$$

Otherwise, if $x_{i_j} = x_0$ then

$$\left(\frac{\partial}{\partial (c,\nu)} \psi_{c_u}(x_1)\right)(e) = 0.$$

From a physical point of view note that the term $\nu \circ c_u$ in the expression for $\partial(c \circ c_u)/\partial(c,\nu)$ above involves the change in the output due exclusively to a change in the plant and *not* any perturbation in the applied input, which is the primary interest here. So in this case it is convenient to *define* $\mathcal{D}_\nu^\ell(c \circ c_u)$ without the first term, which can be handled separately if necessary due to the additivity. This leads to the second main result.

Theorem 3.2: For any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ in Theorem 2.4 with $(c,\nu) \neq 0$, $\nu \in X^*$ and $\ell \geq 0$

$$\mathcal{D}_\nu^\ell(c \circ c_u) = \sum_{\eta = x_{i_1} \cdots x_{i_k} \in X^*} (c,\eta) \sum_{j=1}^{k} \Big(\psi_{c_u}(x_{i_1}) \cdots \psi_{\frac{\partial c_u}{\partial (c,\nu)} \delta_{i_j 1}}(x_{i_j}) \cdots \psi_{c_u}(x_{i_k})(1), x_0^\ell \Big).$$

Again, the corresponding sensitivity function follows immediately from the definition, namely,

$$\mathcal{S}_\nu^\ell(c \circ c_u) := \mathcal{D}_\nu^\ell(c \circ c_u) \frac{(c,\nu)}{(c \circ c_u, x_0^\ell)}.$$

IV. Example

Consider the state space system in Example 4.1.5 of [13]

$$\dot{z} = \begin{bmatrix} z_1 z_2 - z_1^3 \\ z_1 \\ -z_3 \\ z_1^2 + z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 + 2z_3 \\ 1 \\ 0 \end{bmatrix} u, \quad y = z_4 \tag{4}$$

with initial condition $z(0) = z_0$. For any word $\eta = x_{i_k} \cdots x_{i_1} \in X^*$, the corresponding coefficient of c is

$$(c,\eta) = L_{g_\eta} h(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h(z_0),$$

where $L_{g_i} h$ denotes the Lie derivative of output h with respect to the vector field g_i. It follows with some help from Mathematica software that

$$c = 2x_0x_1 + 2x_0x_1^2 - 2x_0x_1x_0x_1 + 2x_0x_1x_0^2x_1 - \cdots$$

when $z_0 = 0$. In which case, the series has relative degree $r = 2$. Since $c_N = 0$, the range of F_c contains all analytic outputs with generating series of the form $c_y = \sum_{k \geq 2} (c_y, x_0^k) x_0^k$. Select $y(t) = t^2/2$ as in [9]. Then $c_y = x_0^2$, $(x_0^2)^{-1}(c - c_y) = -1$, and

$$(x_0x_1)^{-1}(c) = 2 + 2x_1 - 2x_0x_1 + 2x_0^2x_1 - 2x_0^3x_1 + \cdots.$$

TABLE I. DERIVATIVES $D^{\ell}_{\eta}c_u$ IN SECTION IV EXAMPLE

ℓ/η	x_0x_1	$x_0x_1^2$	$x_0x_1x_0x_1$
0	$-\frac{1}{4}$	0	0
1	$\frac{3}{8}$	$-\frac{1}{8}$	0
2	$-\frac{21}{16}$	$\frac{3}{8}$	$-\frac{1}{8}$

TABLE II. SENSITIVITIES $S^{\ell}_{\eta}c_u$ IN SECTION IV EXAMPLE

ℓ/η	x_0x_1	$x_0x_1^2$	$x_0x_1x_0x_1$
0	-1	0	0
1	-3	1	0
2	$-\frac{21}{5}$	$\frac{6}{5}$	$\frac{2}{5}$

TABLE III. DERIVATIVES $D^{\ell}_{\eta}(c\circ c_u)$ IN SECTION IV EXAMPLE

ℓ/η	x_0x_1	$x_0x_1^2$	$x_0x_1x_0x_1$
2	$-\frac{1}{2}$	0	0
3	$\frac{1}{4}$	$-\frac{1}{4}$	0
4	$-\frac{5}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$

TABLE IV. SENSITIVITIES $S^{\ell}_{\eta}(c\circ c_u)$ IN SECTION IV EXAMPLE

ℓ/η	x_0x_1	$x_0x_1^2$	$x_0x_1x_0x_1$
2	-1	–	–
3	∞	∞	–
4	∞	∞	∞

From (2) it follows directly that

$$c_u = \frac{1}{2} - \frac{1}{4}x_0 + \frac{5}{8}x_0^2 - \frac{35}{16}x_0^3 + \frac{307}{32}x_0^4 - \cdots .$$

Applying Theorem 3.1, for example, when $\nu \in \{x_0x_1, x_0x_1^2\}$ and $\ell = 1$ yields

$$\mathcal{D}^1_{x_0x_1}c_u = -\frac{(c_y,x_0^3)-(c,x_0^3)}{(c,x_0x_1)^2} + \frac{2(c,x_0x_1x_0)}{(c,x_0x_1)^3}\cdot [(c_y,x_0^2)-(c,x_0^2)] + \frac{3(c,x_0x_1^2)}{(c,x_0x_1)^4}[(c_y,x_0^2)-(c,x_0^2)]^2$$

$$\mathcal{D}^1_{x_0x_1^2}c_u = \frac{-[(c_y,x_0^2)-(c,x_0^2)]^2}{(c,x_0x_1)^3}.$$

Evaluating all the derivative functions for the given series c and computing $\mathcal{S}^{\ell}_{\nu}c_u$ from the definition gives the values shown in Tables I and II. As a specific example, suppose that the coefficient $(c,x_0x_1)=2$ is perturbed so that $(c,x_0x_1) = 2.1$. In which case, the data in Table I can be use to estimate the change in the first three entries of c_u to determine the perturbed version of c_u, say $\hat{c}_u$. Specifically,

$$(\hat{c}_u,\emptyset) \approx \frac{1}{2} + \left(-\frac{1}{4}\right)0.1 = 0.475$$
$$(\hat{c}_u,x_0) \approx -\frac{1}{4} + \left(\frac{3}{8}\right)0.1 = -0.2125$$
$$(\hat{c}_u,x_0^2) \approx \frac{5}{8} + \left(-\frac{21}{16}\right)0.1 = 0.49375.$$

For comparison, a direct application of the formula for c_u in Theorem 2.4 with the new series c gives

$$\hat{c}_u = 0.47619 - 0.215959x_0 + 0.509782x_0^2 + \cdots ,$$

which compares favorably with the estimate above.

Next apply Theorem 3.2 to determine $\mathcal{D}^{\ell}_{\nu}(c\circ c_u)$ and $\mathcal{S}^{\ell}_{\nu}(c\circ c_u)$. Since the relative degree of c is $r=2$ and $c_N=0$, it is not hard to see that $\mathcal{D}^{\ell}_{\nu}(c\circ c_u)=0$ for $\ell=0,1$. On the other hand, for example, when $\nu = x_0x_1$ and $\ell \geq 2$:

$$\mathcal{D}^{\ell}_{x_0x_1}(c\circ c_u) = (c,x_0x_1)\left(x_0^2\frac{\partial c_u}{\partial(c,x_0x_1)}, x_0^{\ell}\right) + (c,x_0x_1^2)\left(x_0^2\left[\frac{\partial c_u}{\partial(c,x_0x_1)} ⧢ x_0c_u\right], x_0^{\ell}\right) + (c,x_0x_1^2)\left(x_0^2\left[c_u ⧢ x_0\frac{\partial c_u}{\partial(c,x_0x_1)}\right], x_0^{\ell}\right) (c,x_0x_1x_0x_1)\left(x_0^2\left[\frac{\partial c_u}{\partial(c,x_0x_1)} ⧢ x_0^2c_u\right], x_0^{\ell}\right) + (c,x_0x_1x_0x_1)\left(x_0^2\left[c_u ⧢ x_0^2\frac{\partial c_u}{\partial(c,x_0x_1)}\right], x_0^{\ell}\right) + \cdots$$

Again evaluating all these derivative functions for the given series c and computing $\mathcal{S}^{\ell}_{\nu}c_u$ from the definition gives the values shown in Tables III and IV. The fact that some sensitivities are infinite or undefined for $\ell > 2$ is simply due to the fact that $(c_y,x_0^{\ell})=0$ for $\ell>2$. As a check, observe that when $\hat{c}_u$ is applied directly to the unperturbed plant,

$$\hat{c}_y := c\circ\hat{c}_u = 0.952381x_0^2 + 0.0215959x_0^3 - 0.0509782x_0^4 + \cdots$$

Thus, the error introduced in the calculation of c_u has perturbed the coefficient of the quadratic term and introduced small higher order terms. Analogous to the previous analysis, the data from Table III gives the following estimates for the first three terms of $\hat{c}_y$, which are quite accurate:

$$(\hat{c}_y,x_0^2) \approx 1 + \left(-\frac{1}{2}\right)0.1 = 0.95$$
$$(\hat{c}_y,x_0^3) \approx 0 + \left(\frac{1}{4}\right)0.1 = 0.025$$
$$(\hat{c}_y,x_0^4) \approx 0 + \left(-\frac{5}{8}\right)0.1 = -0.0625.$$

Finally, functions u and $\hat{u}$ are plotted in Figure 1 using the truncated Taylor series derived from the first three terms of c_u (see (1)) and $\hat{c}_u$, respectively. Figure 2 shows the output functions y and $\hat{y}$ generated by a MatLab simulation of the state space system (4) driven by u and $\hat{u}$, respectively, and compared against the desired output. In this particular instance, the perturbed output has *improved* tracking performance, but other coefficient perturbations can just as easily degrade the performance.

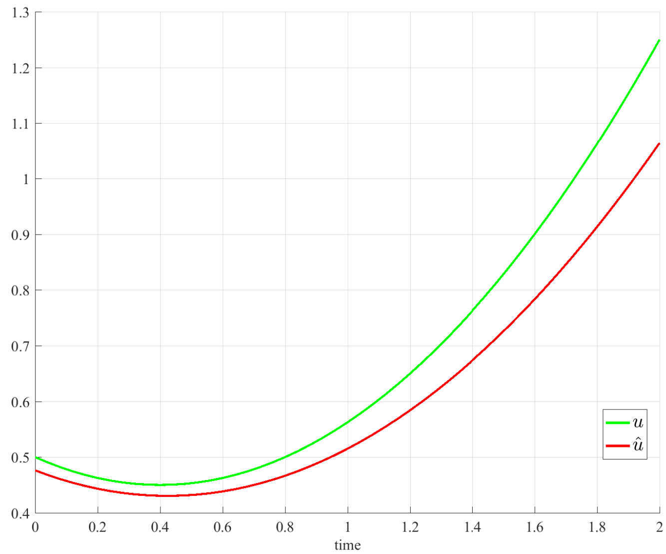

Fig. 1. Input u and $\hat{u}$ in Section IV example

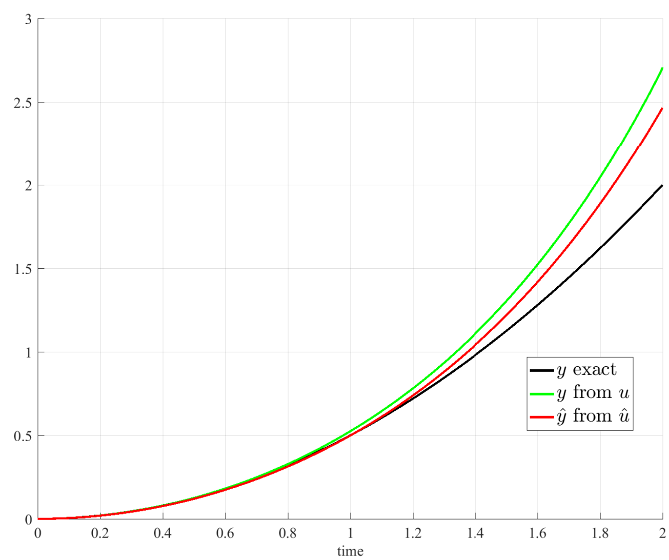

Fig. 2. Exact output $y(t) = t^2/2$, output y from u, and output $\hat{y}$ from $\hat{u}$ in Section IV example

V. Conclusions

The parametric sensitivity functions were determined explicitly for the left inverse of any SISO system written in terms of a Chen-Fliess functional series with respect to the coefficients of its generating series, assuming that this generating series has a well defined relative degree. A formula was also given to describe how this sensitivity propagates to the output. All the results were demonstrated with an example.

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