On Global Convergence of Fractional Fliess Operators with Applications to Bilinear Systems

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***Abstract*—A common representation for the input-output map of a nonlinear control system is the Chen-Fliess functional series or Fliess operator. The objective of this paper is to further develop a generalization of this class of operators, so called *fractional* Fliess operators. These are functional series whose iterated integrals are defined in terms of fractional integrals and where the series coefficients can have a fractional Gevrey growth rate. Here the existing convergence analysis is expanded to handle fractional integrated integrals for a class of L_1 inputs. The results are applied to represent the input-output map of a fractional bilinear state space system.**

Keywords—Fractional systems, Chen-Fliess series, bilinear systems

I. Introduction

A common representation for the input-output map of a nonlinear control system is the Chen-Fliess functional series or Fliess operator [2], [8]. It can be viewed as a noncommutative generalization of a Taylor series, and its algebraic nature is especially well suited for describing system interconnections [4], [6] and solving system inversion problems [5] in a nonlinear setting. The objective of this paper is to further develop a generalization of this class of operators, so called *fractional* Fliess operators. These are functional series whose iterated integrals are defined in terms of fractional integrals and where the series coefficients can have a fractional Gevrey growth rate. The idea was first proposed by the authors in [18] and applied mainly in the context of fractional linear systems. A sufficient condition for a kind of global convergence was given for the special case where only the coefficients have this fractional nature. Here the convergence analysis is expanded to handle fractional integrated integrals of L_1 inputs dominated by any monomial t^ϵ, where $\epsilon > 0$. The results are applied to represent the input-output map of a fractional bilinear state space system. There are in fact two approaches possible, one that is compatible with the Riemann-Liouville fractional derivative and the other with the Caputo fractional derivative. Both approaches are pursued in parallel. In the process, a fractional notion of a causal Volterra series is also identified.

The presentation is organized as follows. In the next section, some fundamentals from fractional calculus are briefly summarized. In Section III, the notion of a fractional Fliess operator is described and the new convergence analysis is presented. The concept is then applied in Section IV to fractional bilinear systems.

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II. Preliminaries: Fractional Calculus

Fractional calculus and fractional system theory are well developed areas with a variety of applications. The main definitions and results concerning fractional calculus presented here are largely based on [1], [10], [14], [15], [17]. Assume $\alpha, \beta \in \mathbb{R}$, and f is a real-valued function defined on $[0, T]$. Under suitable restrictions on f, which will be assumed throughout, the following fractional extensions of integration and differentiation are well defined [1].

Definition 2.1: The *Riemann-Liouville fractional integral* for any $\alpha > 0$ and $t \in [0, T]$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\, \mathrm{d}\tau,$$

where Γ is the gamma function. In addition, $I^0 f(t) := f(t)$.

Definition 2.2: The *Riemann-Liouville fractional derivative* for any $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$ and $t \in [0, T]$ is defined by

$$D^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha} f(t).$$

Definition 2.3: The *Caputo fractional derivative* for any $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$, $t \in [0, T]$, and n times differentiable function f is defined by

$${}^C\!D^\alpha f(t) = I^{n-\alpha}\left(\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right)(t).$$

Lemma 2.1: The Riemann-Liouville and Caputo fractional derivatives are related by

$${}^C\!D^\alpha f(t) = D^\alpha(f(t) - f(0)).$$

Lemma 2.2: The following relations hold when $0 < \alpha$ and f, g are real-valued measurable functions on $[0, T]$:

1) $f(t) = I^{1-\alpha} f(0) \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} + I^\alpha D^\alpha f(t),\ \alpha \leq 1$
2) $D^\alpha (f \circ g)(t) =$
$$\sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{r=0}^{k} \binom{k}{r} (-g(t))^r D^\alpha g^{k-r}(t)\right) \frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)).$$

This last relation above is the chain rule for fractional calculus introduced by Osler [14, equations (3.4)-(3.5) with $z = t$, $g(t) = t$ and $F(t, w) = 1$].

Example 2.1: In the case where $\alpha = 1$ in Lemma 2.2, part 2, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ g)(t)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{r=0}^{k} \binom{k}{r} (-g(t))^r \frac{\mathrm{d}}{\mathrm{d}t} g^{k-r}(t) \right) \frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t))$$
$$= \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{1}(t) \right) f(g(t)) + \left(\frac{\mathrm{d}}{\mathrm{d}t} g(t) - g(t) \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{1}(t) \right) \frac{\mathrm{d}}{\mathrm{d}g} f(g(t))$$
$$+ \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}t} g^2(t) - 2g(t) \frac{\mathrm{d}}{\mathrm{d}t} g(t) + g^2(t) \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{1}(t) \right) \frac{\mathrm{d}^2}{\mathrm{d}g^2} f(g(t))$$
$$+ \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}t} g^3(t) - 3g(t) \frac{\mathrm{d}}{\mathrm{d}t} g^2(t) + 3g^2(t) \frac{\mathrm{d}}{\mathrm{d}t} g(t) \right.$$
$$\left. - g^3(t) \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{1}(t) \right) \frac{\mathrm{d}^3}{\mathrm{d}g^3} f(g(t)) + \cdots ,$$

where $\mathbb{1}$ is the unit step function. Since $\frac{\mathrm{d}}{\mathrm{d}t} g^k(t) = \sum_{r=1}^{k-1} \binom{k}{r} (-g(t))^r \frac{\mathrm{d}}{\mathrm{d}t} g^{k-r}(t)$, $k \in \mathbb{N}$ and $k > 1$ [3], and $\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{1}(t) = 0$ for $t > 0$, all the terms are zero except the one corresponding to $k = 1$. This term gives the usual chain rule. □

Corollary 2.1: [3] If $\alpha = N \in \mathbb{N}$ then Lemma 2.2, part 2 reduces to

$$\frac{\mathrm{d}^N}{\mathrm{d}t^N} (f \circ g)(t) = \sum_{k=0}^{N} \frac{1}{k!} \left(\sum_{r=0}^{k} \binom{k}{r} (-g(t))^r \frac{\mathrm{d}^N}{\mathrm{d}t^N} g^{k-r}(t) \right) \frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)).$$

Let P_q denote the class of polynomials with real coefficients and having degree q.

Corollary 2.2: Suppose $f \in P_q$ then Lemma 2.2, part 2 becomes

$$D^\alpha (f \circ g)(t) = \sum_{k=0}^{q} \frac{1}{k!} \left(\sum_{r=0}^{k} \binom{k}{r} (-g(t))^r D^\alpha g^{k-r}(t) \right) \frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)).$$

Example 2.2: In the case where $f \in P_1$, observe

$$D^\alpha (f \circ g)(t) = (D^\alpha \mathbb{1}(t)) f(g(t)) + (D^\alpha g(t) - g(t) D^\alpha \mathbb{1}(t)) \frac{\mathrm{d}}{\mathrm{d}g} f(g(t)).$$

□

The next definition and lemma are used throughout the paper.

Definition 2.4: [11] The *matrix Mittag-Leffler function* is

$$\mathscr{E}_{\alpha,\beta}(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k + \beta)},$$

where t is real-valued, $A \in \mathbb{R}^{n \times n}$, and α, β are positive real numbers. In particular, $\mathscr{E}_{1,1}(At) = \mathrm{e}^{At}$ is the usual matrix exponential.

Lemma 2.3: The following identities are easily verified for real numbers $\alpha, \beta > 0$ and real-valued function f:

1) $I^\alpha t^\beta = t^{\alpha+\beta} \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$

2) $\displaystyle\sum_{k=0}^{\infty} A^k I^{\alpha k} \mathbb{1}(t) = \mathscr{E}_{\alpha,1}(At^\alpha)$

3) $\displaystyle\sum_{k=1}^{\infty} A^{k-1} I^{\alpha k} f(t) = (t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)) * f(t).$

Here $*$ refers to the usual convolution product.

III. Global Convergence of Fractional Fliess Operators

Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X \rangle\rangle$. The image of η under c is written as (c, η). For each $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, one can formally associate an m input, ℓ output operator, F_c, as described next. Let $\mathfrak{p} \geq 1$ and $T > 0$ be given. For a Lebesgue measurable function $u : [0, T] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[0, T]$. Let $L^m_\mathfrak{p}[0, T]$ denote the set of all measurable functions defined on $[0, T]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[0, T] := \{u \in L^m_\mathfrak{p}[0, T] : \|u\|_\mathfrak{p} \leq R\}$. The next two definitions are fundamental.

Definition 3.1: [18] Let $0 < \alpha \leq 1$ and $T \in \mathbb{R}$ be fixed. The *fractional iterated integral* for any $\eta = x_i \eta' \in X^*$ is the mapping $E^\alpha_\eta : L^m_1[0, T] \to \mathbb{R}[0, T]$ defined by the recursion

$$E^\alpha_{x_i \eta'}[u](t) = I^\alpha [u_i(\tau) \, E^\alpha_{\eta'}[u](\tau)](t),$$

where $E^\alpha_\emptyset := 1$ and

$$u_0(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & : \text{Riemann-Liouville derivative approach} \\ \mathbb{1}(t) & : \text{Caputo derivative approach.} \end{cases}$$

Note that when $\alpha = 1$, the resulting iterated integrals are continuous. But when $0 < \alpha < 1$ this is so if, for example, the functions u_i, $i = 1, 2, \ldots, m$ are all continuous. Weaker sufficient conditions also exist, see [1, Definition 2.1.1].

Definition 3.2: [18] For any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $0 < \alpha \leq 1$ the *fractional Fliess operator* is defined formally by

$$F^\alpha_c[u](t) = \sum_{\eta \in X^*} (c, \eta) \, E^\alpha_\eta[u](t).$$

The operator has *Gevrey order* $s \in \mathbb{R}$ if there exists real constants $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|} (|\eta|!)^s, \quad \forall \eta \in X^*,$$

and s is the smallest number having this property.

The next two technical lemmas will be useful.

Lemma 3.1: [18] For any integer $n \geq 0$ and $0 < r \leq 1$ such that $nr \gg 1$ it follows that

$$\Gamma(rn + 1) \approx \frac{1}{n^{(1-r)/2}} K_r M_r^n (n!)^r,$$

where $K_r := ((2\pi)^{1-r} r)^{1/2}$ and $M_r := r^r$.

Lemma 3.2: For any integer $n, k \geq 0$ and $0 < r \leq 1$ such that $rn, rk \gg 1$ it follows that

$$\Gamma(r(nk + n + k) + 1)\Gamma(r(n + 1) + 1)\Gamma(r(k + 1) + 1) \leq \Gamma(r(n + 1)(k + 1) + 1)\Gamma(rn + 1)\Gamma(rk + 1).$$

Proof: Applying the approximation in Lemma 3.1 for each gamma function on the left-hand side of the proposed inequality gives

$$\Gamma(r(nk + n + k) + 1)\Gamma(r(n + 1) + 1)\Gamma(r(k + 1) + 1)$$

$$\approx \frac{K_r^3 M_r^{nk+2n+2k+2}((nk+n+k)!(n+1)!(k+1)!)^r}{((nk+n+k)(n+1)(k+1))^{(1-r)/2}},$$

where $K_r := ((2\pi)^{1-r}r)^{1/2}$ and $M_r := r^r$. It is clear that

$$(nk+n+k)!(n+1)!(k+1)! = \frac{nk(nk+n+k+1)!n!k!}{nk+n+k+1} \leq (nk+n+k+1)!n!k!$$

and

$$(nk+n+k)(n+1)(k+1) \geq nk(n+1)(k+1).$$

It then follows that

$$\begin{aligned}&\Gamma(r(nk+n+k)+1)\Gamma(r(n+1)+1)\Gamma(r(k+1)+1)\\ &\leq \frac{K_r^3 M_r^{nk+2n+2k+1}((nk+n+k+1)!n!k!)^r}{((n+1)(k+1)nk)^{(1-r)/2}}\\ &\approx \Gamma(r(n+1)(k+1)+1)\Gamma(rn+1)\Gamma(rk+1),\end{aligned}$$

again using Lemma 3.1 in the final step. ■

The next lemma provides upper bounds for certain fractional iterated integrals. Henceforth, the focus is on the single-input, single-output (SISO) case ($m=\ell=1$).

Lemma 3.3: Let $X=\{x_0,x_1\}$. For any $x_i \in X$, integers $k,\gamma \geq 0$ and $0<\alpha\leq 1$

$$E^{\alpha}_{x_i^k}[u](t) \leq \frac{(I^{\alpha}u_i(t))^k}{\Gamma(\alpha k+1)},$$

where $u(t)=t^{\alpha\gamma}$, t is real valued, and the Caputo derivative approach is used.

Proof: The fractional iterated integral when $i=0$ is

$$E^{\alpha}_{x_0{}^k}[t^{\alpha\gamma}](t) = I^{\alpha k}\,\mathbb{1}(t) = \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \leq \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^k \frac{1}{\Gamma(\alpha k+1)},$$

since $\Gamma(\alpha+1) \leq 1$ when $1<\alpha+1\leq 2$. Applying Lemma 2.3, part 1 gives

$$E^{\alpha}_{x_0{}^k}[t^{\alpha\gamma}](t) \leq \frac{(I^{\alpha}\,\mathbb{1}(t))^k}{\Gamma(\alpha k+1)}.$$

On the other hand, when $i=1$ an inductive proof on k can be used. The claim is clearly true when $k=0$. When $k=1$ observe that

$$E^{\alpha}_{x_1}[t^{\alpha\gamma}](t) = I^{\alpha}t^{\alpha\gamma} \leq \frac{I^{\alpha}t^{\alpha\gamma}}{\Gamma(\alpha+1)},$$

since $\Gamma(\alpha+1)\leq 1$. Now suppose the identity in question holds up to some fixed $k\geq 0$. Then

$$E^{\alpha}_{x_1^{k+1}}[t^{\alpha\gamma}](t) = I^{\alpha}[t^{\alpha\gamma}E^{\alpha}_{x_1^k}[t^{\alpha\gamma}]].$$

From the induction hypothesis, it follows that

$$\begin{aligned}E^{\alpha}_{x_1^{k+1}}[t^{\alpha\gamma}](t) &\leq I^{\alpha}\left(t^{\alpha\gamma}\frac{(I^{\alpha}t^{\alpha\gamma})^k}{\Gamma(\alpha k+1)}\right)\\ &= \frac{\Gamma(\alpha\gamma+1)^k}{\Gamma(\alpha k+1)\Gamma(\alpha(\gamma+1)+1)^k} I^{\alpha}t^{\alpha(\gamma k+\gamma+k)}.\end{aligned}$$

Applying Lemma 2.3, part 1 gives

$$\begin{aligned}&E^{\alpha}_{x_1^{k+1}}[t^{\alpha\gamma}](t)\\ &\leq \frac{\Gamma(\alpha\gamma+1)^k\Gamma(\alpha(\gamma k+\gamma+k)+1)t^{\alpha(\gamma+1)(k+1)}}{\Gamma(\alpha k+1)\Gamma(\alpha(\gamma+1)+1)^k\Gamma(\alpha(\gamma+1)(k+1)+1)}.\end{aligned}$$

Finally, employing Lemma 3.2 yields

$$\begin{aligned}E^{\alpha}_{x_1^{k+1}}[t^{\alpha\gamma}](t) &\leq \frac{\Gamma(\alpha\gamma+1)^{k+1}t^{\alpha(\gamma+1)(k+1)}}{\Gamma(\alpha(k+1)+1)\Gamma(\alpha(\gamma+1)+1)^{k+1}}\\ &= \frac{(I^{\alpha}t^{\alpha\gamma})^{k+1}}{\Gamma(\alpha(k+1)+1)}.\end{aligned}$$

Therefore, the inequality holds for all $k\geq 0$. ■

The main result of this section is the following theorem.

Theorem 3.1: Let $X=\{x_0,x_1\}$. Suppose $c\in\mathbb{R}\langle\langle X\rangle\rangle$ is a Gevrey series of order $0\leq s<1$. If $0<s<\alpha\leq 1$ then for any $u\in L^1_{1/\alpha}[0,T]$ where $|u(t)|\leq t^{\alpha\gamma}$ for all $t\in[0,T]$ with $\gamma\in\mathbb{N}$, the series

$$y(t) = F^{\alpha}_c[u](t) = \sum_{\eta\in X^*}(c,\eta)\;E^{\alpha}_{\eta}[u](t),$$

converges absolutely and uniformly on $[0,T]$ for any $T>0$ assuming the Caputo derivative approach is used.

Proof: Fix some $T>0$. Pick any $u\in L^1_{1/\alpha}[0,T]$ where $|u(t)|\leq t^{\alpha\gamma}$, $\forall t\in[0,T]$ with $\gamma\in\mathbb{N}$. Let $R=\max\{C_{\alpha,\gamma}\|t^{\alpha\gamma}\|_1, T^{\alpha}/\Gamma(\alpha+1)\}$ on $[0,T]$ with $C_{\alpha,\gamma}:=(\gamma+1)^{\alpha}\Gamma(\alpha\gamma+1)/\Gamma(\alpha(\gamma+1)+1)$. Observe from Lemma 3.3 it follows that

$$\begin{aligned}F^{\alpha}_c[u](t) &\leq \sum_{\eta\in X^*}|(c,\eta)|\left|E^{\alpha}_{\eta}[u](t)\right|\\ &\leq \sum_{k=0}^{\infty}KM^k(k!)^s\sum_{\eta\in X^k}E^{\alpha}_{\eta}[|u|](t)\\ &\leq \sum_{k=0}^{\infty}KM^k(k!)^s\sum_{\eta\in X^k}E^{\alpha}_{\eta}[t^{\alpha\gamma}]\\ &\leq \sum_{k=0}^{\infty}KM^k(k!)^s\frac{R^k}{\Gamma(\alpha k+1)},\end{aligned}$$

since R can be rewritten as $R=\max\{I^{\alpha}u_0(t), I^{\alpha}u_1(t)\}$ on $[0,T]$ using the identities $I^{\alpha}\,\mathbb{1}(t)=t^{\alpha}/\Gamma(\alpha+1)$ and $I^{\alpha}t^{\alpha\gamma}=C_{\alpha,\gamma}\|t^{\alpha\gamma}\|_1$. In which case,

$$F^{\alpha}_c[u](t) \leq K\sum_{k=0}^{\infty}(MR)^k\frac{(k!)^s}{\Gamma(\alpha k+1)}.$$

Applying the ratio test to the sequence $a_k := (k!)^s/\Gamma(\alpha k+1)$, $k\geq 0$:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\frac{((k+1)!)^s}{\Gamma(\alpha(k+1)+1)}\frac{\Gamma(\alpha k+1)}{(k!)^s}\\ &= \lim_{k\to\infty}\frac{(k+1)^s\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}.\end{aligned}$$

Using Lemma 3.1 for each gamma function and the fact that $0<\alpha-s\leq 1$, it follows that

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \frac{1}{M_{\alpha}}\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{(1-\alpha)/2}\frac{1}{(k+1)^{\alpha-s}}\\ &= \frac{1}{M_{\alpha}}\lim_{k\to\infty}\frac{1}{(k+1)^{\alpha-s}} = 0,\end{aligned}$$

where $M_{\alpha} := \alpha^{\alpha}$. Thus, the series $F^{\alpha}_c[u](t)$ converges absolutely and uniformly on $[0,T]$. ■

The notion of convergence described above is referred to as *global* since there is no a priori upper bound on either γ

or T. As expected, Theorem 3.1 is consistent with the fact that Fliess operators are known to be globally convergent when $\alpha = 1$ and $0 \leq s < 1$ [18]. An analogous theorem holds when the Riemann-Liouville approach is taken in light of Lemma 2.1.

IV. Fractional Bilinear Systems

In this section the Fliess operator representation of the input-output map of a SISO fractional bilinear system is given for the Riemann-Liouville derivative approach and the Caputo derivative approach. The development also suggests the notion of a fractional Volterra series, so the idea is presented in the final subsection.

A. Riemann-Liouville fractional derivative approach

Consider a SISO bilinear system

$$D^\alpha z = N_0 z + N_1 z u,\ z(0) = z_0 \tag{1a}$$

$$y = \lambda z, \tag{1b}$$

where $N_0, N_1 \in \mathbb{R}^{nxn}$, $\lambda \in \mathbb{R}^{1xn}$, and $0 < \alpha \leq 1$. The state equation can be written in integral form using Lemma 2.2, part 1 as

$$z(t) = I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 I^\alpha z(t) + N_1 I^\alpha z(t) u(t). \tag{2}$$

Since integration and differentiation are defined componentwise, there is no loss of generality in the following analysis by assuming $n = 1$. For any $F \in P_1$, the differential chain rule in integral form given in Example 2.2 becomes

$$D^\alpha F(z(t)) = (D^\alpha \mathbb{1}(t)) F(z(t)) + (D^\alpha z(t) - z(t) D^\alpha \mathbb{1}(t)) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t)).$$

Using this result and Lemma 2.2, part 1 gives

$$F(z(t)) = I^{1-\alpha} F(z(0)) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + I^\alpha((D^\alpha \mathbb{1}(t)) F(z(t))) + N_0 I^\alpha \left(z(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t))\right) + N_1 I^\alpha \left(z(t) u(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t))\right) - I^\alpha \left((D^\alpha \mathbb{1}(t)) z(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t))\right).$$

Now let $F(z(t))$ be replaced by $N_i z(t),\ i = 0,1$ and substitute back into (2). This yields

$$\begin{aligned} z(t) = {} & I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 I^{1-\alpha} z(0) I^\alpha \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + N_1 I^{1-\alpha} z(0) I^\alpha \left[u(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0^2 I^{2\alpha} z(t) \\ & + N_0 N_1 I^{2\alpha}(z(t) u(t)) + N_1 N_0 I^\alpha (I^\alpha z(t) u(t)) \\ & + N_1^2 I^\alpha (I^\alpha (z(t) u(t)) u(t)). \end{aligned}$$

Repeating this procedure, let $F(z(t))$ be replaced by $N_i N_j z(t),\ i,j = 0,1$ and substitute back into the equation above. This gives

$$\begin{aligned} z(t) = {} & I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 I^{1-\alpha} z(0) I^\alpha \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + N_1 I^{1-\alpha} z(0) I^\alpha \left[u(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0^2 I^{1-\alpha} z(0) I^{2\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + N_0 N_1 I^{1-\alpha} z(0) I^{2\alpha} \left[u(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_1^2 I^{1-\alpha} z(0) I^\alpha (u(t) I^\alpha u(t)) \\ & + N_1 N_0 I^{1-\alpha} z(0) I^\alpha \left[u(t) I^\alpha \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\right] + R_2(z(t), u(t)), \end{aligned}$$

where $R_2(z(t), u(t))$ contains all the integrals depending explicitly on $z(t)$ and $u(t)$. Continuing in this manner produces in the limit a fractional form of the usual Peano-Baker formula [16]. So the series solution of (1a) is

$$z(t) = \sum_{\eta \in X^*} (c_z, \eta)\ E_\eta^\alpha[u](t), \tag{3}$$

where $u_0(t) := t^{\alpha-1}/\Gamma(\alpha)$ as indicated in Definition 3.1, and the coefficients of the generating series for $y(t) = \lambda z(t) = F_c^\alpha[u](t)$ are

$$(c, \eta) = \lambda N_{i_k} \cdots N_{i_1} I^{1-\alpha} z(0) \tag{4}$$

for $\eta = x_{i_k} \cdots x_{i_1}$. As a check, write (3) in the form

$$\begin{aligned} z(t) = {} & I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + \sum_{i=0}^{1} \sum_{k=0}^{\infty} \sum_{i_0,\ldots,i_k=0}^{1} N_i N_{i_k} \cdots N_{i_1} I^{1-\alpha} z(0) \\ & I^\alpha [u_i(\tau) E_{x_{i_k} \cdots x_{i_1}}^\alpha u(\tau)](t), \end{aligned}$$

and take the Riemann-Liouville derivative of order α so that

$$\begin{aligned} & D^\alpha z(t) \\ & = \sum_{k=0}^{\infty} \sum_{i_0,\ldots,i_k=0}^{1} N_0 N_{i_k} \cdots N_{i_1} I^{1-\alpha} z(0) u_0(t) E_{x_{i_k} \cdots x_{i_1}}^\alpha [u](t) \\ & + \sum_{k=0}^{\infty} \sum_{i_0,\ldots,i_k=0}^{1} N_1 N_{i_k} \cdots N_{i_1} I^{1-\alpha} z(0) u_1(t) E_{x_{i_k} \cdots x_{i_1}}^\alpha [u](t) \\ & = N_0 z(t) + N_1 z(t) u(t), \end{aligned}$$

which is consistent with (1a). Also note that in this Riemann-Liouville approach, the coefficients are not explicit functions of $z(0)$, but rather in terms of $I^{1-\alpha} z(0)$. This *fractional* initial condition can be written as

$$I^{1-\alpha} z(t)\big|_{t=0} = \left.\frac{z(t)\Gamma(\alpha)}{t^{\alpha-1}}\right|_{t=0}$$

[12], [17].

B. Caputo fractional derivative approach

Consider now a SISO bilinear system

$${}^C D^\alpha z = N_0 z + N_1 z u,\ z(0) = z_0 \tag{5a}$$

$$y = \lambda z, \tag{5b}$$

where $N_0, N_1 \in \mathbb{R}^{nxn}$, $\lambda \in \mathbb{R}^{1xn}$, and $0 < \alpha \leq 1$. The state equation can be written in integral form using Lemma 2.1 and Lemma 2.2, part 1 as

$$z(t) = z(0) + N_0 I^\alpha z(t) + N_1 I^\alpha (z(t) u(t)). \tag{6}$$

Following the same procedure as in the previous subsection and assuming $n = 1$ yields

$$F(z(t)) = F(z(0)) + I^\alpha((D^\alpha \mathbb{1}(t))(F(z(t)) - F(z(0)))) + N_0 I^\alpha \left(z(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t))\right) + N_1 I^\alpha \left(z(t) u(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t))\right) - I^\alpha \left((D^\alpha \mathbb{1}(t))(z(t) - z(0)) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t))\right).$$

Now let $F(z(t))$ be replaced by $N_i z(t),\ i = 0,1$ and substitute back into (6). This yields

$$z(t) = z_0 + N_0 z_0 I^\alpha \mathbb{1}(t) + N_1 z_0 I^\alpha u(t) + R_1(z(t), u(t)),$$

where $R_1(z(t),u(t))$ contains all the integrals depending explicitly on $z(t)$ and $u(t)$. Continuing in this way produces the Caputo analogue of (3)-(4), namely, the series solution of (5a) is

$$z(t) = \sum_{\eta\in X^*} (c_z,\eta)\; E^\alpha_\eta[u](t), \tag{7}$$

where now $u_0(t) := \mathbb{1}(t)$ as indicated in Definition 3.1. The coefficients of the generating series for $y(t) = \lambda z(t) = F^\alpha_c[u](t)$ are

$$(c,\eta) = \lambda N_{i_k}\cdots N_{i_1} z_0 \tag{8}$$

for $\eta = x_{i_k}\cdots x_{i_1}$. Again as a check, a straightforward calculation shows that the Caputo derivative of (7) gives (5a) as expected. The main differences between the Caputo and Riemann-Liouville cases are the presence of $z(0)$ instead of $I^{1-\alpha}z(0)$ in the series coefficients and the factor of $\mathbb{1}(t)$ instead of $t^{\alpha-1}/\Gamma(\alpha)$ in the iterated integrals. Henceforth, the focus is mainly on the Caputo approach since the coefficients and the iterated integrals are easier to compute.

C. Fractional Volterra series

A generalization of a Volterra series for a fractional bilinear system is given in the next theorem.

Theorem 4.1: The solution $y(t) = \lambda z(t) = F^\alpha_c[u](t)$ of a bilinear system in the Caputo sense (5) can be written in terms of the matrix Mittag-Leffler function as

$$y(t) = w_0(t) + \sum_{k=1}^\infty \int_0^t\int_0^{\tau_k}\cdots\int_0^{\tau_2} w_k(t,\tau_k,\ldots,\tau_1)\, u(\tau_k)\cdots u(\tau_1)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k,$$

where

$$\begin{aligned} w_0(t) &= \lambda\mathscr{E}_{\alpha,1}(N_0t^\alpha)z_0\\ w_k(t,\tau_k,\ldots,\tau_1) &= \lambda(t-\tau_k)^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau_k)^\alpha)\\ &\quad N_1(\tau_k-\tau_{k-1})^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(\tau_k-\tau_{k-1})^\alpha)\\ &\quad\cdots N_1(\tau_2-\tau_1)^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(\tau_2-\tau_1)^\alpha)\\ &\quad N_1\mathscr{E}_{\alpha,1}(N_0{\tau_1}^\alpha)z_0,\ k\geq 1.\end{aligned}$$

Proof: Since each word $\eta = x_{i_k}\cdots x_{i_1}\in X^*$ can be rewritten uniquely in the form ${x_0}^{n_k}x_1{x_0}^{n_{k-1}}x_1\cdots {x_0}^{n_1}x_1{x_0}^{n_0}$, it follows from (7)-(8) that

$$y(t) = \sum_{k=0}^\infty\sum_{n_0,\ldots,n_k=0}^\infty \lambda {N_0}^{n_k}N_1{N_0}^{n_{k-1}}\cdots N_1{N_0}^{n_0}z_0\, E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t).$$

The claim is that $y = \sum_{k\geq 0} y_k$, where

$$\begin{aligned} y_k(t) &:= \sum_{n_0,\ldots,n_k=0}^\infty \lambda {N_0}^{n_k}N_1{N_0}^{n_{k-1}}\cdots N_1{N_0}^{n_0}z_0\, E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t)\\ &= \sum_{n_0,\ldots,n_k=0}^\infty \int_0^t\int_0^{\tau_k}\cdots\int_0^{\tau_2} w_k(t,\tau_k,\ldots,\tau_1)\, u(\tau_k)\cdots u(\tau_1)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\end{aligned}$$

for $k\geq 1$ and $y_0 := \sum_{n_0\geq 0}\lambda N_0^{n_0}z_0\,E^\alpha_{x_0^{n_0}}[u] = w_0$. The expressions for y_k are proved by induction on k. Observe that when $k=0$

$$y_0(t) = \sum_{n_0=0}^\infty \lambda {N_0}^{n_0}z_0\, I^{\alpha n_0}\,\mathbb{1}(t),$$

and using Lemma 2.3, part 2 yields $y_0(t) = \lambda\mathscr{E}_{\alpha,1}(N_0t^\alpha)z_0 = w_0(t)$. When $k=1$, it follows via Lemma 2.3, part 3 that

$$\begin{aligned} y_1(t) &= \sum_{n_0,n_1=0}^\infty \lambda {N_0}^{n_1}N_1{N_0}^{n_0}z_0\, I^{\alpha(n_1+1)}(u(t)I^{\alpha n_0}\,\mathbb{1}(t))\\ &= \sum_{n_0=0}^\infty (\lambda t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0t^\alpha)N_1{N_0}^{n_0}z_0) * (u(t)I^{\alpha n_0}\,\mathbb{1}(t)).\end{aligned}$$

Applying next Lemma 2.3, part 2 yields

$$\begin{aligned} y_1(t) &= \int_0^t \lambda(t-\tau)^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau)^\alpha)N_1\mathscr{E}_{\alpha,1}(N_0{\tau_1}^\alpha)\, z_0u(\tau)\,\mathrm{d}\tau\\ &= \int_0^t w_1(t,\tau_1)u(\tau_1)\,\mathrm{d}\tau_1.\end{aligned}$$

Now suppose the identity in question holds for all terms up to some fixed $k\geq 0$. Then

$$y_{k+1}(t) = \sum_{n_0,\ldots,n_{k+1}=0}^\infty \lambda {N_0}^{n_{k+1}}N_1{N_0}^{n_k}\cdots N_1{N_0}^{n_0}z_0\, I^{\alpha(n_{k+1}+1)}(u(t)E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t)),$$

and Lemma 2.3, part 3 gives

$$\begin{aligned} y_{k+1}(t) &= \sum_{n_0,\ldots,n_k=0}^\infty \lambda t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0t^\alpha)N_1{N_0}^{n_k}\cdots N_1{N_0}^{n_0}z_0 * (u(t)E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t))\\ &= \sum_{n_0,\ldots,n_k=0}^\infty \int_0^t \lambda(t-\tau_{k+1})^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau_{k+1})^\alpha)\, N_1{N_0}^{n_k}\cdots N_1{N_0}^{n_0}z_0\, u(\tau_{k+1})E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t)\,\mathrm{d}\tau_{k+1}.\end{aligned}$$

Finally, from the induction hypothesis, it follows that

$$\begin{aligned} y_{k+1}(t) &= \int_0^t\int_0^{\tau_{k+1}}\cdots\int_0^{\tau_2} \lambda(t-\tau_{k+1})^{\alpha-1}\, \mathscr{E}_{\alpha,\alpha}(N_0(t-\tau_{k+1})^\alpha)\cdots N_1(\tau_2-\tau_1)^{\alpha-1}\, \mathscr{E}_{\alpha,\alpha}(N_0(\tau_2-\tau_1)^\alpha)N_1\mathscr{E}_{\alpha,1}(N_0{\tau_1}^\alpha)z_0\, u(\tau_{k+1})\cdots u(\tau_1)\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{k+1}\\ &= \int_0^t\int_0^{\tau_{k+1}}\cdots\int_0^{\tau_2} w_{k+1}(t,\tau_{k+1},\ldots,\tau_1)\, u(\tau_{k+1})\cdots u(\tau_1)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{k+1}.\end{aligned}$$

Thus, the expression for y_k holds for all $k\geq 0$. ■

In the case of a bilinear system corresponding to the Riemann-Liouville approach, an analogous Volterra series representation can be directly computed using Lemma 2.1. Recall also in the non-fractional case that there is a dichotomy between causal Volterra series induced by initialized state space realizations [8] and a potentially noncausal variety whose kernel functions are often derived from measurements and represented in terms of their multivariable Laplace transforms [16]. Theorem 4.1 above corresponds

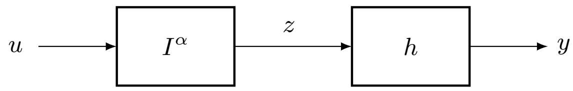

Fig. 1: Fractional Wiener system in Example 4.1

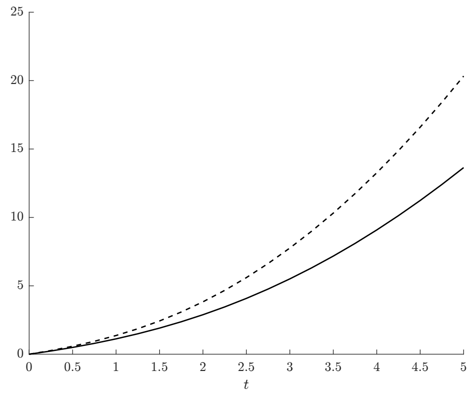

Fig. 2: Response $y(t)$ in Example 4.2 when $u(t) = t^{0.5}$ (solid line) and $\hat{y}(t) := \mathscr{E}_{0.5,1}(I^{0.5}t^{0.5})$ (dotted line) both on a logarithmic scale.

to the former in this fractional setting, while the fractional generalization of the latter has appeared in [13].

Two examples are given to illustrate the main results.

Example 4.1: Consider the scalar bilinear realization

$$^{C}D^{\alpha}z(t) = N_1 z(t)u(t),\ z(0) = z_0 \tag{9a}$$

$$y(t) = \lambda z(t), \tag{9b}$$

where $N_1, \lambda \in \mathbb{R}$, $u(t) = \mathbb{1}(t)$, and $0 < \alpha \leq 1$. From (7) and (8), the coefficients of the generating series are

$$(c, \eta) = \begin{cases} \lambda N_1^k z_0 & : \ \eta = x_1^k,\ k \geq 0 \\ 0 & : \ \text{otherwise.} \end{cases}$$

The generating series $c = \sum_{k\geq 0} x_1^k$ is clearly Gevrey of order $s = 0$ with growth constants $K = 1$ and $M = N_1$. The fractional iterated integral is

$$E^{\alpha}_{x_1{}^k}[\mathbb{1}](t) = I^{\alpha k}\,\mathbb{1}(t).$$

Therefore, the series representation of the output is

$$y(t) = F_c^{\alpha}[\mathbb{1}](t) = \lambda \sum_{k=0}^{\infty} N_1^k z_0\, I^{\alpha k}\,\mathbb{1}(t).$$

Using Lemma 2.3, part 2 gives

$$y(t) = \lambda \mathscr{E}_{\alpha,1}(N_1 t^{\alpha})z_0.$$

This system can be seen as the fractional Wiener system shown in Figure 1, where $h(z) = \mathscr{E}_{\alpha,1}(z)$, and the fractional integrator is initialized so that $z(0) = z_0$. This is the fractional version of the example given in [7] where $h(z) = \mathrm{e}^z$. Also, note that $F_c^{\alpha}[\mathbb{1}](t)$ is well defined for all $t > 0$, which is consistent with Theorem 3.1 when $\gamma = s = 0$. □

Example 4.2: Reconsider the scalar bilinear realization in (9) but now with the input $u \in L^1_{1/\alpha}[0,T]$, where $|u(t)| \leq t^{\alpha\gamma}$ for all $t \in [0,T]$ with $\gamma \in \mathbb{N}$ and $T > 0$ fixed. In this case the series representation of the output is

$$y(t) = F_c^{\alpha}[u](t) = \lambda \sum_{k=0}^{\infty} N_1^k z_0\ E^{\alpha}_{x_1^k}[u](t).$$

Lemma 3.3 yields

$$\begin{aligned} y(t) = F_c^{\alpha}[u](t) &\leq |\lambda| \sum_{k=0}^{\infty} |N_1|^k\, |z_0|\ \frac{(I^{\alpha}t^{\alpha\gamma})^k}{\Gamma(\alpha k + 1)} \\ &= |\lambda|\, \mathscr{E}_{\alpha,1}(|N_1|\, I^{\alpha}t^{\alpha\gamma})\, |z_0| =: \hat{y}(t). \end{aligned} \tag{10}$$

The closed-form version of $y(t)$ can be found in [9, Example 4] and is rather complicated, but it follows from Theorem 3.1 that the series for $F_c^{\alpha}[u](t)$ converges absolutely and uniformly on $[0,T]$ provided $|u(t)| \leq t^{\alpha\gamma}$ for all $t \in [0,T]$ with $\alpha > s = 0$, T, and $\gamma \in \mathbb{N}$ fixed. The bounding function $\hat{y}$ defined in (10) in terms of the Mittag-Leffler function is entire, which is also consistent with Theorem 3.1. A MatLab simulation of (9) with $N_1 = 1$, $\lambda = 1$, $z_0 = 1$, and $u(t) = t^{\alpha\gamma}$, where $\alpha = 0.5$ and $\gamma = 1$, is shown on a logarithmic scale in Figure 2 along with $\hat{y}$ to further confirm the upper bound in (10). □

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