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Fractional Fliess Operators: Two Approaches

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***Abstract*—A useful representation of an input-output map in a nonlinear control system is the Chen-Fliess functional series or Fliess operator. It can be viewed as a noncommutative generalization of a Taylor series, and its algebraic nature is especially well suited for a number of important applications. The objective of this paper is to describe a generalization of this class of operators, so called *fractional* Fliess operators. These are functional series whose coefficients have a certain fractional growth rate (Gevrey series) and whose iterated integrals are defined in terms of fractional integrals. The motivation for this idea is two-fold. First, fractional system theory is a well developed area with a variety of applications, so this concept is a natural generalization in its own right. But even in the classical case it has been observed that the cascade interconnection of two Fliess operators can result in a composite system that has a certain fractional nature. Hence, developing this generalization may also provide some insight into this issue.**

I. Introduction

A useful representation of an input-output map in a nonlinear control system is the Chen-Fliess functional series or Fliess operator [6], [13]. It can be viewed as a noncommutative generalization of a Taylor series, and its algebraic nature is especially well suited for describing system interconnections [10], feedback invariants [8], [11] and solving system inversion problems [9] in a nonlinear setting. The objective of this paper is to describe a generalization of this class of operators, so called *fractional* Fliess operators. These are functional series whose coefficients have a certain fractional growth rate (Gevrey series) and whose iterated integrals are defined in terms of fractional integrals. The motivation for this idea is two-fold. First, fractional calculus and fractional system theory are well developed areas with a variety of applications [2], [16], [20], [22]. So this concept is a natural generalization in its own right. But even in the classical case it has been observed that the cascade interconnection of two Fliess operators can result in a composite system that has a certain fractional nature [23]. Hence, developing this generalization may also provide some insight into this issue, which to date has not been fully explained. This topic, however, is beyond the scope of the present paper.

The presentation is organized as follows. In the first section, the fundamentals needed from fractional calculus are summarized in order to make the paper more self-contained and to establish the notation. In the subsequent section, the concept of a fractional Fliess operator is presented. There are in fact two choices available for the definition, one that is compatible with the Riemann-Liouville fractional derivative and the other with the Caputo fractional derivative. Both approaches are pursued in the context of state space realizations for these operators. At this stage, no requirement is placed on the coefficients of the generating series, so the development is purely formal. In Section IV, the issue of convergence of fractional Fliess operators is addressed by assuming that the generating series of the operator is of Gevrey type. Sufficient conditions for convergence are given for the special case where the iterated integrals are *not* fractional. The general case where both the series coefficients and integrals are fractional is not pursue here since this would require an extension of the shuffle algebra associated with the product of fractional iterated integrals, which is an open problem at present. Nevertheless, this special case gives some insight into the general problem and contains as special cases all the convergence conditions described in [12] for the classical case. The final section provides the conclusions of the paper and some remarks about future work.

II. Preliminaries

The main definitions and results concerning fractional calculus presented here are largely based on [2], [3], [7], [14]–[16], [19], [20]. Assume $\alpha, \beta \in \mathbb{R}$, and f is a real-valued defined on $[a, b]$. It will be assumed throughout that $a = 0$. Under suitable restrictions on f, the following fractional extensions of integration and differentiation are well defined.

Definition 2.1: The *Riemann-Liouville fractional integral* for any $\alpha > 0$ and $t \in [0, b]$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\,\mathrm{d}\tau,$$

where Γ is the gamma function.

Definition 2.2: The *Riemann-Liouville fractional derivative* for any $n-1 < \alpha \le n$ with $n \in \mathbb{N}$ and $t \in [0, b]$ is defined by

$$D^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha} f(t).$$

Definition 2.3: The *Caputo fractional derivative* for any $n-1 < \alpha \le n$ with $n \in \mathbb{N}$, $t \in [0, b]$, and n times differentiable function f is defined by

$$ {}^{C}D^{\alpha} f(t) = I^{n-\alpha}\left(\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right)(t). $$

Lemma 2.1: The Riemann-Liouville and Caputo fractional derivatives are related by

$$ {}^{C}D^{\alpha} f(t) = D^{\alpha}(f(t) - f(0)). $$

Lemma 2.2: The following relations hold when f, g are real-valued measurable functions on $[0, b]$:

1) $f(t) = I^{1-\alpha} f(0)\dfrac{t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha} D^{\alpha} f(t),\ 0 < \alpha \leq 1$
2) $D^{\alpha}(f \circ g)(t) =$
$$ \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r D^{\alpha} g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)), \alpha > 0. $$

This last relation is the chain rule for fractional calculus introduced by Osler [19, equations (3.4)-(3.5) with $z = t$, $g(t) = t$ and $F(t, w) = 1$].

Example 2.1: In the case where $\alpha = 1$ in Lemma 2.2, part 2, it follows that

$$ \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(f \circ g)(t) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r \frac{\mathrm{d}}{\mathrm{d}t} g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right) f(g(t)) + \left(\frac{\mathrm{d}}{\mathrm{d}t} g(t) - g(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)\frac{\mathrm{d}}{\mathrm{d}g} f(g(t)) \\ &+ \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} g^2(t) - 2g(t)\frac{\mathrm{d}}{\mathrm{d}t} g(t) + g^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)\frac{\mathrm{d}^2}{\mathrm{d}g^2} f(g(t)) \\ &+ \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} g^3(t) - 3g(t)\frac{\mathrm{d}}{\mathrm{d}t} g^2(t) + 3g^2(t)\frac{\mathrm{d}}{\mathrm{d}t} g(t)\right. \\ &\left. - g^3(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)\frac{\mathrm{d}^3}{\mathrm{d}g^3} f(g(t)) + \cdots, \end{aligned} $$

where $\mathbb{1}$ is the unit step function. Since $\frac{\mathrm{d}}{\mathrm{d}t} g^k(t) = \sum_{r=1}^{k-1}\binom{k}{r}(-g(t))^r \frac{\mathrm{d}}{\mathrm{d}t} g^{k-r}(t),\ k \in \mathbb{N}$ and $k > 1$ [7], and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t) = 0$ for $t > 0$, all the terms are zero except the one corresponding to $k = 1$. This term gives the usual chain rule.
□

Corollary 2.1: [7] If $\alpha = N \in \mathbb{N}$ then Lemma 2.2, part 2 reduces to

$$ \frac{\mathrm{d}^N}{\mathrm{d}t^N}(f \circ g)(t) = \sum_{k=0}^{N} \frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r \frac{\mathrm{d}^N}{\mathrm{d}t^N} g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)). $$

Let P_q denote the class of polynomials with real coefficients and having degree q.

Corollary 2.2: Suppose $f \in P_q$ then Lemma 2.2, part 2 becomes

$$ D^{\alpha}(f \circ g)(t) = \sum_{k=0}^{q} \frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r D^{\alpha} g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)). $$

Example 2.2: In the case where $f \in P_1$, observe

$$ \begin{aligned} D^{\alpha}(f \circ g)(t) &= (D^{\alpha}\mathbb{1}(t)) f(g(t)) \\ &+ (D^{\alpha} g(t) - g(t) D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}g} f(g(t)). \end{aligned} $$

□

The next definition and lemma are used throughout the paper.

Definition 2.4: [17] The *Mittag-Leffler function* is

$$ \mathscr{E}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, $$

where t, α, β are real numbers and $\alpha, \beta > 0$. In particular, $\mathscr{E}_{1,1}(t) = \mathrm{e}^t$.

Lemma 2.3: The following elementary identities are known to be true for real numbers $\alpha, \beta > 0$:

1) $D^{\alpha}\mathbb{1}(t) = \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$
2) $I^{\alpha k}\mathbb{1}(t) = \dfrac{t^{\alpha k}}{\Gamma(\alpha k + 1)},\ \ k \in \mathbb{N}$
3) $I^{\alpha k}\left(\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}\right) = \dfrac{t^{\alpha k + \alpha - 1}}{\Gamma(\alpha k + \alpha)},\ \ k \in \mathbb{N}$
4) $I^{\alpha k} f(t) = (I^{\alpha k - 1}\mathbb{1}(t)) * f(t),\ \ k \in \mathbb{N}$ and $k > 0$
5) $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\alpha,1}(At^{\alpha}) = At^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha})$
6) $\displaystyle\sum_{k=0}^{\infty} I^{\alpha k}\mathbb{1}(t) = \mathscr{E}_{\alpha,1}(t^{\alpha})$
7) $\mathscr{L}[D^{\alpha} f(t)] = s^{\alpha}\mathscr{L}[f(t)] - I^{1-\alpha} f(0),\ \ 0 < \alpha \leq 1$
8) $\mathscr{L}[t^{\beta-1}\mathscr{E}_{\alpha,\beta}(At^{\alpha})] = (s^{\alpha}\mathbf{I} - A)^{-1} s^{\alpha-\beta}$.

Here $*$ refers to the usual convolution product, A is a square matrix, $\mathbf{I}$ is the identity matrix, and $\mathscr{L}$ is the right-sided Laplace transform.

III. Fractional Fliess Operators

Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^{\ell}$, and the set of all such mappings will be denoted by $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$. The image of η under c is written as (c, η). For each $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, one can formally associate an m input, ℓ output operator, F_c, as described next. Let $\mathfrak{p} \geq 1$ and $a < b$ be given. For a Lebesgue measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_{\mathfrak{p}}^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R)[a, b] := \{u \in L_{\mathfrak{p}}^m[a, b] : \|u\|_{\mathfrak{p}} \leq R\}$. The next two definitions are fundamental.

Definition 3.1: Let $0 < \alpha \leq 1$ and $T \in \mathbb{R}$ be fixed. The *fractional iterated integral* for any $\eta = x_i\eta' \in X^*$ is the mapping $E_{\eta}^{\alpha} : L_1^m[0, T] \to \mathbb{R}[0, T]$ defined by the recursion

$$ E_{x_i\eta'}^{\alpha}[u](t) = I^{\alpha}[u_i(\tau)\, E_{\eta'}^{\alpha}[u](\tau)](t), $$

where $E^\alpha_\emptyset := 1$ and

$$u_0(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & : \text{Riemann-Liouville derivative approach} \\ 1 & : \text{Caputo derivative approach.} \end{cases}$$

Note that when $\alpha = 1$, the resulting iterated integrals are continuous. But when $0 < \alpha < 1$, the corresponding fractional iterated integrals will be continuous if, for example, the functions u_i, $i = 1, 2, \ldots, m$ are all continuous. Weaker sufficient conditions also exist, see [3, Definition 2.1.1].

Definition 3.2: For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $0 < \alpha \leq 1$ the *fractional Fliess operator* is defined formally by

$$F^\alpha_c[u](t) = \sum_{\eta\in X^*} (c, \eta)\, E^\alpha_\eta[u](t).$$

Two interpretations of this definition are presented next, one corresponding to the Riemann-Liouville derivative and the other to the Caputo derivative. To keep the analysis manageable, the focus in these subsections will be on linear, time-invariant (LTI) systems. A more general treatment will be pursued in a later publication.

A. Riemann-Liouville fractional derivative approach

Consider a single-input, single-output LTI system

$$D^\alpha z = Az + Bu,\ z(0) = z_0 \tag{1a}$$

$$y = Cz, \tag{1b}$$

where $A \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nx1}, C \in \mathbb{R}^{1xn}$ and $0 < \alpha \leq 1$. The state equation can be written in integral form using Lemma 2.2, part 1 as

$$z(t) = I^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + AI^\alpha z(t) + BI^\alpha u(t). \tag{2}$$

Since integration and differentiation are defined componentwise, there is no loss of generality in the following analysis by assuming $n = 1$. For any $F \in P_1$, the differential chain rule in integral form given in Example 2.2 becomes

$$D^\alpha F(z(t)) = (D^\alpha \mathbb{1}(t))F(z(t)) + (D^\alpha z(t) - z(t)D^\alpha \mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t)).$$

Using this result and Lemma 2.2, part 1 gives

$$\begin{aligned} F(z(t)) &= I^{1-\alpha}F(z(0))\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + I^\alpha((D^\alpha \mathbb{1}(t))F(z(t))) \\ &\quad + A\, I^\alpha\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) + B\, I^\alpha\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad - I^\alpha\left((D^\alpha \mathbb{1}(t))z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right). \end{aligned}$$

Now let $F(z(t)) = Az(t)$ above and then substitute the resulting equation into (2). This yields

$$\begin{aligned} z(t) &= I^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + AI^{1-\alpha}z(0)I^\alpha\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ &\quad + BI^\alpha u(t) + ABI^{2\alpha}u(t) + A^2I^{2\alpha}z(t). \end{aligned}$$

Repeating this procedure with $F(z(t)) = A^2z(t)$ gives

$$\begin{aligned} z(t) &= I^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + AI^{1-\alpha}z(0)I^\alpha\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ &\quad + A^2I^{1-\alpha}z(0)I^{2\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + BI^\alpha u(t) + ABI^{2\alpha}u(t) \\ &\quad + A^2BI^{3\alpha}u(t) + R_2(z(t)), \end{aligned}$$

where $R_2(z(t))$ contains all the integrals depending explicitly on $z(t)$. Continuing in this manner produces in the limit

$$z(t) = \sum_{k=0}^\infty A^kI^{1-\alpha}z(0)I^{\alpha k}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + \sum_{k=1}^\infty A^{k-1}BI^{\alpha k}u(t), \tag{3}$$

which can be viewed as a fractional form of the usual Peano-Baker formula [4], [21]. Equivalently, the series solution of (1a) is

$$z(t) = \sum_{\eta\in X^*} (c_z, \eta)\, E^\alpha_\eta[u](t), \tag{4}$$

where $u_0(t) := t^{\alpha-1}/\Gamma(\alpha)$ as indicated in Definition 3.1, and the coefficients of the generating series for $y(t) = Cz(t) = F^\alpha_c[u](t)$ are

$$(c, \eta) = \begin{cases} CA^kB & : \eta = x_0{}^kx_1,\ k \geq 0 \\ CA^kI^{1-\alpha}(z_0) & : \eta = x_0{}^k,\ k \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

The fractional iterated integral in (4) is

$$E^\alpha_\eta[u](t) = \begin{cases} I^{\alpha k}u(t) & : \eta = x_0{}^{k-1}x_1,\ k \geq 1 \\ I^{\alpha k}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] & : \eta = x_0{}^k,\ k \geq 0. \end{cases}$$

Observe that using Lemma 2.3, parts 3, 4 on (3) yields

$$\begin{aligned} y(t) &= C\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha k + \alpha)}t^{\alpha-1}I^{1-\alpha}z(0) \\ &\quad + C\sum_{k=1}^\infty A^{k-1}I^{\alpha k-1}\,\mathbb{1}(t)B * u(t) \\ &= C\mathscr{E}_{\alpha,\alpha}(At^\alpha)t^{\alpha-1}I^{1-\alpha}z(0) \\ &\quad + C(t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^\alpha))B * u(t). \end{aligned}$$

As a check, take the Laplace transform of (1) and apply Lemma 2.3, part 7. Then it follows that

$$s^\alpha\mathscr{L}[z(t)] - I^{1-\alpha}z(0) = A\mathscr{L}[z(t)] + B\mathscr{L}[u(t)],$$

and thus,

$$\mathscr{L}[z(t)] = (s^\alpha I - A)^{-1}I^{1-\alpha}z(0) + (s^\alpha I - A)^{-1}B\mathscr{L}[u(t)].$$

Taking the inverse Laplace transform and using Lemma 2.3, parts 3, 8 gives

$$\begin{aligned} y(t) = Cz(t) &= C\mathscr{E}_{\alpha,\alpha}(At^\alpha)t^{\alpha-1}I^{1-\alpha}z(0) \\ &\quad + C(t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^\alpha))B * u(t), \end{aligned} \tag{5}$$

which is the same result obtained using (4). That is, the generating series c can be written is terms of the function $t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^\alpha)$, which is a generalization of the matrix exponential e^{At} when $\alpha = 1$. Also note that in this Riemann-Liouville approach the coefficients are not explicit functions of $z(0)$ but rather $I^{1-\alpha}z(0)$. This *fractional* initial condition can be written as

$$I^{1-\alpha}z(t)\Big|_{t=0} = \frac{z(t)\Gamma(\alpha)}{t^{\alpha-1}}\Big|_{t=0}$$

[18], [22].

B. Caputo fractional derivative approach

Consider the single-input, single-output LTI system related to (1) by Lemma 2.1,

$$^{C}D^{\alpha} z = Az + Bu, \; z(0) = z_0 \tag{6a}$$

$$y = Cz, \tag{6b}$$

where $A \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nx1}, C \in \mathbb{R}^{1xn}$ and $0 < \alpha \leq 1$. The state equation can be written in integral form using Lemma 2.2, part 1 as

$$z(t) = z(0) + AI^{\alpha}z(t) + BI^{\alpha}u(t). \tag{7}$$

Following the same procedure as in the previous section and assuming $n = 1$ yields

$$\begin{aligned}F(z(t)) &= F(z(0)) + I^{\alpha}((D^{\alpha}\,\mathbb{1}(t))(F(z(t)) - F(z(0)))) \\ &+ A\,I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) + B\,I^{\alpha}\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &- I^{\alpha}\left((D^{\alpha}\,\mathbb{1}(t))(z(t) - z(0))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right).\end{aligned}$$

Now let $F(z(t)) = Az(t)$ above and then substitute the result into (7). This gives

$$\begin{aligned}z(t) &= z(0) + Az(0)I^{\alpha}\,\mathbb{1}(t) + BI^{\alpha}u(t) + ABI^{2\alpha}u(t) \\ &+ R_1(z(t)),\end{aligned}$$

where $R_1(z(t))$ contains all the integrals depending explicitly on $z(t)$. Continuing in this way produces the Caputo analogue of (3), namely,

$$z(t) = \sum_{k=0}^{\infty} A^k z(0) I^{\alpha k}\,\mathbb{1}(t) + \sum_{k=1}^{\infty} A^{k-1} B I^{\alpha k} u(t). \tag{8}$$

The main differences in this case are the presence of $z(0)$ instead of $I^{1-\alpha}z(0)$ in the series coefficients and the factor of $\mathbb{1}(t)$ instead of $t^{\alpha-1}/\Gamma(\alpha)$ in the iterated integrals. The series solution of (6a) is then

$$z(t) = \sum_{\eta \in X^*} (c_z, \eta)\; E^{\alpha}_{\eta}[u](t), \tag{9}$$

where $u_0(t) := \mathbb{1}(t)$ as indicated in Definition 3.1. The coefficients of the generating series for $y(t) = Cz(t) = F^{\alpha}_{c}[u](t)$ are

$$(c,\eta) = \begin{cases} CA^k B & : \eta = x_0{}^k x_1, \; k \geq 0 \\ CA^k z_0 & : \eta = x_0{}^k, \; k \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

The fractional iterated integral in (9) is

$$E^{\alpha}_{\eta}[u](t) = \begin{cases} I^{\alpha k} u(t) & : \eta = x_0{}^{k-1} x_1, \; k \geq 1 \\ I^{\alpha k}\,\mathbb{1}(t) & : \eta = x_0{}^k, \; k \geq 0. \end{cases}$$

Observe that using Lemma 2.3, parts 2, 4, 6 on (8) yields

$$\begin{aligned}y(t) &= C\sum_{k=0}^{\infty} \frac{(At^{\alpha})^k}{\Gamma(\alpha k+1)} z_0 + \sum_{k=1}^{\infty} A^{k-1} I^{\alpha k-1}\,\mathbb{1}(t) B * u(t) \\ &= C\mathscr{E}_{\alpha,1}(At^{\alpha})z_0 + C(t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha}))B * u(t)\end{aligned}$$

Again, as a check, take the Laplace transform of (6) and apply Lemma 2.3, part 7. One then has that

$$s^{\alpha}\mathscr{L}[z] - s^{\alpha-1}z_0 = A\mathscr{L}[z] + B\mathscr{L}[u],$$

and thus,

$$\mathscr{L}[z] = (s^{\alpha}I - A)^{-1}s^{\alpha-1}z_0 + (s^{\alpha}I - A)^{-1}B\mathscr{L}[u].$$

Taking the inverse Laplace transform and using Lemma 2.3, parts 2, 8 gives

$$y(t) = C\mathscr{E}_{\alpha,1}(At^{\alpha})z_0 + C(t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha}))B * u(t), \tag{10}$$

which is the same result obtained using (9). Comparing (5) and (10), it is evident that both approaches produce the same impulse response $h(t) = C(t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha}))B$, while the zero-input state responses differ in the way they depend on the initial conditions. Namely, $\mathscr{E}_{\alpha,\alpha}(At^{\alpha})t^{\alpha-1}I^{1-\alpha}z(0)$ in the Riemann-Liouville case versus $\mathscr{E}_{\alpha,1}(At^{\alpha})z_0$ in the Caputo case.

IV. Convergence Conditions

In this section, convergence conditions for the operator F^{α}_{c} are describe for the case where $\alpha = 1$ but c has a *fractional* nature as described next.

Definition 4.1: [1] A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is said to be *Gevrey of order s*, where s is a rational number, if there exists constants $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^s, \;\; \forall \eta \in X^*.$$

In [12], the case $s = 1$ provides for a type of local convergence for Fliess operators, while the stronger condition $s = 0$ provides global convergence. But here the main interest is when $0 < s < 1$. A useful product in the following analysis is the shuffle product, the $\mathbb{R}$-bilinear extension of the shuffle product of two words, which is defined inductively by

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi)$$

with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ for all $\eta, \xi \in X^*$ and $x_i, x_j \in X$ [6]. Henceforth, E^1_{η} is abbreviated as E_{η} and F^1_c as F_c.

Lemma 4.1: [5] Let $X = \{x_0, x_1, \ldots x_m\}$. For any $u \in L^m_1[0,T]$ and $\eta \in X^*$

$$|E_{\eta}[u](t)| \leq E_{\eta}[\bar{u}](t), \; 0 \leq t \leq T, \tag{11}$$

where $\bar{u} \in L^m_1[0,T]$ has components $\bar{u}_j := |u_j|$, $j = 1, 2, \ldots m$. Furthermore, for any integers $r_j \geq 0$ it follows that

$$\left|\prod_{j=0}^{m} E_{x_j{}^{r_j}}[u](t)\right| \leq \prod_{j=0}^{m} \frac{U_j^{r_j}(t)}{r_j!}, \; 0 \leq t \leq T,$$

where $U_j(t) := \int_0^t |u_j(s)|ds$. In particular, if on $[0,T]$ it is assumed that $\max\{\|u\|_1, T\} \leq R$ then

$$\prod_{j=0}^{m} E_{x_j{}^{r_j}}[\bar{u}](t) \leq \frac{R^k}{\prod_{j=0}^{m} r_j!}, \quad 0 \leq t \leq T, \qquad (12)$$

where $k = \sum_j r_j$.

The main result of this section is the following theorem.

Theorem 4.1: Suppose c is a Gevrey series of order $0 \leq s < 1$, then for any $u \in L_1^m[0,T]$ the series

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t)$$

converges absolutely and uniformly on $[0,T]$ for any $T > 0$.

Proof: Fix some $T > 0$. Pick any $u \in L_1^m[0,T]$ and let $R = \max\{\|u\|_1, T\}$ on $[0,T]$. Observe that from (11)

$$\begin{aligned}
F_c[u](t) &= \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t) \\
&\leq \sum_{\eta \in X^*} |(c,\eta)|\, |E_\eta[u](t)| \\
&\leq \sum_{k=0}^{\infty} K M^k (k!)^s \sum_{\eta \in X^k} E_\eta[\bar{u}](t) \\
&= \sum_{k=0}^{\infty} K M^k (k!)^s \sum_{\substack{r_0,r_1,\ldots,r_m \geq 0 \\ r_0+r_1+\cdots+r_m = k}} \prod_{j=0}^{m} E_{x_j{}^{r_j}}[\bar{u}](t),
\end{aligned}$$

where the identities

$$\sum_{\eta \in X^k} \eta = \sum_{\substack{r_0,r_1,\ldots,r_m \geq 0 \\ r_0+r_1+\cdots+r_m = k}} x_0^{r_0} \sqcup\!\sqcup\, x_1^{r_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, x_m^{r_m}, \quad k \geq 0$$

and $E_\eta E_\xi = E_{\eta \sqcup\!\sqcup \xi}$ were used in the last step. Then from (12) it follows that

$$\begin{aligned}
F_c[u](t) &\leq \sum_{k=0}^{\infty} K M^k (k!)^s \sum_{\substack{r_0,r_1,\ldots,r_m \geq 0 \\ r_0+r_1+\cdots+r_m = k}} \frac{R^k}{\prod_{j=0}^{m} r_j!} \\
&= \sum_{k=0}^{\infty} K (MR)^k (k!)^s \sum_{\substack{r_0,r_1,\ldots,r_m \geq 0 \\ r_0+r_1+\cdots+r_m = k}} \frac{1}{\prod_{j=0}^{m} r_j!} \\
&= \sum_{k=0}^{\infty} K (MR)^k (k!)^s \frac{(m+1)^k}{k!} \\
&= \sum_{k=0}^{\infty} K (MR(m+1))^k \frac{1}{(k!)^{1-s}}.
\end{aligned}$$

Applying the ratio test to the sequence $a_k := K(MR(m+1))^k/(k!)^{1-s}$, $k \geq 0$ and using the fact that $0 < 1-s \leq 1$:

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \frac{(MR(m+1))^{k+1}}{((k+1)!)^{1-s}} \frac{(k!)^{1-s}}{(MR(m+1))^k} \\
&= (MR(m+1)) \lim_{k\to\infty} \frac{1}{(k+1)^{1-s}} \\
&= 0.
\end{aligned}$$

Thus, the series $F_c[u](t)$ converges absolutely and uniformly on $[0,T]$ ■

The notion of a *bounding function* has proved useful in computing the radius of convergence for Fliess operators [23]. The next corollary describes a class of bounding functions for generating series with a certain Gevrey degree.

Corollary 4.1: Suppose c is Gevrey of order $0 \leq s < 1$ with growth constants $K, M > 0$. Then for any $u \in L_1^m[0,T]$ and $T > 0$, the function

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t)$$

is bounded pointwise in time by the bounding function

$$B(t) = \sum_{k=0}^{\infty} K(MR(t)(m+1))^k \frac{1}{(k!)^{1-s}},$$

where $R(t) := \max\{\|u\|_{1,[0,t]}, t\}$, $t \in [0,T]$. (Here $\|\cdot\|_{1,[0,t]}$ denotes the 1-norm restricted to the interval $[0,t]$.) Furthermore, $B(t)$ is Gevrey of order $s-1$. Specifically, $B(t) = \sum_{k=0}^{\infty} c_k z^k$, where $c_k = K(k!)^{s-1}$ and $z = MR(t)(m+1)$.

Example 4.1: Consider the single-input, single-output Wiener system, where $\dot{z} = u$ with $z(0) = 0$ and $h(z) = e^z$. In which case, direct substitution of z into h gives

$$\begin{aligned}
y(t) &= \sum_{k=0}^{\infty} \frac{(z(t))^k}{k!} = \sum_{k=0}^{\infty} \frac{(E_{x_1}[u](t))^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{E_{x_1 {\sqcup\!\sqcup}\, k}[u](t)}{k!} = \sum_{k=0}^{\infty} \frac{k! E_{x_1{}^k}[u](t)}{k!} \\
&= \sum_{k=0}^{\infty} E_{x_1{}^k}[u](t) = F_c[u](t).
\end{aligned}$$

Note here that the generating series $c = \sum_{k=0}^{\infty} x_1{}^k$ is Gevrey of order $s = 0$ with growth constants $K = M = 1$. For any $T > 0$ and $u \in L_1[0,T]$, let $R(t) = \max\{\|u\|_{1,[0,t]}, t\}$ on $[0,T]$. It follows from Corollary 4.1 that

$$\begin{aligned}
F_c[u](t) &= \sum_{k=0}^{\infty} E_{x_1{}^k}[u](t) \leq \sum_{k=0}^{\infty} \frac{2^k R(t)^k}{k!} = e^{2R(t)} \\
&= B(t).
\end{aligned}$$

This example is consistent with the fact that globally convergent generating series (i.e, $s = 0$) are known to have exponential bounding functions [12]. □

The final theorem of the paper is a factional generalization of the previous example. That is, in light of Theorem 4.1, when $0 \leq s < 1$, the corresponding Fliess operator is well defined on $[0,T]$ for any $T > 0$. Therefore, as in the $s = 0$ case, it may also have a bounding function which is entire. To develop this result, the following technical lemma is needed first.

Lemma 4.2: For any integer $l \geq 0$ and $0 < r \leq 1$ such that $lr \gg 1$ it follows that

$$(lr)! \leq K_r M_r^l (l!)^r,$$

where $K_r = ((2\pi)^{1-r} r)^{1/2}$ and $M_r = r^r$.

Proof: Using Stirling's formula, $k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$, $k \gg 1$, observe

$$\begin{aligned}(lr)! &\approx \sqrt{2\pi lr}\left(\frac{lr}{e}\right)^{lr} \\ &\approx \frac{\sqrt{2\pi lr}}{\left(\sqrt{2\pi l}\right)^r}(r^r)^l (l!)^r \\ &= \frac{1}{l^{(1-r)/2}}((2\pi)^{1-r} r)^{1/2}(r^r)^l (l!)^r.\end{aligned}$$

Noting that $1/l^{(1-r)/2} \leq 1$, it follows that

$$(lr)! \leq ((2\pi)^{1-r} r)^{1/2}(r^r)^l (l!)^r$$

when $lr \gg 1$. ■

In this theorem, a general bounding function is given in term of a Mittag-Leffler function.

Theorem 4.2: Suppose c is a Gevrey generating series of order $0 \leq s < 1$ with growth constants $K, M > 0$. Let $R(t) = \max\{\|u\|_{1,[0,t]}, t\}$ on the interval $[0,T]$. Then a bounding function for $F_c[u](t)$ is

$$B(t) = KK_{\bar{s}}\mathscr{E}_{-\bar{s},1}(M_{\bar{s}}A(t)),$$

where $\bar{s} = s - 1$ is the Gevrey order of $B(t)$, $A(t) = MR(t)(m+1)$, $K_{\bar{s}} = (-(2\pi)^{1+\bar{s}}\bar{s})^{1/2}$ and $M_{\bar{s}} = (-\bar{s})^{-\bar{s}}$.

Proof: Setting $A(t) = MR(t)(m+1)$, it follows from Corollary 4.1 that the bounding function

$$B(t) = K\sum_{k=0}^{\infty}\frac{1}{k!^{-\bar{s}}}A(t)^k$$

applies. Using Lemma 4.2 with $l = k$ and $r = -\bar{s}$ gives

$$\begin{aligned}B(t) &\leq K\sum_{k=0}^{\infty}\frac{K_{\bar{s}}M_{\bar{s}}^{\;k}}{(-k\bar{s})!}A(t)^k \\ &= KK_{\bar{s}}\sum_{k=0}^{\infty}\frac{1}{\Gamma(-k\bar{s}+1)}(M_{\bar{s}}A(t))^k \\ &= KK_{\bar{s}}\mathscr{E}_{-\bar{s},1}(M_{\bar{s}}A(t)),\end{aligned}$$

where $K_{\bar{s}} = (-(2\pi)^{1+\bar{s}}\bar{s})^{1/2}$ and $M_{\bar{s}} = (-\bar{s})^{-\bar{s}}$. ■

As expected, when $s = 0$ then $KK_{\bar{s}}\mathscr{E}_{-\bar{s},1}(M_{\bar{s}}A(t)) = K\mathscr{E}_{1,1}(A(t)) = K\mathrm{e}^{MR(t)(m+1)}$.

V. Conclusions and Future Work

This paper described two possible fractional generalizations for Fliess operators, one compatible with the notation of the Riemann-Liouville derivative and the other with the Caputo derivative. The linear time-invariant case was fully exercised to pin point the key differences in the corresponding realization theories. Then sufficient conditions for convergence were given for Fliess operators with Gevrey type generating series and non-fractional iterated integrals. A class of so called bounding functions was also described.

Future work will include a systematic development of the full nonlinear case starting with bilinear systems. This will be non-trivial since, for example, the chain rule in fractional calculus involves an infinite series. In addition, the authors plan to explore possible shuffle algebras for fractional iterated integrals as a prelude to system interconnection theory.

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References

[1] Y. Andrè, Arithmetic Gevrey series and transcendence. A survey, *J. Théorie Nombres Bordeaux*, 15 (2003) 1–10 .

[2] D. Baleanu, K. Diethelm, E. Scalas, and J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, Boston, MA, 2012.

[3] B. Bonilla and J. J. Trujillo, Fractional order continuity and some properties about integrability and differentiability of real functions, *J. Math. Anal. Appl.*, 231 (1999) 205–212.

[4] L. A. Duffaut Espinosa, *Interconnections of Nonlinear Systems Driven by L_2-Itô Stochastic Processes*, Old Dominion University, 2009.

[5] L. A. Duffaut Espinosa, W. S. Gray, and O. R. González, On Fliess operators driven by L_2-Itô random processes, *Proc. 48nd IEEE Conf. on Decision and Control*, Shanghai, China, 2009, pp. 7478–7484.

[6] M. Fliess, Fonctionnelles causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. France*, 109 (1981) 3–40.

[7] I. S. Gradshteyn and I. M.Ryzhik, *Table of Integrals, Series, and Products*, 7th Ed., Academic Press, New York, 2007.

[8] W. S. Gray and L. A. Duffaut Espinosa, Feedback transformation group for nonlinear input-output systems, *Proc. 52nd IEEE Conf. on Decision and Control*, Florence, Italy, 2013, pp. 2570–2575.

[9] W. S. Gray, L. A. Duffaut Espinosa, and M. Thitsa, Left inversion of analytic nonlinear SISO systems via formal power series methods, *Automatica*, 50 (2014) 2381–2388.

[10] W. S. Gray and Y. Li, Generating series for interconnected analytic nonlinear systems, *SIAM J. Control Optim.*, 44 (2005) 646–672.

[11] W. S. Gray, M. Thitsa, and L. A. Duffaut Espinosa, Pre-Lie algebra characterization of SISO feedback invariants, *Proc. 53nd IEEE Conf. on Decision and Control*, Los Angeles, CA, 2014, pp. 4807–4813.

[12] W. S. Gray and Y. Wang, Fliess operators on L_p spaces: Convergence and continuity, *Systems Control Lett.*, 46 (2002) 67–74.

[13] A. Isidori, *Nonlinear Control Systems*, 3rd Ed., Springer–Verlag, London, 1995.

[14] S. Kazem, Exact solution of some linear fractional differential equations by Laplace transform, *Int. J. Nonlinear Sci.*, 16 (2013) 3–11.

[15] A. A. Kilbas, M. Rivero, L. Rodríguez-Germá, and J. J. Trujillo, α-Analytic solutions of some linear fractional differential equations with variable coefficients, *Appl. Math. Comput.*, 187 (2007) 239–249.

[16] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[17] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[18] C. Li, D. Qian and Y. Chen, On Riemann-Liouville and Caputo derivatives, *Discrete Dyn. Nat. Soc.*, 2011 (2011) 1–15.

[19] T. Osler, The fractional derivative of a composite function, *SIAM J. Math. Anal.*, 1 (1970) 288–293.

[20] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

[21] W. J. Rugh, *Nonlinear System Theory: The Volterra/Wiener Approach*, The Johns Hopkins University Press, Baltimore, MD, 1981.

[22] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[23] M. Thitsa and W. S. Gray, On the radius of convergence of interconnected analytic nonlinear input-output systems, *SIAM J. Control Optim.*, 50 (2012) 2786–2813.