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Analytic Left Inversion of SISO Lotka-Volterra Models

W. Steven Gray
Department of Electrical and Computer Engineering
Old Dominion University
Norfolk, VA 23529 USA
Email: sgray@odu.edu

Luis A. Duffaut Espinosa
Department of Electrical and Computer Engineering
George Mason University
Fairfax, VA 22030 USA
Email: l.duffautespinosa@gmail.com

Kurusch Ebrahimi-Fard
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
28049 Madrid, Spain
Email: kurusch@icmat.es

***Abstract*—There is great interest in managing populations of animal species such as fish that are vital food sources for humans. A classical population model is the Lotka-Volterra system, which can be viewed as a nonlinear input-output system where time-varying parameters are taken as inputs and the population levels are the outputs. If some of these inputs can be actuated, this sets up an open-loop control problem where a certain population profile as a function of time is desired, and the objective is to determine suitable system inputs to produce this profile. Mathematically, this is a left inversion problem. In this paper, this inversion problem is solved analytically using known methods based on combinatorial Hopf algebras. The focus is on the simplest case, two species models and single-input, single-output (SISO) systems.**

I. Introduction

There is great interest in managing populations of animal species such as fish that are vital food sources for humans [15]. Technological advances make it possible to practically implement some ecological control systems, but without rigorous models and corresponding control strategies, the design of such systems remains a formidable problem. For example, Ainsworth et al. in their recent plea for fishery models stated that: *We are drowning in information and that successful conservation and resource management depend ultimately on the rigorous synthesis of information* [1]. A classical population model is the Lotka-Volterra system, a coupled set of autonomous quadratic ordinary differential equations which describe the interactions between various species in an ecosystem. The coefficients entering the model are a function of both natural environmental conditions and human intervention. For example, populations of fish species can be influenced by a number of factors such as water temperature, the availability of food, the number of predators, harvesting, and the presence of disease and pollution. Many of these parameters are time varying, and a few can be affected by human behavior through certain ecology practices and environmental laws. In which case, the Lotka-Volterra system can also be viewed as a nonlinear input-output system, where the time-varying parameters are the inputs and the population levels are the outputs. Given that some of these inputs can be actuated, this sets up an open-loop control problem where a certain population profile as a function of time is desired, and the objective is to determine suitable system inputs to produce this profile. Mathematically, this is a left inversion problem.

Inversion problems have a long history in nonlinear control theory [4], [10], [11], [16], [17], but their application to population biology as described above is fairly recent. Up to the present, the main focus has been on using methods related to dynamic inversion [3], [19]–[21]. This class of techniques, which is largely based on a geometric view of the system, generates a left inverse by driving a certain inverse dynamical system with the output of the original system. [6], [12]. While it is quite effective in certain situations, it does not provide any explicit representation of the input function one seeks. Therefore, it is difficult to use as an analytical design tool. In this paper, a distinctly different approach is taken, one that is purely algebraic and relies on the underlying combinatorial structures of the nonlinear input-output system. The theory behind this approach is described in [8], [9] and uses concepts from combinatorial Hopf algebras. The basic idea is that if an input-output map $F : u \mapsto y$ has a kind of analytic representation known as a Chen-Fliess series or Fliess operator, and an analytic output in the range of F is selected, then under certain conditions the left inverse can be computed explicitly by a formula. From this formula, it is easier to see how to do open-loop/off-line control, which is quite natural for many ecological control problems. As the theory is still evolving, the focus here will be on the simplest case, two species models and the single-input, single-output (SISO) systems they produce.

The paper is organized as follows. In the next section some preliminaries are given to make the presentation self-contained. In particular, the Lotka-Volterra model is described, and the basic facts about Fliess operators are presented, including the main inversion theorem. In the subsequent section, the inversion tool is applied to an input-output system derived from the Lotka-Volterra system to produce its left inverse. These results are then verified in Section IV by numerical simulation.

II. Preliminaries

A. Lotka-Volterra Model for Population Dynamics

A fundamental mathematical model for describing population dynamics is the n species competitive Lotka-Volterra model

$$\dot{z}_i = \beta_i z_i + \sum_{j=1}^{n} \alpha_{ij} z_i z_j, \quad i = 1, 2, \ldots, n, \tag{1}$$

where z_i denotes the biomass of the i-th species (roughly proportional to the population), the parameter β_i is the intrinsic growth rate for the i-th population (the Malthus

TABLE I. FOUR SISO PREDATOR-PREY SYSTEMS WITH PREY AS THE OUTPUT $y = z_1$

I/0 map	state space realization	relative degree	range restrictions
$F_{c_1} : \beta_1 \mapsto y$	$g_0(z) = \begin{bmatrix} -\alpha_{12}z_1z_2 \\ -\beta_2z_2 + \alpha_{21}z_1z_2 \end{bmatrix},\ g_1(z) = \begin{bmatrix} z_1 \\ 0 \end{bmatrix}$	1	$(c_y, \emptyset) = (c_1, \emptyset)$
$F_{c_2} : \alpha_{12} \mapsto y$	$g_0(z) = \begin{bmatrix} \beta_1z_1 \\ -\beta_2z_2 + \alpha_{21}z_1z_2 \end{bmatrix},\ g_1(z) = \begin{bmatrix} -z_1z_2 \\ 0 \end{bmatrix}$	1	$(c_y, \emptyset) = (c_2, \emptyset)$
$F_{c_3} : \beta_2 \mapsto y$	$g_0(z) = \begin{bmatrix} \beta_1z_1 - \alpha_{12}z_1z_2 \\ \alpha_{21}z_1z_2 \end{bmatrix},\ g_1(z) = \begin{bmatrix} 0 \\ -z_2 \end{bmatrix}$	2	$(c_y, \emptyset) = (c_3, \emptyset)$, $(c_y, x_0) = (c_3, x_0)$
$F_{c_4} : \alpha_{22} \mapsto y$	$g_0(z) = \begin{bmatrix} \beta_1z_1 - \alpha_{12}z_1z_2 \\ -\beta_2z_2 \end{bmatrix},\ g_1(z) = \begin{bmatrix} 0 \\ z_1z_2 \end{bmatrix}$	2	$(c_y, \emptyset) = (c_4, \emptyset)$, $(c_y, x_0) = (c_4, x_0)$

parameter), and the competition coefficient α_{ij} models the effect of the j-th species on the i-th species (usually $\alpha_{ii} = 0$). This model is known under realistic conditions to have stable equilibria, stable limit cycles, and can exhibit chaotic behavior, depending largely on the number of species which are interacting [13], [18]. As described in the introduction, a number of environmental factors and human behaviors are known to influence the dynamics of animal populations. This can be modeled by introducing time dependence in the parametrization of (1), namely $\beta_i(t)$ and $\alpha_{ij}(t)$ are taken as integrable functions of time in some sense. In many applications, it is not possible or practical to estimate the size of each population directly, so output functions are often introduced to represent measurement processes

$$y_i = h_i(z), \ \ i = 1, 2, \ldots, \ell, \tag{2}$$

where $z = [z_1 \cdots z_n]^T$. It is clear that since the inputs enter the dynamics linearly, (1)-(2) can be rewritten in the standard form

$$\begin{aligned} \dot{z} &= g_0(z) + \sum_{i=1}^{m} g_i(z)u_i, \ \ z(0) = z_0 \\ y &= h(z), \end{aligned}$$

where each g_i and h_i is an analytic vector field and function, respectively, on some neighborhood $W \subseteq \mathbb{R}^n$, and u_i, $i = 1, 2, \ldots, m$ are the m time dependent parameter functions. In the special case where $n = 2$, this model reduces to the classical predator-prey system, where here z_1 will be assumed to be the prey species. That is,

$$\dot{z}_1 = \beta_1z_1 - \alpha_{12}z_1z_2 \tag{3a}$$
$$\dot{z}_2 = -\beta_2z_2 + \alpha_{21}z_1z_2, \tag{3b}$$

where the model has been re-parameterized so that $\beta_i, \alpha_{ij} > 0$. When the parameters are constant, this system has exactly two equilibria, a stable equilibrium at $(0, 0)$ and a saddle point at $z_e = (\beta_2/\alpha_{21}, \beta_1/\alpha_{12})$. The vector fields are complete within the first quadrant giving concentric periodic trajectories about z_e. If it is assumed that $y = z_1$, then there are four possible SISO maps to consider as shown in Table I.

B. Fliess Operators and Their Inverses

The inversion formula to be employed requires that each input-output system be written as a Fliess operator. Some elements of this theory are outlined next. More complete treatments can be found in [8], [9].

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*}(c, \eta)\eta$. If the *constant term* $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by $\sqcup\!\sqcup$. The latter is the $\mathbb{R}$-bilinear extension of the shuffle product of two words, which is defined inductively by

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi)$$

with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ for all $\eta, \xi \in X^*$ and $x_i, x_j \in X$.

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : \ 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta\in X^*} (c,\eta)\, E_\eta[u](t,t_0).$$

If there exist real numbers $K_c, M_c > 0$ such that

$$|(c,\eta)| \le K_c M_c^{|\eta|}|\eta|!, \quad \eta\in X^*,$$

then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0, t_0{+}T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0{+}T]$ for sufficiently small $R,T>0$, where the numbers $\mathfrak{p},\mathfrak{q}\in[1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}+1/\mathfrak{q}=1$. (Here, $|z|:=\max_i |z_i|$ when $z\in\mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$.

When Fliess operators F_c and F_d are connected in a parallel-product fashion, it is known that $F_cF_d = F_{c \sqcup\!\sqcup\, d}$. If F_c and F_d with $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade manner, the composite system $F_c\circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where $c\circ d$ denotes the *composition product* of c and d as described in [8]. This product is associative and $\mathbb{R}$-linear in its left argument c. In the event that two Fliess operators are interconnected to form a feedback system, the closed-loop system has a Fliess operator representation whose generating series is the *feedback product* of c and d, denoted by $c@d$. This product can be explicitly computed via Hopf algebra methods. The basic idea is to consider the set of operators

$$\mathscr{F}_\delta = \{I + F_c \;:\; c\in\mathbb{R}^m\langle\langle X\rangle\rangle\},$$

where I denotes the identity map, as a group under composition. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ is denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. This set also forms a group under the composition product induced by operator composition, namely, $c_\delta\circ d_\delta := \delta + d + c\,\tilde{\circ}\, d$, where $\tilde{\circ}$ denotes the *modified composition product* [8]. The group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle,\circ,\delta)$ has coordinate functions that form a Faà di Bruno type Hopf algebra. In which case, the group (composition) inverse $c_\delta^{\circ -1}$ can be computed efficiently via the antipode of this Hopf algebra [2], [7]. This inverse also describes the feedback product as $c@d = c\circ(\delta - d\circ c)^{\circ -1}$ and can be used for input-output inversion as described next.

It was shown in [22] that F_c will map every input which is analytic at t_0 to an output which is also analytic at t_0 provided $c\in\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. The problem considered here is computing a left inverse of F_c given a real analytic function in its range. The treatment is restricted to the SISO case and without loss of generality assume $t_0=0$. Note that every $c\in\mathbb{R}\langle\langle X\rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k\ge 0}(c,x_0^k)x_0^k$ and $c_F := c - c_N$. The following definition will provide a sufficient condition under which the left inverse of F_c exists.

Definition 2.1: Given $c\in\mathbb{R}\langle\langle X\rangle\rangle$, let $r\ge 1$ be the largest integer such that $\mathrm{supp}(c_F)\subseteq x_0^{r-1}X^*$. Then c has ***relative degree*** r if the linear word $x_0^{r-1}x_1\in\mathrm{supp}(c)$, otherwise it is not well defined.

It can be shown directly that this notion of relative degree coincides with the usual definition given in a state space setting [12], and that $c\in\mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r only if $(x_0^{r-1}x_1)^{-1}(c)$ is non proper. Here the left-shift operator for any $x_i\in X$ is defined on X^* by $x_i^{-1}(x_i\eta)=\eta$ with $\eta\in X^*$ and zero otherwise. Higher order shifts are defined inductively via $(x_i\xi)^{-1}(\cdot) = \xi^{-1}x_i^{-1}(\cdot)$, where $\xi\in X^*$. The left-shift operator is assumed to act linearly on $\mathbb{R}\langle\langle X\rangle\rangle$. It is easy to verify that each input-output map derived from (3) has well defined relative degree as shown in Table I at every point in the strict first quadrant of the state space, i.e., where neither population is extinct ($z_1,z_2>0$). In the last two cases, the systems F_{c_3} and F_{c_4} have full relative degree ($r=n=2$) and hence are differentially flat [5]. So their left inverses are more easily computed than in the first two cases.

The next theorem provides the main analytical tool used in the paper. Let $X_0 := \{x_0\}$, and $\mathbb{R}[[X_0]]$ denotes the set of all commutative series over X_0. When $c\in\mathbb{R}[[X_0]]$, $F_c[u](t)$ reduces to the Taylor series $\sum_{k\ge0}(c,x_0^k)E_{x_0^k}[u](t) = \sum_{k\ge 0}(c,x_0^k)t^k/k!$.

Theorem 2.1: [9] Suppose $c\in\mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r. Let y be analytic at $t=0$ with generating series $c_y\in\mathbb{R}_{LC}[[X_0]]$ satisfying $(c_y,x_0^k)=(c,x_0^k)$, $k=0,\ldots,r-1$. Then the input

$$u(t) = \sum_{k=0}^{\infty}(c_u,x_0^k)\frac{t^k}{k!}, \tag{4}$$

where

$$c_u = \left(\left(\frac{(x_0^r)^{-1}(c-c_y)}{(x_0^{r-1}x_1)^{-1}(c)}\right)^{\circ -1}\right)_N, \tag{5}$$

is the unique real analytic solution to $F_c[u]=y$ on $[0,T]$ for some $T>0$.

From this theorem one can deduce the restrictions on the ranges of the Lotka-Volterra input-output maps as shown in Table I. In the next section, this result is applied to determine the corresponding left inverse of one of the non-flat systems.

III. Left Inversion of Lotka-Volterra Input-Output Systems

The goal in this section is two-fold. First, the left inverse of the first system in Table I is explicitly computed as an example. Then an input design strategy is devised using this inverse to perform a kind of orbit transfer in the dynamics. The system under study is

$$\begin{aligned}\begin{bmatrix}\dot z_1(t)\\ \dot z_2(t)\end{bmatrix} &= \begin{bmatrix}-\alpha_{12}z_1(t)z_2(t)\\ -\beta_2 z_2(t)+\alpha_{21}z_1(t)z_2(t)\end{bmatrix}\\ &\quad + \begin{bmatrix} z_1(t)\\ 0\end{bmatrix}u(t) \\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}\end{aligned} \tag{6}$$

with initial conditions $z_1(0)=z_{1,0}$ and $z_2(0)=z_{2,0}$. Normalizing all the system parameters to unity, driving the system with the constant input $u(t)=\beta_1(t)=1$, and starting the model at $[z_{1,0},z_{2,0}]^T=[1/6,1/6]^T$ produces the trajectory shown in Figure 1. This will serve as the initial orbit. The goal is to determine a function $\beta_1(t)$ in order to achieve a final orbit corresponding to $\beta_1=1.5$ in

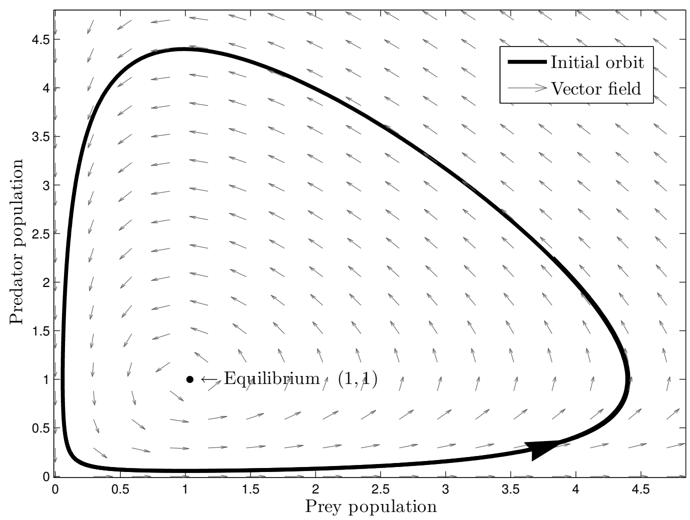

Fig. 1. Initial trajectory for normalized version of (6).

a controlled fashion. More precisely, given a transition time interval $[t_1, t_2]$, $\beta_1(t_1) = 1$, $\beta_1(t_2) = 1.5$, a fixed exit point from the initial orbit $[z_1(t_1), z_2(t_1)]^T$, and a prey population output value $y(t_2)$, find an analytic $\beta_1(t)$ for $t \in (t_1, t_2)$ so that all stated constraints are satisfied.

The first step is to compute the generating series of system (6) via iterated Lie derivatives of the output function [12]. Using the software package *Mathematica* and the NCAlgebra suite [14], it follows that

$$\begin{aligned}
c = {} & z_{1,0} - \alpha_{12} z_{2,0} z_{1,0} x_0 + z_{1,0} x_1 + \big(\alpha_{12}{}^2 z_{2,0}{}^2 z_{1,0} \\
& -\alpha_{12}\alpha_{21} z_{2,0} z_{1,0}{}^2 + \alpha_{12}\beta_2 z_{2,0} z_{1,0}\big) x_0^2 \\
& - \alpha_{12} z_{2,0} z_{1,0} x_0 x_1 - \alpha_{12} z_{2,0} z_{1,0} x_1 x_0 + z_{1,0} x_1^2 \\
& + \big(-\alpha_{12}\alpha_{21}{}^2 z_{2,0} z_{1,0}{}^3 + 4\alpha_{12}{}^2 \alpha_{21} z_{2,0}{}^2 z_{1,0}{}^2 \\
& + 2\alpha_{12}\beta_2\alpha_{21} z_{2,0} z_{1,0}{}^2 - \alpha_{12}{}^3 z_{2,0}{}^3 z_{1,0} \\
& - 3\alpha_{12}{}^2 \beta_2 z_{2,0}{}^2 z_{1,0} - \alpha_{12}\beta_2{}^2 z_{2,0} z_{1,0}\big) x_0^3 \\
& - \alpha_{12} z_{2,0} z_{1,0} x_0 x_1^2 - \alpha_{12} z_{2,0} z_{1,0} x_1 x_0 x_1 \\
& - \alpha_{12} z_{2,0} z_{1,0} x_1^2 x_0 + z_{1,0} x_1^3 + \cdots
\end{aligned}$$

In order to compute c_u via (4)-(5), one must select an output function

$$y(t) = \sum_{k=0}^{\infty} (c_y, x_0^k) \frac{t^k}{k!}, \tag{7}$$

where c_y is the corresponding generating series of y. It is sufficient to consider only polynomial outputs up to degree 3, so let $(c_y, \emptyset) = v_0$ and $(c_y, x_0^i) = v_i$ for $i = 1, 2, 3$. Thus,

$$\begin{aligned}
d := {} & \frac{(x_0^r)^{-1}(c - c_y)}{(x_0^{r-1} x_1)^{-1}(c)} \\
= {} & -\alpha_{12} z_{2,0} - \frac{v_1}{z_{1,0}} + \Bigg(\alpha_{12}\beta_2 z_{2,0} - \frac{v_2}{z_{1,0}} \\
& -\alpha_{12}\alpha_{21} z_{1,0} z_{2,0} - \frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}}\Bigg) x_0 + \frac{v_1}{z_{1,0}} x_1 \\
& + \Bigg(-\frac{v_1 \alpha_{12}{}^2 z_{2,0}{}^2}{z_{1,0}} + \frac{v_1 \alpha_{12}\beta_2 z_{2,0}}{z_{1,0}} - v_1 \alpha_{12}\alpha_{21} z_{2,0} \\
& - \frac{2 v_2 \alpha_{12} z_{2,0}}{z_{1,0}} + \alpha_{12}{}^2 \alpha_{21} z_{1,0} z_{2,0}{}^2 + 2\alpha_{12}\beta_2\alpha_{21} z_{1,0} z_{2,0} \\
& - \frac{v_3}{z_{1,0}} - \alpha_{12}\beta_2{}^2 z_{2,0} - \alpha_{12}\alpha_{21}{}^2 z_{1,0}{}^2 z_{2,0}\Bigg) x_0^2 \\
& + \left(\frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} + \frac{v_2}{z_{1,0}} - \alpha_{12}\alpha_{21} z_{1,0} z_{2,0}\right) x_0 x_1 \\
& + x_1 x_0 \left(\frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} + \frac{v_2}{z_{1,0}}\right) - \frac{v_1}{z_{1,0}} x_1 x_1 + \cdots
\end{aligned}$$

In which case,

$$\begin{aligned}
& c_u = \left(d^{\circ -1}\right)_N = \\
& \frac{v_1}{z_{1,0}} + \alpha_{12} z_{2,0} + \Bigg(\frac{v_1\left(-\frac{v_1}{z_{1,0}} - \alpha_{12} z_{2,0}\right)}{z_{1,0}} + \frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} \\
& + \frac{v_2}{z_{1,0}} - \alpha_{12}\beta_2 z_{2,0} + \alpha_{12}\alpha_{21} z_{1,0} z_{2,0}\Bigg) x_0 \\
& + \Bigg(-\frac{v_1{}^2\left(-\frac{v_1}{z_{1,0}} - \alpha_{12} z_{2,0}\right)}{z_{1,0}{}^2} \\
& + \frac{v_1\left(-\frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} - \frac{v_2}{z_{1,0}} + \alpha_{12}\beta_2 z_{2,0} - \alpha_{12}\alpha_{21} z_{1,0} z_{2,0}\right)}{z_{1,0}} \\
& - \left(\frac{v_1}{z_{1,0}} + \alpha_{12} z_{2,0}\right)\left(\frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} + \frac{v_2}{z_{1,0}} - \alpha_{12}\alpha_{21} z_{1,0} z_{2,0}\right) \\
& - \left(\frac{v_1}{z_{1,0}} + \alpha_{12} z_{2,0}\right)\left(\frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} + \frac{v_2}{z_{1,0}}\right) \\
& + \frac{v_1 \alpha_{12}{}^2 z_{2,0}{}^2}{z_{1,0}} - \frac{v_1 \alpha_{12}\beta_2 z_{2,0}}{z_{1,0}} + v_1 \alpha_{12}\alpha_{21} z_{2,0} \\
& + \frac{v_1\left(-\frac{v_1}{z_{1,0}} - \alpha_{12} z_{2,0}\right)^2}{z_{1,0}} + \frac{2 v_2 \alpha_{12} z_{2,0}}{z_{1,0}} + \frac{v_3}{z_{1,0}} \\
& -\alpha_{12}{}^2 \alpha_{21} z_{1,0} z_{2,0}{}^2 + \alpha_{12}\beta_2{}^2 z_{2,0} - 2\alpha_{12}\beta_2\alpha_{21} z_{1,0} z_{2,0} \\
& + \alpha_{12}\alpha_{21}{}^2 z_{1,0}{}^2 z_{2,0}\Bigg) x_0^2 + \cdots
\end{aligned} \tag{8}$$

This provides the most general formula for the input u to make the system output follow the path specified by the coefficients of c_y.

The next step is to determine the specific c_y which satisfies the constraints of the desired orbit transfer. The procedure is as follows:

1) Choose the time interval $[t_1, t_2]$ with $t_2 > t_1 > 0$ for the transition path. The transition duration is $t_f := t_2 - t_1$.
2) Choose the initial input value $u(t_1) = \beta_1(t_1)$ and the final input value $u(t_2) = \beta_1(t_2) =: u_f$.
3) Choose the exit point of the initial orbit $[z_1(t_1), z_2(t_1)]^T$, and pick the prey population value $y(t_2) = z_1(t_2) =: y_f$ for the entry point of the final orbit.
4) Solve the system of nonlinear algebraic equations:

$$\begin{aligned}
c_y(v_1, v_2, v_3)\Big|_{x_0^k \to t_f^k/k!,\, k>0} &= y_f \\
c_u(v_1, v_2, v_3)\Big|_{x_0^k \to t_f^k/k!,\, k>0} &= u_f.
\end{aligned} \tag{9}$$

The dependence of c_y and c_u on $\{v_1, v_2, v_3\}$ is evident in (7) and (8), respectively. Observe that v_0 is not included in (9) as a free parameter since it must equal $(c, \emptyset) = z_{1,0}$ as stated in Theorem 2.1. In general, system (9) is under determined. So one could set up an optimization problem to compute the v_i's. However, this is beyond the scope of the present paper. The objective here is to simply show that the problem is tractable. Also note that in this procedure there is no direct control over the predator population. This can be observed in step 3, where only the prey population of the final orbit entry point is specified. Therefore, the predator value during the transition is allowed to evolve freely. Finally, single-input, single-output design is often appropriate due to the limitations encountered in complex ecosystems, where relatively few parameters can be changed over time. But a multivariable procedure would likely allow one to also specify the predator population at the entry point of the final orbit as a design parameter.

IV. Numerical Simulations

In this section, an orbit transfer is numerically simulated based on the open-loop control procedure described in Section III. Assume system (6) is on the initial orbit shown in Figure 1. The generating series c of the normalized system (6) is then

$$\begin{aligned}
c = {} & \\
& 3.82405 - 8.60733x_0 + 3.82405x_1 - 4.93383x_0^2 \\
& - 8.60733x_0x_1 - 8.60733x_1x_0 + 3.82405x_1^2 \\
& + 125.97x_0^3 - 37.8487x_0^2x_1 - 4.93383x_0x_1x_0 \\
& - 8.60733x_0x_1^2 - 4.93383x_1x_0^2 - 8.60733x_1x_0x_1 \\
& - 8.60733x_1^2x_0 + 3.82405x_1^3 + 250.087x_0^4 \\
& + 236.408x_0^3x_1 + 181.189x_0^2x_1x_0 - 103.678x_0^2x_1^2 \\
& + 125.97x_0x_1x_0^2 - 37.8487x_0x_1x_0x_1 - 4.93383x_0x_1^2x_0 \\
& - 8.60733x_0x_1^3 + 125.97x_1x_0^3 - 37.8487x_1x_0^2x_1 \\
& - 4.93383x_1x_0x_1x_0 - 8.60733x_1x_0x_1^2 - 4.93383x_1^2x_0^2 \\
& - 8.60733x_1^2x_0x_1 - 8.60733x_1^3x_0 + 3.82405x_1^4 + \cdots
\end{aligned} \tag{10}$$

At time $t_1 = 12.5$ sec., the system starts transferring to a new and final orbit with a different vector field. That is, the exit point in the phase portrait is $[z_1(t_1), z_2(t_1)]^T = [3.82, 2.25]^T$, and the parameter β_1 of the population model changes from $\beta_1(t_1) = 1$ to $\beta_1(t_2) = 1.5$, which moves the equilibrium point to $(1, 1.5)$. Following the procedure introduced in Section III:

1) The transition time interval is $[t_1, t_2] = [12.5, 12.7]$, so $t_f = 0.2$.
2) The input constraints are $u(12.5) = \beta_1(12.5) = 1.0$ and $u(12.7) = \beta_1(12.7) = 1.5$.
3) The exit point of the initial orbit is $[z_1(12.5), z_2(12.5)]^T = [3.82, 2.25]^T$. The target prey population at $t_2 = 12.7$ is chosen to be $y(t_2) = z_1(t_2) = 2.0$.
4) Set $v_2 = 0$ in (9) and then solve

$$c_y(v_1, 0, v_3)\Big|_{x_0^k \to (0.2)^k/k!,\, k>0} = 2.0$$

$$c_u(v_1, 0, v_3)\Big|_{x_0^k \to (0.2)^k/k!,\, k>0} = 1.5.$$

Using the Mathematica function *Solve* gives the values

$$v_1 = -11.8 \quad \text{and} \quad v_3 = 407.6. \tag{11}$$

The target orbit having these parameters and passing through the point $z_1(12.7) = 2$ is shown in Figure 2. The other entry point coordinate is $z_2(12.7) = 3.21$, which was obtained by allowing the predator population to evolved unconstrained since it cannot be freely chosen. The fact that system (6) uses

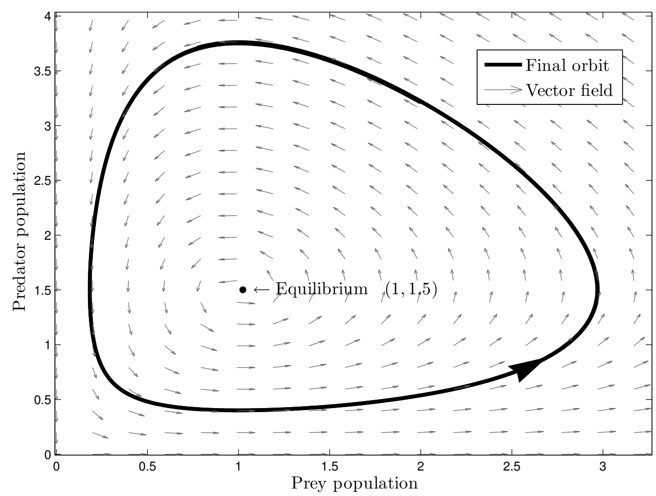

Fig. 2. Final target Orbit.

as input the parameter β_1 restricts the equilibrium points that the system can have. Here the equilibrium point can only move along the vertical line $z_1 = 1$ as β_1 is varied. Again setting Substituting (11) into (8) and setting $x_0^k = (t - t_1)^k/k!$ gives u up to order 6 as

$$\begin{aligned}
u(t) = {} & -0.844733 - 3.22608(t - t_1) + 19.2847(t - t_1)^2 \\
& + 98.9718(t - t_1)^3 + 483.681(t - t_1)^4 \\
& + 1476.69(t - t_1)^5 + 2818.13(t - t_1)^6
\end{aligned}$$

over $[t_1, t_2]$. It can be checked that for (11) and $t = t_2$ one has $u(t_2) = 1.5$ and $y(t_2) = 2.0$ as desired.

While the procedure computes c_u exactly, in practice the series has to be truncated to a polynomial approximation $c_{\hat{u}}$ for implementation. Thus, the actual orbit transfer is described by $c_{\hat{y}} = c \circ c_{\hat{u}}$, and the error is

$$\begin{aligned}
& c_y - c_{\hat{y}} = \\
& \left(-0.146745{v_1}^6 - 1.26308{v_1}^5\right) x_0^6 \\
& + \big(0.111925{v_1}^7 - 42.3975{v_1}^5 + 1.47304{v_1}^4 v_3 \\
& -183.973{v_1}^4 + 4455.45{v_1}^3 - 117.77{v_1}^2 v_3 \\
& -165.506{v_1}^2 + 4.78686 v_1 {v_3}^2 - 667.431 v_1 v_3 - 265267 v_1 \\
& +4930.26 v_3 - 559505\big) x_0^7 + \big(-0.10244{v_1}^8 + 34.2465{v_1}^6 \\
& -1.71202{v_1}^5 v_3 + 169.545{v_1}^5 - 174.566{v_1}^4 \\
& +224.964{v_1}^3 v_3 + 37513.9{v_1}^3 - 10.0142{v_1}^2 {v_3}^2 \\
& +2456.05{v_1}^2 v_3 - 187575{v_1}^2 - 19197.9 v_1 v_3
\end{aligned}$$

$-3.90735 \times 10^6 v_1 + 22.5084{v_3}^2 - 95419.3 v_3$
$-9.30057 \times 10^6\big)\, x_0^8 + \cdots$

Again setting $x_0^k = (t-t_1)^k/k!$ and substituting (11) gives

$$y(t) - \hat{y}(t) = -153.04(t-t_1)^6 - 269.34(t-t_1)^7 - 1610.74(t-t_1)^8 + \cdots$$

for $t \in [t_1, t_2]$. So the computed orbit is accurate up to order 6 as well. In Figure 3, one can see the effect of the applied u on the prey's trajectory. The predator trajectory is also shown for comparison. Finally, Figure 4 shows how u forces the system dynamics to reach the target orbit characterized by $\beta_1 = 1.5$. Moreover, it shows that the entry point is exactly at $y(12.7) = z_1(12.7) = 2.0$.

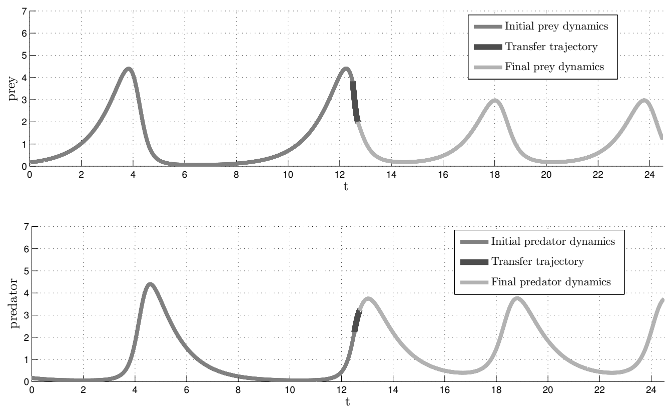

Fig. 3. Prey (top) and predator (bottom) populations as a function of time.

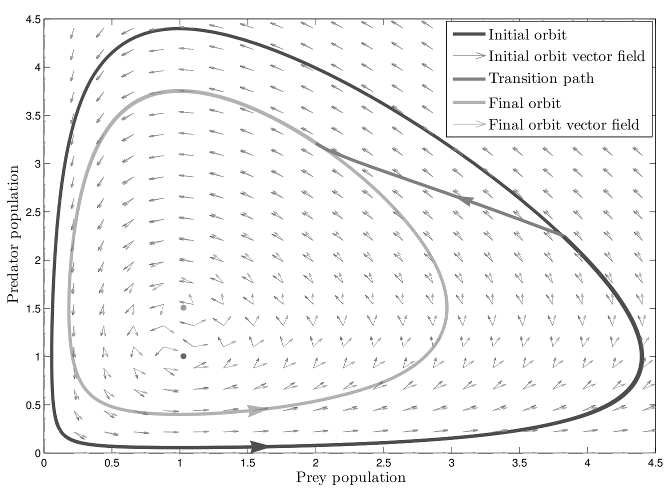

Fig. 4. Simulation of the designed orbit transfer.

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