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Analytic Left Inversion of Multivariable Lotka-Volterra Models

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Abstract— **There is great interest in managing populations of animal species that are vital food sources for humans. A classical population model is the Lotka-Volterra system, which can be viewed as a nonlinear input-output system when time-varying parameters are taken as inputs and the population levels are the outputs. If some of these inputs can be actuated, this sets up an open-loop control problem where a certain population profile as a function of time is desired, and the objective is to determine suitable system inputs to produce this profile. Mathematically, this is a left inversion problem. In this paper, the general left inversion problem is solved for multivariable input-output systems that can be represented in terms of Chen-Fliess series using concepts from combinatorial Hopf algebras. The method is then applied to a three species, two-input, two-output Lotka-Volterra system. The biological goal is to change the population dynamics of the top-level predator species in a food chain in order to prevent extinction.**

Index Terms— **Lotka-Volterra models, system inversion, Chen-Fliess series, formal power series**

I. Introduction

There is great interest in managing populations of animal species that are vital food sources for humans [15]. Technological advances make it possible to practically implement some ecological control systems, but without rigorous models and corresponding control strategies, the design of such systems remains a formidable problem. A classical population model is the Lotka-Volterra system, a coupled set of autonomous quadratic ordinary differential equations which describe the interactions between various species in an ecosystem. The coefficients entering the model are a function of both natural environmental conditions and human intervention. For example, a population of fish species is known to be controlled by a number of factors such as water temperature, the availability of food, the number of predators, harvesting, and the presence of disease and pollution. Many of these parameters are time varying, and a few can be affected by human behavior through certain ecology practices and environmental laws. In which case, the Lotka-Volterra system can also be viewed as a nonlinear input-output system, where the time-varying parameters are the inputs and the population levels are the outputs. Given that some of these inputs can be actuated, this sets up an open-loop control problem where a certain population profile as a function of time is desired, and the objective is to determine suitable system inputs to produce this profile. Mathematically, this is a left inversion problem.

Inversion problems have a long history in nonlinear control theory [3], [10], [11], [16], [17], but their application to population biology as described above is fairly recent. Up to the present, the main focus has been on methods related to dynamic inversion [19]. This class of techniques, which is largely based on a geometric view of the system, generates a left inverse by driving a certain inverse dynamical system with the output of the original system [5], [12]. While this approach is quite effective in certain situations, it does not provide any explicit representation of the input function one seeks. Therefore, it is difficult to use as an analytical design tool. In this paper, a distinctly different approach is taken, one that is purely algebraic and relies on the underlying combinatorial structures of the nonlinear input-output system. The theory behind this approach is described in [8], [9] for single-input, single-output (SISO) systems and uses concepts from combinatorial Hopf algebras. The basic idea is that if an input-output map $F : u \mapsto y$ has an analytic representation known as a Chen-Fliess series or Fliess operator, and an analytic output in the range of F is selected, then under certain conditions the left inverse can be computed explicitly by a single formula. The technique was demonstrated for SISO Lotka-Volterra systems in [7]. Here the general multivariable extension of the method in [9] is described and then applied to a three species, two-input, two-output Lotka-Volterra system. The biological goal is to change the population dynamics of the top-level predator species in a food chain in order to prevent extinction.

The paper is organized as follows. In the next section some preliminaries are given to make the presentation self-contained. In particular, the Lotka-Volterra model is described, and the basic facts about Fliess operators are presented. In the subsequent section the general multivariable inversion theorem is presented. The new inversion tool is applied in Section IV to an input-output system derived from a Lotka-Volterra system to produce its left inverse. These results are then verified in Section V by numerical simulation.

II. Preliminaries

A. Lotka-Volterra Model for Population Dynamics

A fundamental mathematical model for describing the population dynamics of a predator-prey system is the n species competitive Lotka-Volterra model

$$\dot{z}_i = \beta_i z_i + \sum_{j=1}^{n} \alpha_{ij} z_i z_j, \quad i = 1, 2, \ldots, n, \tag{1}$$

where z_i denotes the biomass of the i-th species (roughly proportional to the population), the parameter β_i is the intrinsic growth rate for the i-th population (the Malthus parameter), and the competition coefficient α_{ij} models the effect of the j-th species on the i-th species (usually $\alpha_{ii} = 0$). This model is known under realistic conditions to have stable equilibria, stable limit cycles, and can exhibit chaotic behavior, depending largely on the number of species which are interacting [13], [18]. As described in the introduction,

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a number of environmental factors and human behaviors are known to influence the dynamics of animal populations. This can be modeled by introducing time dependence in the parametrization of (1), namely some $\beta_i(t)$ and $\alpha_{ij}(t)$ are taken as integrable functions of time. In many applications, it is not possible or practical to estimate the size of each population directly, so output functions are introduced to represent measurement processes

$$y_i = h_i(z), \quad i = 1, 2, \ldots, \ell, \tag{2}$$

where $z = [z_1, \ldots, z_n]^T$. It is clear that since the inputs enter the dynamics linearly, (1)-(2) can be rewritten in the standard form

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z)u_i, \quad z(0) = z_0, \quad y = h(z),$$

where each g_i and h_i is an analytic vector field and function, respectively, on some neighborhood $W \subseteq \mathbb{R}^n$, and u_i are the time dependent parameter functions.

In this paper, the main focus is on the three species model

$$\dot{z}_1 = \beta_1 z_1 - \alpha_{12} z_1 z_2 \tag{3a}$$

$$\dot{z}_2 = -\beta_2 z_2 + \alpha_{21} z_2 z_1 - \alpha_{23} z_2 z_3 \tag{3b}$$

$$\dot{z}_3 = -\beta_3 z_3 + \alpha_{32} z_3 z_2, \tag{3c}$$

where the model has been re-parameterized so that $\beta_i, \alpha_{ij} > 0$. Here z_1 is the lowest level prey species, z_2 is the mid-level predator species, and z_3 is the top-level predator species. This system models a food chain in that the top predator preys only on the mid-level predator, and the mid-level predator preys only on the lowest level prey. When the parameters are constant, this system has exactly two equilibria in the first octant, a stable equilibrium at $(0, 0, 0)$ and a saddle point at

$$z_e = (\beta_2/\alpha_{21}, \beta_1/\alpha_{12}, 0).$$

The vector fields are complete within the first octant giving qualitatively three types of behavior [1]. If $\beta_1\alpha_{32} = \beta_3\alpha_{12}$, then the system exhibits periodic solutions. If

$$\beta_1\alpha_{32} < \beta_3\alpha_{12}, \tag{4}$$

then the top-level predator becomes extinct, and the system reduces to a two-species predator-prey system. If $\beta_1\alpha_{32} > \beta_3\alpha_{12}$ then the top-level predator and the lowest-level prey populations grow (non-monotonically) without bound, while the mid-level species experiences larger and larger fluctuations. Somewhat counter-intuitive is the fact that the parameters $\beta_2, \alpha_{21}, \alpha_{23}$ for the mid-level species have *no effect* on the ultimate (steady-state) survival of the top-level predator. However, altering these parameters does affect the transient response of the system.

B. Fliess Operators and Their Interconnections

The inversion formula to be employed requires that the input-output system be written as a Fliess operator. Some elements of this theory are outlined next. More complete treatments can be found in [8], [9].

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*}(c, \eta)\eta$. If the *constant term* $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by $ш$. The latter is the $\mathbb{R}$-bilinear extension of the shuffle product of two words, which is defined inductively by $(x_i\eta) ш (x_j\xi) = x_i(\eta ш (x_j\xi)) + x_j((x_i\eta) ш \xi)$ with $\eta ш \emptyset = \emptyset ш \eta = \eta$ for all $\eta, \xi \in X^*$ and $x_i, x_j \in X$.

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L_\mathfrak{p}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B_\mathfrak{p}^m(R)[t_0, t_1] := \{u \in L_\mathfrak{p}^m[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output system corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0).$$

If there exist real numbers $K_c, M_c > 0$ such that $|(c, \eta)| \leq K_c M_c^{|\eta|}|\eta|!, \quad \eta \in X^*$, then F_c constitutes a well defined mapping from $B_\mathfrak{p}^m(R)[t_0,\, t_0 + T]$ into $B_\mathfrak{q}^\ell(S)[t_0,\, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$.

When Fliess operators F_c and F_d are connected in a parallel-product fashion, it is known that $F_c F_d = F_{c\, ш\, d}$. If F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade manner, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where $c \circ d$ denotes the *composition product* of c and d as described in [8]. This product is associative and $\mathbb{R}$-linear in its left argument c. In the event that two Fliess operators are interconnected to form a feedback system, the closed-loop system has a Fliess operator representation whose generating series is the *feedback product* of c and d, denoted by $c@d$. This product can be explicitly computed via Hopf algebra methods. The basic idea is to consider the set of operators $\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{R}^m\langle\langle X\rangle\rangle\}$, where I denotes the identity map, as a group under composition. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta + c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ is denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. This set

TABLE I

VECTOR RELATIVE DEGREE r FOR THREE MULTIVARIABLE LOTKA-VOLTERRA SYSTEMS WITH $y_1 = z_2$ AND $y_2 = z_3$

I/0 map	r	range restrictions
$F_c : \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	not defined	–
$F_c : \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	[1 1]	$(c_{y_1}, \emptyset) = (c_1, \emptyset)$ $(c_{y_2}, \emptyset) = (c_2, \emptyset)$
$F_c : \begin{bmatrix} \beta_1 \\ \beta_3 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	[2 1] (full)	$(c_{y_1}, \emptyset) = (c_1, \emptyset)$ $(c_{y_1}, x_0) = (c_1, x_0)$ $(c_{y_2}, \emptyset) = (c_2, \emptyset)$

also forms a group under the composition product induced by operator composition, namely, $c_\delta \circ d_\delta := \delta + d + c \,\tilde{\circ}\, d$, where $\tilde{\circ}$ denotes the *modified composition product* [8]. The group $(\mathbb{R}\langle\langle X_\delta \rangle\rangle, \circ, \delta)$ has coordinate functions that form a Faà di Bruno type Hopf algebra. In which case, the group (composition) inverse $c_\delta^{\circ -1} := \delta + c^{\circ -1}$ can be computed efficiently via the antipode of this Hopf algebra [2], [6], [8]. This inverse also describes the feedback product as $c@d = c \,\tilde{\circ}\, (-d \circ c)^{\circ -1}$ and can be used for input-output inversion as described next.

III. LEFT INVERSION OF MULTIVARIABLE FLIESS OPERATORS

It was shown in [20] that F_c will map every input which is analytic at t_0 to an output which is also analytic at t_0 provided $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$. In [9] an explicit formula was given for computing the left inverse of a SISO mapping F_c given a real analytic function in its range. Here the multivariable version of this method is given. Without loss of generality assume $t_0 = 0$. Note that every $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$. The following definition will provide a sufficient condition under which the left inverse of F_c exists.

Definition 1: Given $c \in \mathbb{R}^m\langle\langle X \rangle\rangle$, let $r_i \geq 1$ be the largest integer such that $\mathrm{supp}(c_{i,F}) \subseteq x_0^{r_i - 1}X^*$, where $i = 1, 2, \ldots, m$. Then the component series c_i has ***relative degree*** r_i if the linear word $x_0^{r_i-1}x_j \in \mathrm{supp}(c_i)$ for some $j \in \{1, \ldots, m\}$, otherwise it is not well defined. In addition, c has ***vector relative degree*** $r = [r_1\ r_2\ \cdots\ r_m]$ if each c_i has relative degree r_i and the $m \times m$ matrix

$$A = \begin{bmatrix} (c_1, x_0^{r_1-1}x_1) & (c_1, x_0^{r_1-1}x_2) & \cdots & (c_1, x_0^{r_1-1}x_m) \\ (c_2, x_0^{r_2-1}x_1) & (c_2, x_0^{r_2-1}x_2) & \cdots & (c_2, x_0^{r_2-1}x_m) \\ \vdots & \vdots & \vdots & \vdots \\ (c_m, x_0^{r_m-1}x_1) & (c_m, x_0^{r_m-1}x_2) & \cdots & (c_m, x_0^{r_m-1}x_m) \end{bmatrix}$$

has full rank. Otherwise, c does not have vector relative degree.

It can be shown directly that this notion of vector relative degree coincides with the usual definition given in a state space setting [12], and that c has vector relative degree r only if for each i the series $(x_0^{r_i-1}x_j)^{-1}(c_i)$ is non proper for some j. Here the left-shift operator for any $x_i \in X$ is defined on X^* by $x_i^{-1}(x_i\eta) = \eta$ with $\eta \in X^*$ and zero otherwise. Higher order shifts are defined inductively via $(x_i\xi)^{-1}(\cdot) = \xi^{-1}x_i^{-1}(\cdot)$, where $\xi \in X^*$. The left-shift operator is assumed to act linearly and componentwise on $\mathbb{R}^m\langle\langle X \rangle\rangle$. In Table I, three examples of input-output systems induced by (3) are shown. In the octant of the state space where $z_i > 0$, $i = 1, 2, 3$, they range from having no vector relative degree to having full vector relative degree ($r_1 + r_2 = n = 3$), which corresponds to differential flatness via static state feedback [4].

To develop the main inversion theorem, the following two lemmas are needed. Their proofs are similar to their single variable counterparts given in [9].

Lemma 1: The set of series $\mathbb{R}^{m\times m}\langle\langle X \rangle\rangle$ having invertible constant terms is a group under the shuffle product. In particular, the shuffle inverse of any such series C is

$$C^{\,\sqcup\!\sqcup\, -1} = ((C, \emptyset)(I - C'))^{\,\sqcup\!\sqcup\, -1} = (C, \emptyset)^{-1}(C')^{\,\sqcup\!\sqcup\, *},$$

where $C' = I - (C, \emptyset)^{-1}C$ is proper, i.e., $(C', \emptyset) = 0$, and $(C')^{\,\sqcup\!\sqcup\, *} := \sum_{k\geq 0}(C')^{\,\sqcup\!\sqcup\, k}$.

Lemma 2: For any $C \in \mathbb{R}^{m\times m}\langle\langle X \rangle\rangle$ with an invertible constant term, F_C, which is defined componentwise by $[F_C]_{i,j} = F_{C_{i,j}}$, has a well defined multiplicative inverse given by $(F_C)^{-1} = F_{C^{\,\sqcup\!\sqcup\, -1}}$.

The next theorem provides the main inversion tool used in the paper. Let $X_0 := \{x_0\}$, and $\mathbb{R}[[X_0]]$ denotes the set of all commutative series over X_0. Note that when $c \in \mathbb{R}[[X_0]]$, $F_c[u](t)$ reduces to the Taylor series $\sum_{k\geq 0}(c, x_0^k)E_{x_0^k}[u](t) = \sum_{k\geq 0}(c, x_0^k)t^k/k!$.

Theorem 1: Suppose $c \in \mathbb{R}^m\langle\langle X \rangle\rangle$ has vector relative degree r. Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}^m_{LC}[[X_0]]$ satisfying $(c_{y_i}, x_0^k) = (c_i, x_0^k)$, $k = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$. Then the input

$$u(t) = \sum_{k=0}^{\infty}(c_u, x_0^k)\frac{t^k}{k!}, \tag{5}$$

is the unique real analytic solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$, where

$$c_u = \left(\left[C^{\,\sqcup\!\sqcup\, -1} \,\sqcup\!\sqcup\, (x_0^r)^{-1}(c - c_y)\right]^{\circ -1}\right)_N, \tag{6}$$

the i-th row of $(x_0^r)^{-1}(c - c_y)$ is $(x_0^{r_i})^{-1}(c_i - c_{y_i})$, and the (i, j)-th entry of C is $(x_0^{r_i-1}x_j)^{-1}(c_i)$.

Proof: First observe that the identity $dy_i/dt = \sum_{j=0}^{m} u_j F_{x_j^{-1}(c_i)}[u]$ gives

$$y_i = F_{c_i}[u] \tag{7a}$$

$$\dot{y}_i = F_{x_0^{-1}(c_i)}[u] \tag{7b}$$

$$\vdots$$

$$y_i^{(r_i-1)} = F_{(x_0^{r_i-1})^{-1}(c_i)}[u] \tag{7c}$$

$$y_i^{(r_i)} = F_{(x_0^{r_i})^{-1}(c_i)}[u] + \sum_{j=1}^{m} u_j F_{(x_0^{r_i-1}x_j)^{-1}(c_i)}[u], \tag{7d}$$

which can be written in matrix notation as

$$y^{(r)} = F_{(x_0^r)^{-1}(c)}[u] + F_C[u]u. \tag{8}$$

In light of the fact that $(C, \emptyset) = A$, and A is nonsingular by assumption, it follows from Lemma 2 that

$$\begin{aligned} u &= -(F_C[u])^{-1}F_{(x_0^r)^{-1}(c-c_y)}[u] \\ &= -F_{C^{\,\sqcup\!\sqcup\, -1}}[u]F_{(x_0^r)^{-1}(c-c_y)}[u] = -F_d[u], \end{aligned}$$

where $d := C^{\,\sqcup\!\sqcup\, -1} \,\sqcup\!\sqcup\, (x_0^r)^{-1}(c - c_y)$. Finally, the theorem is proved by using the composition inverse in [8] to solve the equation $u + F_d[u] = 0$, that is,

$$F_{c_u}[0] = u = F_{d^{\circ -1}}[0] = F_{(d^{\circ -1})_N}[0].$$

The fact that the generating series of a Fliess operator is unique gives $c_u = \left(d^{\circ -1}\right)_N$ as desired. The constraints on the coefficients (c_{y_i}, x_0^k) come from evaluating (7a)-(7c) at $t = 0$. ■

From this theorem one can deduce the restrictions on the ranges of the Lotka-Volterra input-output maps as shown in Table I. In the next section, this result is applied to explicitly determine a left inverse.

IV. Left Inversion of Multivariable Lotka-Volterra Input-Output Systems

The goal of this section is two-fold. First, the left inverse of the second system in Table I is explicitly computed as an example. Then an input design strategy is devised using this inverse to perform an orbit transfer in the dynamics. In particular, the top-level predator in the system tends to extinction when condition (4) is satisfied. So an orbit transfer is designed to avoid this condition. The input-output system in Table I having full vector relative degree does not need the full treatment since its differential flatness property can be exploited directly. That is, by direct differentiation one gets an expression analogous to (8)

$$\begin{bmatrix} \ddot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} [-\beta_2 + \alpha_{21} z_1 + \alpha_{23} y_2]\dot{y}_2 - \alpha_{32} y_1 y_2 \\ [\alpha_{12}\alpha_{21} z_1 + \alpha_{23}\alpha_{32} y_2] y_2^2 \end{bmatrix} + \begin{bmatrix} \alpha_{21} y_1 z_1 & \alpha_{23} y_1 y_2 \\ 0 & -y_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Therefore, if y_1, y_2 and z_1 are given and always positive then $u_1 = \beta_2$ and $u_2 = \beta_3$ can be solved for directly.

Instead, the system under study is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \beta_1 z_1 - \alpha_{12} z_1 z_2 \\ \alpha_{21} z_2 z_1 - \alpha_{23} z_2 z_3 \\ \alpha_{32} z_3 z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -z_2 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ -z_3 \end{bmatrix} u_2 \tag{9a}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} \tag{9b}$$

with initial conditions $z_i(0) = z_{i,0} > 0$, $i = 1, 2, 3$. Normalizing all parameters to 1, driving the system with constant inputs $u_1 = \beta_2 = 1$ and $u_2 = \beta_3 = 1.2$, and initializing the model at $[z_{1,0}, z_{2,0}, z_{3,0}]^T = [4, 4, 4]^T$ produces the trajectory shown in Figure 1. It is clear that the system satisfies condition (4), so the system's trajectory converges to a periodic orbit in the plane formed by the mid-level predator and prey populations. This will serve as the initial trajectory. The goal is to determine input functions $u_1(t) = \beta_2(t)$ and $u_2(t) = \beta_3(t)$ in order to achieve a stable periodic final orbit corresponding to $\beta_2 = \beta_3 = 1$ in a controlled fashion over an interval $[t_1, t_2]$. Therefore, the top-level predator population does not reach extinction as shown in Figure 2. More precisely, given $u_1(t_1) = 1$, $u_2(t_1) = 1.2$, $u_1(t_2) = 1$, $u_2(t_2) = 1$, a fixed exit point from the initial orbit $[z_1(t_1), z_2(t_1), z_3(t_1)]^T$, and mid-level and top-level predator population output values $y_1(t_2)$ and $y_2(t_2)$, find analytic $u_1(t)$ and $u_2(t)$ over $[t_1, t_2]$, using inversion formula (5)-(6), so that the orbit transfer is achieved over some transition time $\Delta t := t_1 - t_2$.

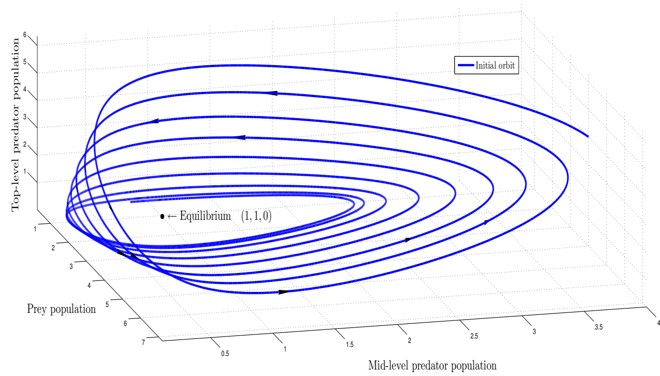

Fig. 1. Initial trajectory for normalized version of (9).

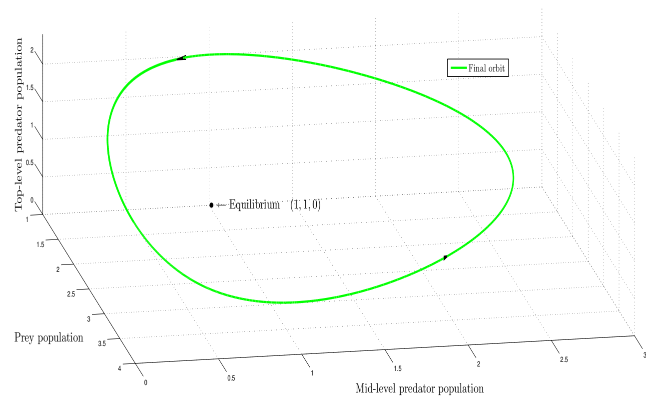

Fig. 2. Final target orbit.

The first step is to compute the generating series of system (9) via iterated Lie derivatives of the output function [12]. For brevity, only the first component of each result is shown throughout. Using the software package *Mathematica* and the NCAlgebra suite [14], it follows that the first component of $c = [c_1, c_2]^T$ is

$$\begin{aligned} c_1 &= z_{2,0} + (\alpha_{21} z_{1,0} z_{2,0} - \alpha_{23} z_{2,0} z_{3,0}) x_0 - z_{2,0} x_1 \\ &+ (\beta_1 \alpha_{21} z_{1,0} z_{2,0} + \alpha_{21}^2 z_{1,0}^2 z_{2,0} - \alpha_{12}\alpha_{21} z_{1,0} z_{2,0}^2 \\ &- 2\alpha_{21}\alpha_{23} z_{1,0} z_{2,0} z_{3,0} - \alpha_{23}\alpha_{32} z_{2,0}^2 z_{3,0} + \alpha_{23}^2 z_{2,0} z_{3,0}^2) x_0^2 \\ &+ (\alpha_{23} z_{2,0} z_{3,0} - \alpha_{21} z_{1,0} z_{2,0}) x_0 x_1 + \alpha_{23} z_{2,0} z_{3,0} x_0 x_2 \\ &+ (-\alpha_{21} z_{1,0} z_{2,0} + \alpha_{23} z_{2,0} z_{3,0}) x_1 x_0 + z_{2,0} x_1^2 + \cdots . \end{aligned}$$

In order to compute $c_u = [c_{u_1}, c_{u_2}]^T$ via (6), one must select an output function

$$y(t) = \sum_{k=0}^{\infty} (c_y, x_0^k) \frac{t^k}{k!}, \tag{10}$$

where $c_y = [c_{y_1}, c_{y_2}]^T$ is the corresponding generating series of y. It is sufficient to consider polynomial outputs up to degree 4, so let $(c_{y_j}, x_0^i) = v_{ij}$ for $i = 0, 1, 2, 3, 4$ and $j = 1, 2$. Thus,

$$A = (C, \emptyset) = \begin{bmatrix} -z_{2,0} & 0 \\ 0 & -z_{3,0} \end{bmatrix}$$

is, as expected, full rank and

$$d = C^{\sqcup\!\sqcup -1} {}_{\sqcup\!\sqcup} (x_0^r)^{-1}(c - c_y) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

The first component of d is found to be

$$\begin{aligned} d_1 &= \alpha_{23}z_{3,0} + \frac{v_{11}}{z_{2,0}} - \alpha_{21}z_{1,0} + \frac{v_{11}}{z_{2,0}}x_1 - \alpha_{23}z_{3,0}x_2 \\ &+ \left(\frac{v_{11}\alpha_{23}z_{3,0}}{z_{2,0}} - \frac{v_{11}\alpha_{21}z_{1,0}}{z_{2,0}} + \frac{v_{12}}{z_{2,0}} - \beta_1\alpha_{21}z_{1,0} \right. \\ &\left. + \alpha_{12}\alpha_{21}z_{1,0}z_{2,0} + \alpha_{23}\alpha_{32}z_{2,0}z_{3,0} \right) x_0 + \left(\frac{v_{11}{\alpha_{21}}^2{z_{1,0}}^2}{z_{2,0}} \right. \\ &+ v_{11}\alpha_{12}\alpha_{21}z_{1,0} - \frac{v_{11}\beta_1\alpha_{21}z_{1,0}}{z_{2,0}} - \frac{2v_{11}\alpha_{21}\alpha_{23}z_{1,0}z_{3,0}}{z_{2,0}} \\ &+ \frac{v_{11}{\alpha_{23}}^2{z_{3,0}}^2}{z_{2,0}} + v_{11}\alpha_{23}\alpha_{32}z_{3,0} - \frac{2v_{12}\alpha_{21}z_{1,0}}{z_{2,0}} \\ &+ \frac{2v_{12}\alpha_{23}z_{3,0}}{z_{2,0}} + \frac{v_{31}}{z_{2,0}} - {\beta_1}^2\alpha_{21}z_{1,0} + 2\beta_1\alpha_{12}\alpha_{21}z_{1,0}z_{2,0} \\ &- {\alpha_{12}}^2\alpha_{21}z_{1,0}{z_{2,0}}^2 + \alpha_{12}{\alpha_{21}}^2{z_{1,0}}^2z_{2,0} \\ &- \alpha_{12}\alpha_{21}\alpha_{23}z_{1,0}z_{2,0}z_{3,0} + \alpha_{21}\alpha_{23}\alpha_{32}z_{1,0}z_{2,0}z_{3,0} \\ &\left. - {\alpha_{23}}^2\alpha_{32}z_{2,0}{z_{3,0}}^2 + \alpha_{23}{\alpha_{32}}^2{z_{2,0}}^2z_{3,0} \right) x_0^2 + \cdots . \end{aligned}$$

The composition inverse of d is computed componentwise using the recursive method in [6], [8]. In which case, the explicit formula for c_{u_1} is

$$\begin{aligned} c_{u_1} &= \left(d_1^{\circ -1}\right)_N = \\ &- \frac{v_{11}}{z_{2,0}} + \alpha_{21}z_{1,0} - \alpha_{23}z_{3,0} + \left(\frac{{v_{11}}^2}{{z_{2,0}}^2} - \frac{v_{21}}{z_{2,0}} - v_{12}\alpha_{23} \right. \\ &\left. + \beta_1\alpha_{21}z_{1,0} - \alpha_{12}\alpha_{21}z_{1,0}z_{2,0} \right) x_0 + \left(\frac{3v_{11}v_{21}}{{z_{2,0}}^2} - \frac{2{v_{11}}^3}{{z_{2,0}}^3} \right. \\ &- v_{11}\alpha_{12}\alpha_{21}z_{1,0} - \frac{v_{31}}{z_{2,0}} - v_{22}\alpha_{23} + {\beta_1}^2\alpha_{21}z_{1,0} - \\ &\left. 2\beta_1\alpha_{12}\alpha_{21}z_{1,0}z_{2,0} + {\alpha_{12}}^2\alpha_{21}z_{1,0}{z_{2,0}}^2 \right) x_0^2 + \cdots . \quad (11) \end{aligned}$$

This provides the most general formula for the input u to make the system output follow the path specified by the coefficients of the series c_y.

The next step is to determine the specific c_y satisfying the constraints of the desired orbit transfer. The procedure is as follows:

1) Choose the initial trajectory exit time t_1, which is the time the transition path starts. Define the final transition time $t_2 = t_1 + \Delta t$, where Δt is to be determined.
2) Choose the initial input values $u_1(t_1) = \beta_2(t_1) =: u_{1i}$ and $u_2(t_1) = \beta_3(t_1) =: u_{2i}$, and the final input values $u_1(t_2) = \beta_2(t_2) =: u_{1f}$ and $u_2(t_2) = \beta_3(t_2) =: u_{2f}$.
3) Choose the exit point of the initial trajectory $[z_1(t_1), z_2(t_1), z_3(t_1)]^T$, and select the final output values $y_1(t_2) = z_2(t_2) =: y_{1f}$ and $y_2(t_2) = z_3(t_2) =: y_{2f}$ for the entry point of the final orbit.
4) Let $v := \{v_{ij}, i = 1, 2, 3, 4 \text{ and}, j = 1, 2\}$, and find the values of Δt and v minimizing the system energy and such that the stated constraints are satisfied. That is, for a suitable norm $||\cdot||$, solve the optimization problem

$$\min_{\Delta t, v} \left\{ \int_{t_1}^{t_2} \|u(s)\|^2 + \|y(s)\|^2 \, ds \right\} \quad (12)$$

subject to the following constraints:

$$\begin{aligned} c_u(v) \Big|_{x_0^k \to 0, k>0} &= \begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix}, \\ c_u(v) \Big|_{x_0^k \to (\Delta t)^k/k!, k>0} &= \begin{bmatrix} u_{1f} \\ u_{2f} \end{bmatrix}, \quad (13) \\ c_y(v) \Big|_{x_0^k \to (\Delta t)^k/k!, k>0} &= \begin{bmatrix} y_{1f} \\ y_{2f} \end{bmatrix}. \end{aligned}$$

The dependence of c_y and c_u on v is evident in (10) and (11). Observe that v_{0j} is not free in (13) since $(c, \emptyset) = [v_{01}, v_{02}]^T = [z_{2,0}, z_{3,0}]^T$ as stated in Theorem 1. The optimization problem will be solved numerically in the next section. Note that in this procedure there is no direct control over the prey population. This can be seen in step 3 of the procedure, where only the top-level and mid-level predator populations of the final orbit entry point are specified. Therefore, the prey population during the transition is allowed to evolve freely. Not having control over all the populations is often realistic due to the limitations encountered in real ecosystems, where relatively few system parameters can be controlled over time.

V. Numerical Simulations

In this section, the orbit transfer problem is numerically solved and simulated based on the open-loop control procedure described in the previous section. Assume system (9) is on the initial trajectory shown in Figure 1. The first component of the generating series c of system (9) is

$$\begin{aligned} c_1 &= 0.779265 - 1.29109x_0 - 0.779265x_1 + 0.698465x_0^2 \\ &+ 1.29109x_0x_1 + 2.06908x_0x_2 + 1.29109x_1x_0 \\ &+ 0.779265x_1^2 + \cdots . \end{aligned}$$

At time $t_1 = 9$ sec. the system starts transferring to a periodic, non decreasing final orbit. The exit point in the trajectory is $[z_1(t_1), z_2(t_1), z_3(t_1)]^T = [0.99, 0.78, 2.65]^T$, and the parameter β_3 of the population model will change from $\beta_3(t_1) = 1.2$ to $\beta_3(t_2) = 1$. In the first octant, the equilibrium point is $(1, 1, 0)$ at time t_1 since $\beta_2(t_1) = 1$, and it returns to this position after Δt seconds since $\beta_2(t_2) = 1$. Following the procedure outlined in Section IV:

1) The transition initial time is $t_1 = 9$, and the transition time Δt is a free variable.
2) Defining $t_2 = t_1 + \Delta t$, the input constraints are $u_1(t_1) = \beta_2(t_1) = 1$, $u_2(t_1) = \beta_3(t_1) = 1.2$, $u_1(t_2) = \beta_2(t_2) = 1$ and $u_2(t_2) = \beta_3(t_2) = 1$.
3) The exit point of the initial orbit is
$$[z_1(t_1), z_2(t_1), z_3(t_1)]^T = [0.99, 0.78, 2.65]^T.$$
The target prey population at t_2 is chosen to be $y_1(t_2) = z_2(t_2) = 1$ and $y_2(t_2) = z_3(t_2) = 2$.
4) Find Δt and v such that (12) is minimized and
$$\begin{aligned} c_u(v) \Big|_{x_0^k \to 0, k>0} &= \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}, \\ c_u(v) \Big|_{x_0^k \to (\Delta t)^k/k!, k>0} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ c_y(v) \Big|_{x_0^k \to (\Delta t)^k/k!, k>0} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$
are satisfied.

Using the Mathematica optimization function `NMinimize`, the following results are obtained:

$$\begin{aligned} &\Delta t = 0.294, \\ &v_{11} = -2.07036, \quad v_{12} = -1.11712, \\ &v_{21} = 20.3119, \quad v_{22} = -24.2761, \\ &v_{31} = -13.3511, \quad v_{32} = 169.307, \\ &v_{41} = 16.8201, \quad v_{42} = 19.505. \end{aligned} \tag{14}$$

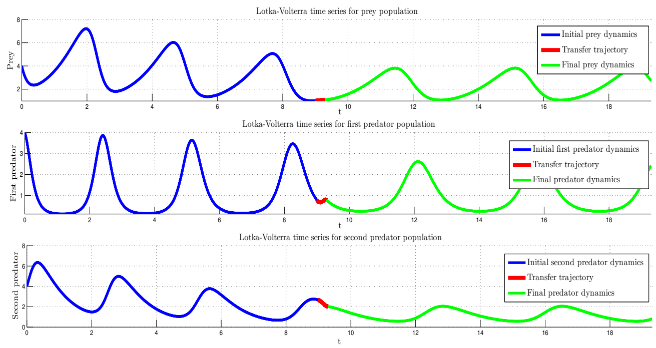

Fig. 3. Populations as a function of time.

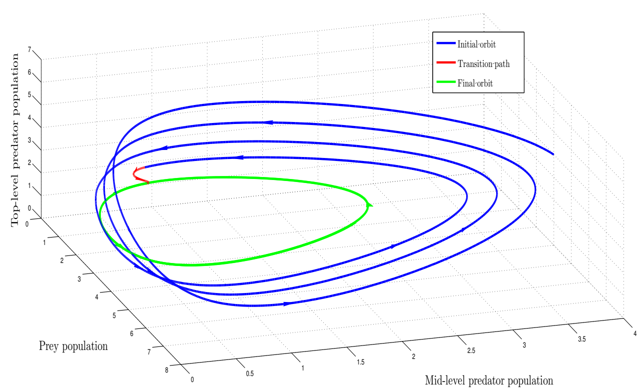

Fig. 4. Orbit transfer.

The target orbit as shown in Figure 2 passes through $z_2(t_2) \approx 1$ and $z_3(t_2) \approx 2$, which constitutes two coordinates of the entry point at $t_2 = 9.294$. The other entry point coordinate is $z_1(t_2) = 1.08$, which was not constrained. Substituting (14) into (11), and setting $x_0^k = (t - t_1)^k/k!$ gives u_1 up to order 6 as

$$\begin{aligned} u_1(t) = 1 &- 17.6693(t-t_1) - 63.3607(t-t_1)^2 \\ &+ 16.9358(t-t_1)^3 + 1068.97(t-t_1)^4 \\ &+ 1927.6(t-t_1)^5 - 3181.75(t-t_1)^6 \end{aligned}$$

over $[t_1, t_2]$. It can be checked that for t_2 one has $u_1(t_2) = u_2(t_2) = 1$.

While Theorem 1 determines c_u exactly, in practice the series has to be truncated to a polynomial approximation $c_{\hat{u}}$ for implementation. Thus, the actual orbit transfer is described by $c_{\hat{y}} = c \circ c_{\hat{u}}$, which corresponds to $\hat{y} = [\hat{y}_1, \hat{y}_2]^T$ when x_0^k is substituted with $(t - t_1)^k/k!$. Given that y is completely specified by c_y in (10), the output error due to truncation of the input components is

$$\begin{aligned} y_1(t) - \hat{y}_1(t) = &-17.19(t-t_1)^6 + 1054.57(t-t_1)^7 \\ &+ 2037.95(t-t_1)^8 + \cdots \end{aligned}$$

for $t \in [t_1, t_2]$. In Figure 3, one can see the response of the applied $\hat{u}$ for the three species trajectories. While $u_1(t) = \beta_2(t)$ does not affect the steady state behavior as per (4), it is clearly employed during the orbit transfer. Finally, Figure 4 shows how $\hat{u}$ changes the system dynamics to avoid the extinction of the top-level predator population. In particular, it shows that the entry point for the final orbit reaches $y_1(t_2) = z_2(t_2) \approx 1$ and $y_2(t_2) = z_3(t_2) \approx 2$ as desired.

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