

Dendriform-Tree Setting for Fully Non-commutative Fliess Operators*

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Abstract—This paper provides a dendriform-tree setting for Fliess operators with matrix-valued inputs. This class of analytic nonlinear input-output systems is convenient, for example, in quantum control. In particular, a description of such Fliess operators is provided using planar binary trees. Sufficient conditions for convergence of the defining series are also given.

I. INTRODUCTION

Fliess operators provide a general framework under which analytic nonlinear input-output systems can be studied [10]–[12]. Let $X = \{x_0, x_1, \dots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word \emptyset) under the catenation product. A formal power series c in X is any mapping of the form $X^* \rightarrow \mathbb{R}^\ell$: $\eta \mapsto (c, \eta)$. The set of all such mappings will be denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$. The support of an arbitrary series c is $\text{supp}(c) = \{\eta \in X^*, (c, \eta) \neq 0\}$. A series having finite support is called a *polynomial*, and the set of all polynomials is represented by $\mathbb{R}^\ell \langle X \rangle$. For a measurable function $u : [a, b] \rightarrow \mathbb{R}^m$ define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \leq i \leq m\}$, where $\|u_i\|_{L_p}$ is the usual L_p -norm for a measurable real-valued component function u_i . Define iteratively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \rightarrow C[t_0, t_0 + T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i \eta'}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_{\eta'}[u](\tau, t_0) d\tau, \quad (1)$$

where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$. The input-output operator corresponding to c is then

$$F_c[u](t) := \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t),$$

which is called a *Fliess operator*. If the generating series c is *locally convergent*, i.e., there exists constants $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|} |\eta|!$ for all $\eta \in X^*$, where $|\eta|$ denotes the number of letters in η , then $F_c[u]$ converges absolutely and uniformly on $[t_0, t_0 + T]$ if T and $\|u\|_{L_p}$ are sufficiently small. In general, the input-output map $F_c : u \rightarrow y$ need not have a state space realization, however, many familiar and relevant examples are obtained from the state space setting.

A tacit assumption in the standard theory of Fliess operators is that the inputs are mutually commutative, i.e., the functions associated with each letter of X commute pointwise in time. The proposition here is that this assumption results in a great deal of simplification and hides

certain underlying algebraic structures that are important in applications like control on Lie groups [3] and quantum control [1].

As a motivating example, consider a bilinear system

$$\dot{z}(t) = Az(t) + B(t)z(t)u(t), \quad (2)$$

where B is a smooth function on $[0, T]$. One can view u as the user controlled input and B as a disturbance input. Let z_i be the solution of (2) when $z(0) = e_i = [0, \dots, 0, 1, 0, \dots, 0]^T$ with the 1 in the i -th position and define $Z(t) = [z_1(t), \dots, z_n(t)]$, where n is the dimension of the system. Then

$$\dot{Z}(t) = (A + B(t)u(t))Z(t) =: U(t)Z(t), \quad (3)$$

where in general $U(t_1)U(t_2) \neq U(t_2)U(t_1)$. This is, for example, the setting of a regulator problem in which the input-output map from the disturbance to some output $y(t) = CZ(t)$ needs to be determined when $u(t) = u_0 \in \mathbb{R}$. Equation (3) is also the usual starting point for control theory on Lie groups. Systems such as in (3) are ubiquitous in quantum mechanics. Take for instance the case of a spin particle in a magnetic field B_m whose direction changes in time. The function U is proportional to the scalar product $S \cdot B_m$, where S represents the spin vector. Now suppose the magnetic field at $t = t_1$ is parallel to the x -axis, and at $t = t_2$ to the y -axis, then $U(t_1) \propto |B_m| S_x$, $U(t_2) \propto |B_m| S_y$, and $[U(t_1), U(t_2)] \propto B_m^2 [S_x, S_y] \propto B_m^2 S_z \neq 0$. Moreover, systems of the form $\dot{Z} = U(t)F(Z(t))$ can be considered where the coordinate change $\bar{Z} = F(Z)$ is valid on a neighborhood of $Z(0) = I$. In which case,

$$\dot{\bar{Z}}(t) = \left(\frac{dF^{-1}(\bar{Z})}{d\bar{Z}} \bigg|_{\bar{Z}=\bar{Z}(0)} \right)^{-1} U(t)\bar{Z}(t) =: W(t)\bar{Z}(t),$$

is in the same class as (3).

A series representation of the solution of (3) can be obtained by successive iterations. That is,

$$Z(t) = I + \sum_{n=1}^{\infty} \int_0^t U(t_1) dt_1 \cdots \int_0^{t_{n-1}} U(t_n) dt_n. \quad (4)$$

This series has an artificial exponential representation in terms of the *time ordered operator*

$$\mathcal{T}(U(t_1) \cdots U(t_n)) := \sum_{\sigma \in S_n} \Theta_n^\sigma U(t_{\sigma(1)}) \cdots U(t_{\sigma(n)}),$$

where $\Theta_n^\sigma = \prod_{i=1}^{n-1} \Theta(t_{\sigma(i)} - t_{\sigma(i+1)})$, Θ is the Heaviside step function, σ is a permutation, and S_n is the group of all permutations of order n [2]. Because of the symmetry of the simplex consisting of all ordered n -tuples (t_1, t_2, \dots, t_n) in the integration limits, this operator satisfies:

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n U(t_1)U(t_2) \cdots U(t_n)$$

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$$= \frac{1}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \mathcal{T}(U(t_1)U(t_2)\cdots U(t_n)).$$

The solution is thus written as the time ordered exponential

$$Z(t) = I + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^t \cdots \int_{t_0}^{t_{n-1}} \mathcal{T}(U(t_1)\cdots U(t_n)) dt_1 \cdots dt_n \\ =: \mathcal{T} \exp \left(\int_{t_0}^t U(s) ds \right). \quad (5)$$

Expression (5) disregards the algebra provided by the non-commutative iterated integrals in (4). However, it is known that by systematically keeping track of the non-commutative orderings of the iterated integrals, a true exponential (*Magnus expansion*) can be derived. That is, $Z(t) = \exp(\Omega(U(t)))$, where Ω is obtained via a recursion [8], [9], [15]. In the case of commutative inputs, the algebra provided by the iterated integrals is the *shuffle algebra*, which is based on the integration by parts formula [17], [18]. The non-commutative version of this formula is

$$\int_0^t u_i(s) ds \int_0^t u_j(s) ds = \int_0^t u_i(s) \left(\int_0^s u_j(r) dr \right) ds \\ + \int_0^t \left(\int_0^s u_i(r) dr \right) u_j(s) ds.$$

Note that the second summand on the right-hand side above cannot be generated recursively as in (1). Moreover, products of iterated integrals are fundamental when the system's state is filtered by an analytic output function [10], in the computation of bounds for iterated integrals [5] and the characterization of system interconnections such as the product, cascade and feedback connections [11]. The first goal of this paper is to provide a fully non-commutative extension of the theory of Fliess operators in the context of *dendriform/tree* algebras. They will be referred to as *dendriform Fliess operators*. The second goal is to give sufficient conditions under which dendriform Fliess operators with non-commutative inputs converge.

The paper is organized as follows. Section II provides a tutorial treatment of dendriform algebras. In Section III, planar binary trees are presented as a tool to keep track of the non-commutativity of iterated integrals. Also, the non-commutative version of the shuffle product is given. These results are then applied in Section IV to define dendriform Fliess operators. Then sufficient conditions for the convergence of dendriform Fliess operators are provided. Proofs of all results are included in the expanded version of the paper [4].

II. DENDRIFORM ALGEBRAS

The goal of this section is to introduce parenthesis words and their relationship to dendriform algebras. The concepts here can be found in [6], [14] and references therein.

Let X be a finite alphabet and $\mathfrak{P}X = X \cup \{[,]\}$. The free semigroup under catenation generated by $\mathfrak{P}X$ is denoted $\mathfrak{P}X^*$. For $\eta = q_1 q_2 \cdots q_n \in \mathfrak{P}X^*$, let $s(\eta)_i$ denote the number of '['s in $q_1 \cdots q_i$ minus the number of ']'s in $q_1 \cdots q_i$.

Definition 1: A word $\eta = q_1 q_2 \cdots q_n \in \mathfrak{P}X^*$ is called a *parenthesis word* if its parenthesization is balanced, i.e., it satisfies:

- i. $s(\eta)_i \geq 0$ for $i = 1, \dots, n-1$ and $s(\eta)_n = 0$.
- ii. $q_i q_{i+1} \neq x_i x_{i_2}$ for $x_{i_1}, x_{i_2} \in X$ and $i = 1, \dots, n-1$.

iii. $q_i q_{i+1} \neq [,]$ for $i = 1, \dots, n-1$.

iv. $q_1 = [$ and $q_n =]$ cannot occur at the same time.

v. There are no sub-words in η of the form $\xi[\nu]\kappa$ or $[[\xi]]$ for $\xi, \nu, \kappa \in \mathfrak{P}X^*$.

Parenthesis words are such that $x_i[x_j] \neq [x_i]x_j$ for $x_i, x_j \in X$. This set of parenthesis words constitutes a *free Magma* under balanced parenthesization [6], [7], [16]. The set of parenthesis words including the empty word \emptyset is denoted by $\mathfrak{P}X^*$. In Section III, the operation in this magma is better understood in terms of the grafting operation on trees. A formal power series in $\mathfrak{P}X$ is any mapping of the form $\mathfrak{P}X^* \rightarrow \mathbb{R}^{\ell \times n} : \eta \mapsto (c, \eta)$. The set of all such mappings will be denoted by $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{P}X \rangle \rangle$, which forms an \mathbb{R} -vector space.

An alternative to parenthesization of words is to encode the order in which balanced parentheses appear by using two different products, say \prec and \succ . For example,

$$x_i[x_j] \equiv x_i \prec x_j \text{ and } [x_i]x_j \equiv x_i \succ x_j. \quad (6)$$

Using these products the induced algebraic structure on $\mathfrak{P}X^*$ is described next.

Definition 2: A *dendriform algebra* is an \mathbb{R} -vector space, $(D, +, \cdot)$, endowed with products \prec and \succ such that for $a, b, c \in D$ the following axioms are satisfied:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \quad (7a)$$

$$(a \succ b) \prec c = a \succ (b \prec c), \quad (7b)$$

$$a \succ (b \succ c) = (a \prec b + a \succ b) \succ c. \quad (7c)$$

If $D = X$, then (X, \prec, \succ) forms a dendriform algebra. Similar to (6), for every $\eta \in \mathfrak{P}X^*$ there is a corresponding dendriform product in (X, \prec, \succ) . This is made explicit by the injection $\delta : \mathfrak{P}X^* \rightarrow (X, \prec, \succ)$, which is defined inductively by

$$\delta(\eta) = \begin{cases} x_i \prec \delta(\eta'), & \text{if } \eta = x_i[\eta'], \\ \delta(\eta') \succ x_i, & \text{if } \eta = [\eta']x_i, \\ \delta(\eta') \succ x_i \prec \delta(\eta''), & \text{if } \eta = [\eta']x_i[\eta''], \end{cases}$$

where $x_i \in X$, $\eta', \eta'' \in \mathfrak{P}X^*$, $\delta(\emptyset) = \emptyset$, and $\delta(x_j) = x_j$ for all $x_j \in X$. For example,

$$\delta(x_i[[x_j]x_k]) = x_i \prec (\delta([x_j]x_k)) = x_i \prec (x_j \succ x_k).$$

Define $\mathfrak{T}X^* = \delta(\mathfrak{P}X^*)$, the image of $\mathfrak{P}X^*$ under δ . Any element of $\mathfrak{T}X^*$ is called a *dendriform word*.

The set of formal power series on dendriform words is denoted by $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle$, and it is also an \mathbb{R} -vector space. An element of $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle$ can be viewed as a mapping $c : \mathfrak{T}X^* \rightarrow \mathbb{R}^{\ell \times n} : \eta \mapsto (c, \eta)$. The set of all series in $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle$ having finite support is denoted by $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle$. In addition, for any dendriform word there is only one corresponding word in X^* given by the projection $\varphi : \mathfrak{T}X^* \rightarrow X^* : \eta \mapsto (c, \eta)$. For example, $\varphi(x_i \prec (x_j \prec x_k)) = x_i x_j x_k \in X^*$.

Next define the product $\diamond : \mathfrak{T}X^* \times \mathfrak{T}X^* \rightarrow \mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle : (\eta, \xi) \mapsto \eta \prec \xi + \eta \succ \xi$. This product is the non-commutative counterpart of the *shuffle product* [10], and it is extended bilinearly on $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle \times \mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle$.

Lemma 1: [9], [14], [16] $(\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{T}X \rangle \rangle, \diamond)$ is an associative \mathbb{R} -algebra.

An important characteristic of the commutative shuffle product is that it can be defined recursively, which is convenient for computer implementations. For the non-commutative shuffle product such a recursive definition is

only available when the words to be *shuffled* have single letters. In this regard, the notion of planar binary trees plays a key role as described next.

III. TREES, DENDRIFORM WORDS AND ITERATED INTEGRALS

The objective of this section is to describe the one-to-one correspondence between planar binary trees and dendriform words. Then their relationship to non-commutative iterated integrals is described. The majority of concepts presented in subsection III-A can be found in [9], [14], [16] and references therein.

A. Trees and dendriform words

A tree is a non-cyclic connected graph (V, Γ) , where V denotes the vertices of the graph and Γ the edges. A *leaf* is defined as a vertex that is the endpoint of only one edge. The n leaves of a tree are labeled from left to right as $1, 2, \dots, n$. A *planar rooted tree* is a tree embedded in the plane in which one vertex (with no incoming edges) is labeled as the *root*. The *interior* vertices of a rooted planar tree is the set V minus the root and the leaves. A planar n -ary tree is a planar rooted tree where every interior vertex has one root and n leaves. *Order* is defined by the number of interior vertices. This paper is concerned with *planar binary trees*, so every interior vertex has one root and two leaves. The set of all planar binary trees is denoted by \mathfrak{T} , and \mathfrak{T}_n denotes the set of planar binary trees of order n . The planar binary trees up to order three are:

$$\mathfrak{T}_0 = \{ | \}, \mathfrak{T}_1 = \{ \vee \}, \mathfrak{T}_2 = \{ \vee \vee, \vee \vee \},$$

$$\mathfrak{T}_3 = \{ \vee \vee \vee, \vee \vee \vee, \vee \vee \vee, \vee \vee \vee, \vee \vee \vee \}.$$

The tree $|$ is the *trivial tree*. A well-known fact about planar binary trees is their cardinality $\#(\mathfrak{T}_n) = C_n := \frac{1}{n+1} \binom{2n}{n}$, which is the n -th Catalan number. It is also known that the number of ways of associating n applications of a binary operator (e.g., balanced parenthesization) is C_n . If trees are *suitably decorated* with a set of symbols, then there is a one-to-one correspondence between trees and dendriform words.

Definition 3: Let V_{int} be the set of interior vertices of tree $\tau \in \mathfrak{T}$ and D a finite set of symbols. A *decoration* of τ is any injection $\rho : V_{int} \rightarrow D$.

Example 1: Let $\tau = \vee \vee$, $D = \{x, y, z\}$ and $V_{int} = \{v_1, v_2, v_3\}$, where v_i is the vertex where the paths starting from leaves i and $i+1$ join together. Figure 1 shows the decoration of τ by ρ , where $\rho(v_1) = x$, $\rho(v_2) = y$ and $\rho(v_3) = z$. \square

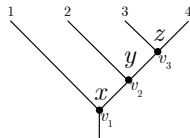


Fig. 1: Tree decoration

In general, any $\tau \in \mathfrak{T}_n$ can be decorated by the letters in the word $\eta = x_{i_1} \cdots x_{i_n} \in X^n$, i.e., $\rho(v_j) = x_{i_j}$. The set of all trees decorated by X^* is denoted as $\mathfrak{TD}X^*$, $|\tau|$ is the order of $\tau \in \mathfrak{TD}X^*$, and the *foliation* of τ is the mapping

$\psi : \mathfrak{TD}X^* \rightarrow X^*$. The decoration of a tree with the letters of a word is defined by the operation $(\cdot; \cdot) : X^* \times \mathfrak{T} \rightarrow \mathfrak{TD}X^* : (\eta; \tau) \mapsto \tau_\eta$. The notation τ_η makes explicit the fact that a tree $\tau \in \mathfrak{T}$ is being decorated by the word $\eta \in X^*$. For example,

$$(x_i x_j x_k; \vee \vee) = \begin{array}{c} \vee \\ / \quad \backslash \\ x_i \quad x_j \\ \vee \\ / \quad \backslash \\ x_i \quad x_k \end{array}$$

A formal power series on decorated trees is any mapping $c : \mathfrak{TD}X^* \rightarrow \mathbb{R}^{\ell \times n} : \eta \mapsto (c, \eta)$. The set of formal power series on decorated trees is denoted by $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{TD}X^* \rangle \rangle$, and forms an \mathbb{R} -vector space. The subset of series with finite support is $\mathbb{R}^{\ell \times n} \langle \mathfrak{TD}X^* \rangle$. In this context, the decoration of trees is a bilinear operation.

One way of constructing new trees from a given set of trees (usually called a *forest*) is by the operation of *grafting*.

Definition 4: The *grafting* of trees is an n -ary operation \vee consisting of joining together n trees to the same root to form a new tree. More precisely, $\vee : \underbrace{\mathfrak{T} \times \cdots \times \mathfrak{T}}_{n \text{ times}} \rightarrow \mathfrak{T}$ such

that

$$\vee(\tau^1, \dots, \tau^n) = \begin{array}{c} \tau^1 \quad \tau^2 \quad \tau^3 \quad \dots \quad \tau^{n-1} \quad \tau^n \\ \vee \\ \vee \end{array}$$

Grafting is for trees what catenation is for words. For example,

$$| \vee | = \vee, \quad \vee \vee \vee = \vee \vee \vee.$$

Observe that if $\tau = \vee(\tau^1 \cdots \tau^m) \in \mathfrak{T}_n$, then $\tau^i \in \mathfrak{T}_{m_i}$ with $\sum_i m_i = n-1$. In this paper, the focus is on binary grafting, i.e., $m = 2$. Any planar binary tree can be decomposed uniquely as $\tau = \tau^1 \vee \tau^2$ since by definition any planar binary tree interior vertex is trivalent (has one root and two leaves). Other tree decompositions such as the ones used in Hopf algebras of trees are not unique [13]. The tree τ^1 (respectively τ^2) is the left part (respectively the right part) of τ . Further decompositions allow one to write any planar binary tree in terms of the trivial tree $|$. The grafting operation \vee makes \mathfrak{T} the free magma algebra with one generator. It is neither commutative nor associative but is of order one with respect to the grading in terms of internal vertices. That is, for two trees τ^1, τ^2 of order n_1, n_2 , respectively, the product $\tau^1 \vee \tau^2$ is of order $n_1 + n_2 + 1$. Particular types of trees that allow an easy decomposition are the so-called right-comb and left-comb as shown in Figure 2.

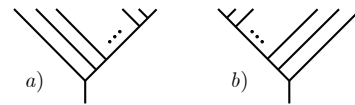


Fig. 2: a) left comb, b) right comb

Clearly for a right-comb (respectively left-comb) $\tau_r^n = \tau_r^{n-1} \vee |$ (respectively $\tau_l^n = | \vee \tau_l^{n-1}$), where τ_r^k denotes the k -th order right-comb (respectively τ_l^k denotes the k -th order left-comb).

One way of realizing the decoration of a tree is by attaching a letter from the alphabet X to every grafting operation used in the construction, say \vee_{x_i} . For example,

$$\left| \vee_{x_i} \right| = \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \quad (8a)$$

$$\left| \vee_{x_i} \left(\left| \vee_{x_j} \right| \right) \right| = \left| \vee_{x_i} \begin{array}{c} \diagup \\ x_j \\ \diagdown \end{array} \right| = \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \quad (8b)$$

$$\left(\left| \vee_{x_i} \right| \right) \vee_{x_j} \left(\left| \vee_{x_k} \right| \right) = \begin{array}{c} \diagup \\ x_j \\ \diagdown \\ \diagup \\ x_k \\ \diagdown \end{array} \quad (8c)$$

The grafting operation allows an explicit description of the correspondence between the sets \mathfrak{TX}^* and \mathfrak{TDX}^* . This is provided by the isomorphism $\Phi : \mathfrak{TX}^* \rightarrow \mathfrak{TDX}^*$ with the inductive definition

$$\Phi(\eta_\tau) = \begin{cases} \left| \vee_{x_i} \Phi(\eta'_{\tau'}) \right|, & \text{if } \eta_\tau = x_i \prec \eta'_{\tau'}, \\ \Phi(\eta'_{\tau'}) \vee_{x_i} \left| \right|, & \text{if } \eta_\tau = \eta'_{\tau'} \succ x_i, \\ \Phi(\eta'_{\tau'}) \vee_{x_i} \Phi(\eta''_{\tau''}), & \text{if } \eta_\tau = \eta'_{\tau'} \succ x_i \prec \eta''_{\tau''}, \end{cases}$$

where $x_i \in X$, $\eta'_{\tau'}, \eta''_{\tau''} \in \mathfrak{TX}^*$, $\Phi(\emptyset) = \left| \right|$ and $\Phi(x_j) = \left| \vee_{x_j} \right|$ for $x_j \in X$.

In [16], the free magma \mathfrak{TX}^* is defined directly as the set of all planar binary trees whose leaves are decorated with the letters in X . Any $\eta \in \mathfrak{TX}^*$ will be denoted as η_τ , where it is made explicit the fact that for any dendriform word there exist a decorated tree $\tau \in \mathfrak{TDX}^*$ providing the order in which the products \prec and \succ appear. The corresponding tree is then obtained as $\Phi(\eta_\tau) = \tau_\eta \in \mathfrak{TDX}^*$, and its inverse satisfies $\Phi^{-1}(\tau_\eta) = \eta_\tau \in \mathfrak{TX}^*$. Moreover, the foliation of τ_η can be written in terms of the map φ of dendriform words as $\psi(\tau_\eta) = \varphi(\Phi^{-1}(\tau_\eta)) = \eta \in X^*$. The isomorphism Φ is extended linearly in the natural way over $\mathbb{R}^{\ell \times n} \langle \langle \mathfrak{TX}^* \rangle \rangle$. Hereafter, due to the isomorphism between \mathfrak{TX}^* and \mathfrak{TDX}^* , no notational distinction will be made between products in \mathfrak{TX}^* and products in \mathfrak{TDX}^* .

The grafting operation allows one to define the *shuffle product of trees* in an inductive manner.

Definition 5: [14] Let $\tau^1 = \tau^{11} \vee_{x_i} \tau^{12}$ and $\tau^2 = \tau^{21} \vee_{x_j} \tau^{22}$ with $\tau^{k1}, \tau^{k2} \in \mathfrak{TDX}^*$. The recursive definition of \diamond : $\mathfrak{TDX}^* \times \mathfrak{TDX}^* \rightarrow \mathbb{R} \langle \mathfrak{TDX} \rangle$ is

$$\tau^1 \diamond \tau^2 = \tau^{11} \vee_{x_i} (\tau^{12} \diamond \tau^2) + (\tau^1 \diamond \tau^{21}) \vee_{x_j} \tau^{22}, \quad (9)$$

where $\left| \diamond \tau \right| = \tau \diamond \left| \right| = \tau$ for any $\tau \in \mathfrak{TDX}^*$.

Example 2: The shuffle product

$$\begin{aligned} \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \diamond \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} &= \left(\left| \vee_{x_i} \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \right| \right) \diamond \left(\left| \vee_{x_k} \right| \right) \\ &= \left| \vee_{x_i} \left(\begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \diamond \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \right) \right| + \left(\begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \diamond \left| \right| \right) \vee_{x_k} \left| \right| \\ &= \left| \vee_{x_i} \left(\left| \vee_{x_j} \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \right| + \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \vee_{x_k} \left| \right| \right) \right| + \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \vee_{x_k} \left| \right| \\ &= \begin{array}{c} \diagup \\ x_j \\ \diagdown \\ \diagup \\ x \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ x \\ \diagdown \\ \diagup \\ x_k \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ x \\ \diagdown \end{array} \vee_{x_k} \left| \right|. \end{aligned}$$

□

Lemma 2: [14] The shuffle product \diamond of trees is non-commutative and associative.

The dendriform products for trees are given next.

Definition 6: [14] Let $\tau^1 = \tau^{11} \vee_{x_i} \tau^{12}$ and $\tau^2 = \tau^{21} \vee_{x_j} \tau^{22}$. The dendriform products for decorated trees are:

$$\tau^1 \prec \tau^2 = \tau^{11} \vee_{x_i} (\tau^{12} \prec \tau^2) \quad (10a)$$

$$\tau^1 \succ \tau^2 = (\tau^1 \succ \tau^{21}) \vee_{x_j} \tau^{22}. \quad (10b)$$

Theorem 1: [14] The products \prec and \succ in (10) satisfy the axioms of dendriform algebras given in (7).

From Theorem 1, (9) and (10), it is clear that $\prec + \succ = \diamond$ as in Section II, and therefore, with the help of the mapping Φ , the relationship between the dendriform and shuffle products on \mathfrak{TX}^* and \mathfrak{TDX}^* is:

$$\Phi(\eta_{\tau^1} \prec \xi_{\tau^2}) = \Phi(\eta_{\tau^1}) \prec \Phi(\xi_{\tau^2}) = \tau^1 \prec \tau^2$$

$$\Phi(\eta_{\tau^1} \succ \xi_{\tau^2}) = \Phi(\eta_{\tau^1}) \succ \Phi(\xi_{\tau^2}) = \tau^1 \succ \tau^2$$

$$\Phi(\eta_{\tau^1} \diamond \xi_{\tau^2}) = \Phi(\eta_{\tau^1}) \diamond \Phi(\xi_{\tau^2}) = \tau^1 \diamond \tau^2$$

$$\Phi(\eta_{\tau^1} \diamond \emptyset) = \Phi(\emptyset \diamond \eta_{\tau^1}) = \tau^1 \diamond \left| \right|.$$

This subsection ends with two key lemmas employed in Section III-B to characterize the grouping of non-commutative iterated integrals.

Lemma 3: For any $n \geq 0$,

$$\text{char}(\mathfrak{T}_{n+1}) := \sum_{\tau \in \mathfrak{T}_{n+1}} \tau = \sum_{i=0}^n \text{char}(\mathfrak{T}_{n-i}) \vee \text{char}(\mathfrak{T}_i).$$

The collection of all trees of a certain order can be described in terms of the non-commutative shuffle product.

Lemma 4: The summation of all undecorated trees of order $n \geq 0$ is given by

$$\text{char}(\mathfrak{T}_n) = \begin{array}{c} \diagup \\ \diagdown \end{array} \diamond^n,$$

where $\begin{array}{c} \diagup \\ \diagdown \end{array} \diamond^{n+1} = (\begin{array}{c} \diagup \\ \diagdown \end{array} \diamond^n) \diamond \begin{array}{c} \diagup \\ \diagdown \end{array}$ and $\begin{array}{c} \diagup \\ \diagdown \end{array} \diamond^0 = \emptyset$.

B. Non-commutative iterated integrals

For a matrix-valued measurable function $u : [0, T] \rightarrow \mathbb{R}^{n \times q}$, define $\|u\|_{L_1} = \int_0^T \|u(s)\|_1 ds$. Note that $\|u(s)\|_1 = \max_j \{ \sum_i |u(s)_{ij}| \}$. Now let instead $u = (u_1, \dots, u_m)$, where each $u_i : [0, T] \rightarrow \mathbb{R}^{n \times q}$. The norm for this u is $\|u\| := \max_i \|u_i\|_{L_1}$. The set $L_1^{m \times (n \times q)}[0, T]$ contains all measurable functions defined on $[0, T]$ having finite $\|\cdot\|$ norm, and $B_1^{m \times (n \times q)}(R)[0, T] := \{u \in L_1^{m \times (n \times q)}[0, T], \|u\| \leq R\}$.

Definition 7: Let $u \in B_1^{m \times (n \times n)}(R)[0, T]$. The non-commutative iterated integral corresponding to $\eta_\tau \in \mathfrak{TX}^*$ for $t \in [0, T]$ is defined inductively by $E_\emptyset[u] = I$, and

$$E_{\eta_\tau}[u](t) = \int_0^t E_{\xi_{\tau^1}}[u](s) u(s) E_{\nu_{\tau^2}}[u](s) ds,$$

where $x_i \in X$, $\eta_\tau = \xi_{\tau^1} \vee_{x_i} \nu_{\tau^2}$ with $\xi_{\tau^1}, \nu_{\tau^2} \in \mathfrak{TX}^*$, $\tau^1, \tau^2 \in \mathfrak{T}$, $u_0 = I$, and I denotes the identity matrix.

The mapping E_{η_τ} is extended linearly on $\mathbb{R}^{n \times n} \langle \mathfrak{TX} \rangle$ in the natural way. For example, the iterated integrals corresponding to (8) are, respectively,

$$E_{x_i}[u](t) = \int_0^t u_i(s) ds,$$

$$E_{x_i \prec x_j}[u](t) = \int_0^t u_i(s) \int_0^s u_j(\tau) d\tau ds,$$

$$\begin{aligned} E_{x_i \succ (x_j \prec x_k)}[u](t) &= E_{(x_i \succ x_j) \prec x_k}[u](t) \\ &= \int_0^t \left(\int_0^s u_i(\tau) d\tau \right) u_j(s) \left(\int_0^s u_k(\tau) d\tau \right) ds. \end{aligned}$$

Some useful results concerning non-commutative iterated integrals and their bounds are given below [4].

Lemma 5: Let τ be an arbitrary tree in \mathfrak{T}_n , τ_l^n the left-comb tree in \mathfrak{T}_n , $x_i \in X$ and $\eta \in X^n$ (all words in X^* of length n). Then

$$\begin{aligned} i. & \|E_{(x_i^n)_\tau}[u](t)\|_1 \leq \frac{\bar{U}_i^{|\tau|}(t)}{\gamma(\tau)}, \\ ii. & \|E_{\eta_{\tau_l^n}}[u](t)\|_1 \leq \prod_{j=1}^n \frac{\bar{U}_j^{n_j}(t)}{n_j!}, \end{aligned}$$

where $\bar{U}_j(t) := \int_0^t \bar{u}_i(s) ds$, $\bar{u}_i(s) := \|u_j(s)\|_1$, $n_j = |\eta|_{x_j}$ for $j = 0, \dots, m$, $|\eta|_{x_j}$ denotes the number of x_j letters in $\eta \in X^*$ and $\gamma(\tau) = (|\tau^1| + |\tau^2| + 1)\gamma(\tau^1)\gamma(\tau^2)$ with $\gamma(\emptyset) = 0$.

Lemma 6: Let $\eta_\tau \in \mathfrak{TX}^*$, and $u \in B_1^{m \times (n \times n)}(R)[0, T]$. The iterated integral corresponding to η_τ satisfies

$$\|E_{\eta_\tau}[u](t)\|_1 \leq E_{\eta_\tau}[\bar{u}](t), \quad \forall t \in [0, T],$$

where $\bar{u}(t) = (\bar{u}_1(t), \dots, \bar{u}_m(t))^T$.

The correspondence between the commutative shuffle product, \sqcup , and the product of commutative iterated integrals generalizes in the non-commutative setting as follows.

Theorem 2: Let $u \in B_1^{m \times (n \times n)}(R)[0, T]$ and $\eta_{\tau_1}, \xi_{\tau_2} \in \mathfrak{TX}^*$. Then

$$E_{\eta_{\tau_1}}[u](t)E_{\xi_{\tau_2}}[u](t) = E_{\eta_{\tau_1} \curvearrowright \xi_{\tau_2}}[u](t).$$

Lemma 7: Let $\tau = \sphericalangle$. The following identity holds when u_i is replaced with \bar{u}_i :

$$E_{(x_i)_\tau \curvearrowright^n}[\bar{u}](t) = n!E_{x_i^n}[\bar{u}](t).$$

IV. DENDRIFORM FLIESS OPERATORS AND THEIR CONVERGENCE

In this section dendriform Fliess operators are defined, and sufficient conditions for their convergence are provided.

A. Dendriform Fliess operators

The definition of a dendriform Fliess operator is given first.

Definition 8: Let $u \in B_1^{m \times (n \times n)}(R)[0, T]$ and $c \in \mathbb{R}^{\ell \times n} \langle \langle \mathfrak{TX} \rangle \rangle$. A *dendriform Fliess operator* with generating series c is defined by the following summation

$$F_c[u](t) = \sum_{\eta_\tau \in \mathfrak{TX}^*} (c, \eta_\tau) E_{\eta_\tau}[u](t).$$

The operator in (4) is a special case of a dendriform Fliess operator. The support of its generating series contains only left-comb trees. This is purely a consequence of the iterative procedure used to derive it. However, defining Fliess operators as a summation comprised of only left-comb trees limits its application as shown in the next example.

Example 3: Suppose two Fliess operators F_c and F_d have generating series in terms of left-combs. Assume $c = c'I$ and $d = d'I$ in $\mathbb{R}^{n \times n} \langle \langle \mathfrak{TX} \rangle \rangle$ with c', d' being scalar-valued series, and $u \in B_1^{1 \times (n \times n)}(R)[0, T]$ for some $R, T > 0$. Since $\sum_{\eta_\tau \in \mathfrak{TX}^*} \eta_\tau = \sum_{\eta \in X^*} \sum_{\tau \in \mathfrak{T}} \eta_\tau$, their product connection is described by

$$F_c[u]F_d[u] = \sum_{n_1, n_2=0}^{\infty} \sum_{\substack{\eta \in X^{n_2} \\ \xi \in X^{n_1}}} (c, \eta_{\tau_1^{n_1}})(d, \xi_{\tau_2^{n_2}}) E_{\eta_{\tau_1^{n_1}}}[u] E_{\xi_{\tau_2^{n_2}}}[u].$$

Recall that $E_{\eta_{\tau_1^{n_1}}}[u] E_{\xi_{\tau_2^{n_2}}}[u] = E_{\eta_{\tau_1^{n_1}} \curvearrowright \xi_{\tau_2^{n_2}}}[u]$, where \curvearrowright generates *more* than just left-combs as shown in Example 2.

Therefore, Definition 8 is general enough to characterize such interconnections in the non-commutative framework. \square

B. Convergence of dendriform Fliess operators

The next theorem addresses the convergence of dendriform Fliess operators by considering bounds on the coefficients of the corresponding generating series. The final three lemmas and theorem in Section III were specifically developed for proving this theorem (see [4]).

Theorem 3: Let $c \in \mathbb{R}^{\ell \times n} \langle \langle \mathfrak{TX} \rangle \rangle$ with coefficients satisfying the growth condition

$$\|(c, \eta_\tau)\|_1 \leq KM^{|\tau|}, \quad \forall \eta_\tau \in \mathfrak{TX}^* \quad (11)$$

for some constants $K, M > 0$. Then there exist $R, T > 0$ such that for each $u \in B_1^{m \times (n \times n)}(R)[0, T]$ the series

$$y(t) = F_c[u](t) = \sum_{\eta_\tau \in \mathfrak{TX}^*} (c, \eta_\tau) E_{\eta_\tau}[u](t)$$

converges absolutely and uniformly on $[0, T]$.

Coefficients bounded as in (11) give convergence of a local nature whereas in the commutative case such bounds provide a type of global convergence [12]. The reason for this discrepancy is that in addition to summing over all possible permutations of letters in X , which is the commutative case, the bounds for non-commutative iterated integrals also require the summation over all trees. This fact contributes an extra $k!$ factor, in the commutative case, coming directly from the integrals.

A *left-comb dendriform Fliess operator* is a dendriform Fliess operator whose generating series support only have dendriform words corresponding to left-combs. The convergence of such operators is addressed in the next theorem.

Theorem 4: Let $c \in \mathbb{R}^{\ell \times n} \langle \langle \mathfrak{TX} \rangle \rangle$ with coefficients satisfying the growth condition

$$\|(c, \eta_\tau)\|_1 \leq KM^{|\tau|} |\tau|!, \quad \forall \eta_\tau \in \mathfrak{TX}^*$$

for some constants $K, M > 0$ and $\text{supp}(c) \subseteq \{\eta_\tau \in \mathfrak{TX}^*, \tau = \tau_l^k, k > 0\}$. Then there exist $R, T > 0$ such that for each $u \in B_1^{m \times (n \times n)}(R)[0, T]$ the series

$$y(t) = F_c[u](t) = \sum_{k=0}^{\infty} \sum_{\eta \in X^k} (c, \eta_{\tau^k}) E_{\eta_{\tau^k}}[u](t)$$

converges absolutely and uniformly on $[0, T]$.

Example 4: Consider $c = \sum_{k=0}^{\infty} x_1^k \tau_1^k \in \mathbb{R}^{n \times n} \langle \langle \mathfrak{TX} \rangle \rangle$. This series is the generating series corresponding to (4), which is the solution of (3). Recall that (3) can represent the evolution of a closed quantum system (all quantum constants normalized to 1). In the commutative case, it is known that $Z = \exp(\Omega)$, where $\Omega(t) = \int_0^t U(s) ds$. From the Fliess operator point of view,

$$Z = F_c[U] = \sum_{n=0}^{\infty} E_{x_1^n}[U],$$

which by the properties of the commutative shuffle product gives

$$Z = F_c[U] = \sum_{k=0}^{\infty} \frac{(E_{x_1}[U])^k}{k!} = \exp(E_{x_1}[U]), \quad (12)$$

where obviously $E_{x_1}[U] = \Omega$. Suppose now U is non-

commutative. Then

$$F_c[U] = \sum_{k=0}^{\infty} E_{x_1 \tau_1^k} [U], \quad (13)$$

which by Theorem 4 with $K = M = 1$ is well defined. Assume now that $F_c[U]$ has an exponential representation similar to the commutative case. That is, $F_c[U] = \exp(\Omega)$ with $\Omega = F_d[U]$ for some $d \in \mathbb{R}^{n \times n} \langle \langle \mathfrak{S}X \rangle \rangle$. Unfortunately, the identities used to obtain (12) cannot be used to find the expression for d . But Lemma 4 provides an inductive way to compute it. Assume that $\Omega = E_{x_1 \tau_1^1} [U]$. Then expanding $\exp(\Omega)$ gives

$$\begin{aligned} \exp(\Omega) &= I + E_{x_1 \tau_1^1} [U] + \frac{1}{2!} \left(E_{x_1 \tau_1^1} [U] \right)^2 + \dots \\ &= I + E_{x_1 \tau_1^1} [U] + \frac{1}{2!} E_{\Phi^{-1}(x_1^2; \vee_+ \vee_+)} [U] + \dots \\ &= I + E_{x_1 \tau_1^1} [U] + E_{x_1^2 \tau_1^2} [U] \\ &\quad - \frac{1}{2!} E_{\Phi^{-1}(x_1^2; \vee_+ \vee_-)} [U] + \frac{1}{2!} E_{\Phi^{-1}(x_1^2; \vee_- \vee_-)} [U] + \dots \end{aligned}$$

Observe that the expansion produces more terms than needed. Therefore, a correction term must be used in order to cancel the extra second order terms. So redefine Ω as

$$\Omega = E_{x_1 \tau_1^1} [U] - \frac{1}{2} E_{\Phi^{-1}(x_1^2; \vee_- \vee_-)} [U].$$

It follows then that the first and second order terms are

$$\begin{aligned} \exp(\Omega) &= I + E_{x_1 \tau_1^1} [U] - \frac{1}{2} E_{\Phi^{-1}(x_1^2; \vee_- \vee_-)} [U] + \\ &\quad + \frac{1}{2!} E_{\Phi^{-1}(x_1^2; \vee_+ \vee_+)} [U] + \frac{1}{2!} E_{\Phi^{-1}(x_1^2; \vee_+ \vee_-)} [U] + \dots \\ &= I + E_{x_1 \tau_1^1} [U] + E_{x_1^2 \tau_1^2} [U] + \dots \end{aligned}$$

Observe that the second order correction term in Ω can be written in the following form

$$E_{\Phi^{-1}(x_1^2; \vee_- \vee_-)} [U] = E_{x_1 \prec x_1} [U] - E_{x_1 \succ x_1} [U].$$

In fact, defining the product $\triangleright = \prec - \succ$, it follows that

$$\begin{aligned} E_{x_1 \triangleright x_1} [U](t) &= E_{x_1 \prec x_1} [U](t) - E_{x_1 \succ x_1} [U](t) \\ &= \int_0^t \left[U(s), \int_0^s U(r) dr \right] ds, \end{aligned}$$

where $[\cdot, \cdot]$ denotes the commutator. The non-associative product \triangleright is an example of a *pre-Lie product* [8]. This correction procedure can be applied successively at every order. At order 3, the correction terms for Ω are

$$\begin{aligned} \Omega &= E_{x_1} [U] - \frac{1}{2} E_{x_1 \triangleright x_1} [U] + \frac{1}{4} E_{(x_1 \triangleright x_1) \triangleright x_1} [U] \\ &\quad + \frac{1}{12} E_{x_1 \triangleright (x_1 \triangleright x_1)} [U], \end{aligned}$$

which gives

$$\exp(\Omega) = I + E_{x_1 \tau_1^1} [U] + E_{x_1^2 \tau_1^2} [U] + E_{x_1^3 \tau_1^3} [U] + \dots,$$

where the generating series d of Ω satisfies the recursion

$$d^{[k]} = \sum_{n \geq 0} \frac{B_n}{n!} L_{d^{[k-1] \triangleright}}^{(n)}(x_1)$$

with $d^{[1]} = x_1$, $\lim_{k \rightarrow \infty} d^{[k]} = d$, $L_{d \triangleright}^{(n)}(x) = d \triangleright (L_{d \triangleright}^{(n-1)}(x))$, $L_{d \triangleright}^{(0)}(x) = d$, and B_n denotes the n -th Bernoulli number.

Thus, the limit of $\exp(F_d^{[k]}[U])$ as $k \rightarrow \infty$ agrees with (13). This is the well-known *Magnus expansion*. The more familiar expression for the Magnus expansion is obtained by noting that

$$E_{L_{d \triangleright}^{(n)}(x_1)} [U](t) = \int_0^t ad_{\Omega(s)}^{(n)}(U(s)) ds,$$

and $ad_{\Omega}^{(n)}(U) = [\Omega, ad_{\Omega}^{(n-1)}(U)]$ with $ad_{\Omega}^{(0)}(U) = U$. Compared to the ordered exponential presented in the introduction, this is a true exponential. In quantum mechanics this is one way to show that the evolution operator is unitary for all times. Finally, the Fliess operator $F_c[U]$ in (13) provides an input-output map that encodes in the iterated integrals the underlying algebraic structure of the system. \square

REFERENCES

- [1] D. D'Alessandro, *Introduction to quantum control and dynamics*, Chapman & Hall/CRC Applied Mathematics & Nonlinear Science, Hoboken, NJ, 2007.
- [2] M. Bauer, R. Chetrite, K. Ebrahimi-Fard, F. Patras, Time-ordering and a generalized Magnus expansion, *Lett. Math. Phys.*, **103**, no. 3 (2013) 331–350.
- [3] R. Brockett, Lie theory and control systems defined on spheres, *SIAM J. Appl. Math.*, **25**, no. 2, (1973) 213–225.
- [4] L. A. Duffaut Espinosa, W. S. Gray, K. Ebrahimi-Fard, Dendriform-tree setting for fully non-commutative Fliess operators, arXiv:1409.0059.
- [5] L. A. Duffaut Espinosa, W. S. Gray, O. R. González, *On Fliess operators driven by L_2 -Itô processes*, *Stochastics*, **84**, no. 4, (2012) 507–532.
- [6] K. Ebrahimi-Fard, L. Guo, Rota-Baxter algebras and dendriform algebras, *J. Pure Appl. Algebra*, **212**, no. 2, (2008) 320–339.
- [7] K. Ebrahimi-Fard, L. Guo, Free Rota-Baxter algebras and rooted trees, *J. Algebra Appl.*, **7**, no. 2, (2008) 167–194.
- [8] K. Ebrahimi-Fard, D. Manchon, A Magnus- and Fer-type formula in dendriform algebras, *Found. Comput. Math.*, **9**, no. 3, (2009) 295–316.
- [9] K. Ebrahimi-Fard, D. Manchon, The Magnus expansion, trees and Knuth's rotation correspondence, arXiv:1203.2878v1, 2012.
- [10] M. Fliess, Fonctions causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. France*, **109**, (1981) 3–40.
- [11] W. S. Gray and Y. Li, Generating series for interconnected analytic nonlinear systems, *SIAM J. Control Optim.*, **44**, no. 2, (2005) 646–672.
- [12] W. S. Gray, Y. Wang, Fliess operators on L_p spaces: Convergence and continuity, *Systems Control Lett.*, **46**, no. 2, (2002) 67–74.
- [13] R. Holtkamp, Rooted trees appearing in products and co-products, *Contemporary Mathematics*, K. Ebrahimi-Fard (ed.) et al., Amer. Math. Soc., vol. 539, 2011, pp. 153–169.
- [14] J. Loday, M. Ronco Hopf algebra of the planar binary trees, *Adv. Math.*, **139**, no. 2, (1998) 293–309.
- [15] W. Magnus, On the exponential solution of differential equations for a linear operator, *Commun. Pure Appl. Math.*, **7**, no. 4, (1954) 649–673.
- [16] G. Melançon, Combinatorics of Hall trees and Hall words, *J. Comb. Theory, Series A* **59**, no. 2, (1992) 285–308.
- [17] R. Ree, Lie elements and an algebra associated with shuffles, *Ann. Math.*, **68**, no. 2, (1958) 210–220.
- [18] C. Reutenauer, *Free Lie algebras*, Oxford University Press, London, 1993.