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Pre-Lie Algebra Characterization of SISO Feedback Invariants

W. Steven Gray† Makhin Thitsa‡ Luis A. Duffaut Espinosa§

***Abstract*— Transformation groups have been used extensively in system theory since its inception. Recently, a feedback transformation group for a nonlinear input-output system characterized by a Chen-Fliess functional expansion was described by the authors. In particular, an algorithm was given to identify a class of feedback invariant series. Their relationship to series having well-defined relative degree was also developed. This paper is a continuation of that work, but with three innovations. First, newly developed algebraic tools from the field of pre-Lie algebras are applied to give more insight into the invariance theory. The role of relative degree in this paper is diminished in favor of systems with arbitrary generating series. Finally, dynamic output feedback, as described by the feedback product, is considered explicitly.**

I. Introduction

Let G be a group and S a given set. G is said to act as a *transformation group* on the right of S if there exists a mapping $\varphi : S \times G \to S : (h, g) \mapsto hg$ such that:

i. $h1 = h$, 1 is the identity element of G;
ii. $h(g_1g_2) = (hg_1)g_2$ for all $g_1, g_2 \in G$.

The action is said to be *free* if $hg = h$ implies that $g = 1$. Transformation groups have been used extensively in system theory since its inception. The early work of Brockett, Krener and others in the case of linear systems [3], [4] and nonlinear state space systems [2], [23] has been important in understanding the role of invariance under feedback and coordinate transformations. More recently in [13], a feedback transformation group for a nonlinear input-output system characterized by a Chen-Fliess functional expansion was described. In particular, an algorithm was given to identify a class of feedback invariant series. Their relationship to series having well-defined relative degree was also developed. This paper is a continuation of that work, but with three innovations. First, newly developed algebraic tools from the field of pre-Lie algebras are applied to give more insight into the invariance theory [10]. The role of relative degree in this paper is diminished in favor of systems with arbitrary generating series. Finally, dynamic output feedback, as described by the *feedback product*, is considered explicitly.

The specific problems to be addressed are most clearly articulated when put in the context of linear, time-invariant (LTI) systems, which from an algebraic point of view is fully commutative, and therefore, much simpler. Consider the field of (irreducible) rational functions in s with real coefficients denoted by $\mathbb{R}(s)$. Let $G = \mathbb{R}_{p\bar{0}}(s)$ be the subfield of proper elements g of $\mathbb{R}(s)$ with the defining property that $g(+\infty)$ exists and is not zero. Let $S = \mathbb{R}_{p0}(s)$ denote the ring of strictly proper elements h of $\mathbb{R}(s)$. Therefore, $h(+\infty) = 0$. Observe that $g \in \mathbb{R}_{p\bar{0}}(s)$ if and only if $g = K + h$ for some scalar $K \neq 0$ and $h \in \mathbb{R}_{p0}(s)$. In this situation, $\mathbb{R}_{p\bar{0}}(s)$ acts freely from the right as a group on $\mathbb{R}_{p0}(s)$, where the product hg is defined in the usual fashion when $h, g \in \mathbb{R}(s)$. Such a transformation group appears naturally in the theory of LTI systems when feedback is applied. For example, when a single-input, single-output (SISO) plant is modeled by a transfer function $h \in \mathbb{R}_{p0}(s)$, and output feedback $h' \in \mathbb{R}_{p0}(s)$ is applied, the closed-loop system has the transfer function hg, where $g = (1-h'h)^{-1}$. As a group action, g acts linearly on h. But in the nonlinear setting this is *not* the case. So the first problem considered in this paper is how to linearize the feedback group action (not the systems) described in [13], which is a nonlinear generalization of the feedback action hg. This leads directly to the pre-Lie algebra defined on a set of formal power series described in Section III and then related to the group action in Section IV. Linearizing a group action is a well established idea in the theory of Lie groups [19]. It is often easier to understand the nature of an action by examining its linear component. That property is exploited in Section V to characterize a class of invariant series under the feedback group described in [13]. Of course, the group element $g = (1-h'h)^{-1}$ has a specific structure. (As explained in [13], other types of group elements can represent static state feedback, for example.) Therefore, one can also view in some sense the feedback h' acting on h, and this is a *nonlinear* action even in the LTI setting. This action can be linearized about $h' = 0$ as follows:

$$\begin{aligned} h_{cl} &= h(1-h'h)^{-1} = h(1 + h'h + (h'h)^2 + \cdots) \qquad (1) \\ &= h(1 + h'h + \mathcal{O}((h'h)^2)) \approx h(1 + h'h). \end{aligned}$$

Following Kalman in [22] (see also [20]), it is not difficult to isolate the *maximal* invariant series of this feedback action. First write $h = b/a$, where (b, a) are polynomials satisfying $\deg(b) < \deg(a)$. Then by Euclidean division, there always exists polynomials (q, p) such that $a = bp+q$ with $\deg(q) < \deg(b)$ and $\deg(p) = r := \deg(a) - \deg(b) \geq 1$. In which case, $h = 1/(p + q/b)$, and

$$h_{cl} = \frac{1}{p + \underbrace{\frac{q}{b} - h'}_{:=-h''}} = \left(\frac{1}{p}\right)\left(1 - h''\left(\frac{1}{p}\right)\right)^{-1}$$

†Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, 28049 Madrid, Spain, on leave from Old Dominion University, Norfolk, VA 23529, USA, sgray@odu.edu
‡School of Engineering, Mercer University, Macon, GA 31207, USA, thitsa_m@mercer.edu
§Department of Electrical and Computer Engineering, George Mason University, Fairfax, VA 22030, USA, lduffaut@gmu.edu

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$$= \frac{1}{p}\left(1 + \left(h''\frac{1}{p}\right) + \mathcal{O}\left(\left(h''\frac{1}{p}\right)^2\right)\right). \tag{2}$$

Since h'' is strictly proper, the series $1/p$ in s^{-1} having order r is clearly invariant under the dynamic output feedback law represented by h'. It will in general not be equivalent to the transfer function of a Brunovsky form, namely s^{-r}, which is the feedback invariant under the action hg for arbitrary $g \in \mathbb{R}_{p0}(s)$. For example, $h = (s+2)/(s^2+3s+3)$ has the dynamic output feedback invariant series $1/p = (s+1)^{-1} = \sum_{k\geq 1}(-1)^k s^{-k}$. Finally, observe that h_{cl} is on an orbit of $1/p$ under the action $(1/p)g$ with $g = (1 - h''(1/p))^{-1}$ and $h'' \in \mathbb{R}_{p0}(s)$ arbitrary. A nonlinear generalization of (2) is developed in Section VI.

II. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1}\cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. $|\eta|_{x_i}$ is the number of times the letter x_i appears in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c,η) and called the *coefficient* of η in c. If $(c,\emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. The *order* of a series is the length of the shortest word in its support ($\mathrm{ord}(0) := +\infty$) Typically, c is represented as the formal sum $c = \sum_{\eta\in X^*}(c,\eta)\eta$. The set of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The subset of series with finite support, i.e., polynomials, is represented by $\mathbb{R}^\ell\langle X\rangle$. Each set forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by $\sqcup\!\sqcup$ [8]. For any $x_i \in X$, the left-shift operator, $x_i^{-1}(\cdot)$, is defined on X^* by $x_i^{-1}(x_i\eta) = \eta$ with $\eta \in X^*$ and zero otherwise. Higher order shifts are defined inductively via $(x_i\xi)^{-1}(\cdot) = \xi^{-1}x_i^{-1}(\cdot)$, where $\xi \in X^*$. The left-shift operator is assumed to act linearly on $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. An analogous right-shift operator, $(\cdot)x_i^{-1}$, satisfying $(\eta x_i)x_i^{-1} = \eta$ and zero otherwise will also be useful.

A. Fliess Operators

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0,t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0,t_1]$. Let $L_{\mathfrak{p}}^m[t_0,t_1]$ denote the set of all measurable functions defined on $[t_0,t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R)[t_0,t_1] := \{u \in L_{\mathfrak{p}}^m[t_0,t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Assume $C[t_0,t_1]$ is the subset of continuous functions in $L_1^m[t_0,t_1]$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0,t_1] \to C[t_0,t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar\eta}[u](t,t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar\eta}[u](\tau,t_0)\,d\tau,$$

where $x_i \in X$, $\bar\eta \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta\in X^*}(c,\eta)\,E_\eta[u](t,t_0)$$

[8], [9]. Assuming certain growth conditions on the coefficients of c, F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,\, t_0+T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0+T]$ for sufficiently small $R,T > 0$, where the numbers $\mathfrak{p},\mathfrak{q} \in [1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [17]. Otherwise, F_c can only be interpreted in a formal sense [18].

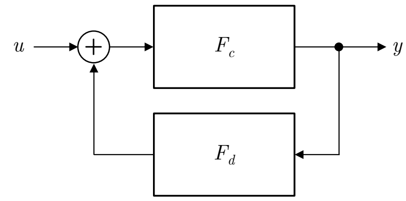

Fig. 1. Feedback connection of Fliess operators

B. Feedback Connection of Fliess Operators

When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta\in X^*}(c,\eta)\,\psi_d(\eta)(1)$$

[6], [7]. Here ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$ uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i)\circ\psi_d(\eta)$ with

$$\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup\, e) \tag{3}$$

for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where $d_0 := 1$ and $\psi_d(\emptyset)$ denotes the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. The composition product is associative and $\mathbb{R}$-linear in its left argument c. It is linear in its right argument if and only if its left argument is a *linear series*, that is, $\mathrm{supp}(c) \subseteq L$, where

$$L := \{\eta \in X^* : \eta = x_0^{n_1}x_ix_0^{n_0},\ i \in \{1,2,\ldots,m\},\ n_j \geq 0\}$$

is the set of *linear words*.

In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 1, the closed-loop system has a Fliess operator representation whose generating series is the *feedback product* of c and d, denoted by $c@d$ [16], [18]. Consider, for example, the SISO case, where $X = \{x_0, x_1\}$ and $\ell = 1$. Define the set of operators

$$\mathscr{F}_\delta = \{I + F_c\ :\ c \in \mathbb{R}\langle\langle X\rangle\rangle\},$$

where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. $\mathscr{F}_\delta$ forms a group under composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c\,\tilde{\circ}\,d$, and $\tilde{\circ}$ denotes the *modified* composition product. That is, the product

$$c\,\tilde{\circ}\,d = \sum_{\eta\in X^*} (c,\eta)\,\phi_d(\eta)(1),$$

where ϕ_d is analogous to ψ_d in (3) except here $\phi_d(x_i)(e) = x_i e + x_0(d_i \sqcup\!\sqcup\, e)$ with $d_0 := 0$. It is of central importance that the corresponding group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ has a dual that forms a Faà di Bruno Hopf algebra with antipode, α, satisfying

$$c_\delta^{\circ -1} = \delta + c^{\circ -1} = \delta + \sum_{\eta\in X^*} (\alpha\, a_\eta)(c)\,\eta,$$

where $c^{\circ -1}$ denotes the composition inverse of c,

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c,\eta)$$

is the coordinate function for $\eta \in X^*$, and $a_\delta(c_\delta) := 1$ [12]. The antipode can be computed recursively [14]. The following theorem gives a nonlinear generalization of (1).

Theorem 1: [12] Let $X = \{x_0, x_1\}$. For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that

$$c@d = c\,\tilde{\circ}\,(-d \circ c)^{\circ -1} = c \circ (\delta - d \circ c)^{\circ -1}.$$

III. Pre-Lie Algebra of Foissy

A (right) pre-Lie algebra over $\mathbb{R}$ is an $\mathbb{R}$-vector space A with a bilinear product $\bullet$ satisfying

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b)$$

for all $a, b, c \in A$. Such algebras constitute an important new tool in combinatorics and differential geometry [5], [24]. It is easily shown that $[a, b] := b \bullet a - a \bullet b$ satisfies the Jacobi identity, and therefore $(A, [\cdot,\cdot])$ represents a special class of Lie algebras. The following pre-Lie product on $\mathbb{R}\langle X\rangle$ was defined by Foissy [10].

Definition 1: Let $X = \{x_0, x_1\}$. Given $c, d \in \mathbb{R}\langle X\rangle$, the pre-Lie product $c \bullet d$ is the $\mathbb{R}$-bilinear product defined inductively by

$$(x_0\eta) \bullet d = x_0(\eta \bullet d)$$
$$(x_1\eta) \bullet d = x_1(\eta \bullet d) + x_0(\eta \sqcup\!\sqcup\, d)$$

with $\eta \in X^*$ and $\emptyset \bullet d = 0$.

The product is also well defined on $\mathbb{R}\langle X\rangle \times \mathbb{R}\langle\langle X\rangle\rangle$ since it involves computing only a finite number of shuffle products between words and series. But to properly extend the definition to $\mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle$, the following lemma is needed.

Lemma 1: For any $\eta \in X^*$ and $d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that $\mathrm{ord}(\eta \bullet d) \geq |\eta| + \mathrm{ord}(d)$, where equality holds if $|\eta|_{x_1} > 0$ or $d = 0$.

Proof: Since the pre-Lie product is right linear, $\eta\bullet 0 = 0$, and therefore, $\mathrm{ord}(\eta\bullet 0) = |\eta| + \mathrm{ord}(0) = +\infty$. Assume hereafter that $d \neq 0$ and without loss of generality $\eta = x_0^n \eta_k$, where $\eta_0 = \emptyset$ and $\eta_{k+1} = x_1 x_0^{n_k}\eta_k$, $k \geq 0$. From the definition, observe that $(x_0^{n_k}\eta_k) \bullet d = x_0^{n_k}(\eta_k \bullet d)$. Thus, the claim is immediate when $k = 0$. For $k > 0$ it is useful to first show that

$$\eta_k \bullet d = \sum_{j=0}^{k-1} (\eta_k)\eta_{j+1}^{-1} x_0 (x_0^{n_j}\eta_j \sqcup\!\sqcup\, d).$$

The $k = 1$ case is straight forward. Suppose the equality holds up to some fixed $k \geq 1$. Then,

$$\begin{aligned}\eta_{k+1} \bullet d &= (x_1 x_0^{n_k}\eta_k) \bullet d \\ &= x_1 x_0^{n_k}(\eta_k \bullet d) + x_0(x_0^{n_k}\eta_k \sqcup\!\sqcup\, d) \\ &= x_1 x_0^{n_k} \sum_{j=0}^{k-1} (\eta_k)\eta_{j+1}^{-1} x_0(x_0^{n_j}\eta_j \sqcup\!\sqcup\, d) \\ &\quad + x_0(x_0^{n_k}\eta_k \sqcup\!\sqcup\, d) \\ &= \sum_{j=0}^{k} (\eta_{k+1})\eta_{j+1}^{-1} x_0(x_0^{n_j}\eta_j \sqcup\!\sqcup\, d),\end{aligned}$$

using the fact that $x_1 x_0^{n_k}(\eta_k)\eta_{j+1}^{-1} = (\eta_{k+1})\eta_{j+1}^{-1}$. Hence, the identity in question holds for all $k \geq 0$. Now to prove the lemma, note that $|\eta_k| = \left|(\eta_k)\eta_{j+1}^{-1} x_0 x_0^{n_j}\eta_j\right|$, $j = 0, 1, \ldots, k-1$. Therefore, $\mathrm{ord}(\eta_k \bullet d) = \mathrm{ord}(\eta_k) + \mathrm{ord}(d)$. Finally, $\mathrm{ord}(\eta \bullet d) = \mathrm{ord}(x_0^n(\eta_k \bullet d)) = \mathrm{ord}(\eta) + \mathrm{ord}(d)$ when $k \geq 1$. ■

Theorem 2: The pre-Lie product in Definition 1 is well defined on $\mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle$ by $c \bullet d = \sum_{\eta\in X^*}(c,\eta)\eta \bullet d$.

Proof: If suffices to show that the family of series $\{\eta \bullet d\}_{\eta\in X^*}$ is locally finite for any $d \in \mathbb{R}\langle\langle X\rangle\rangle$, and hence summable [1]. For a fixed $\xi \in X^*$ and d, define the set of words $I_d(\xi) = \{\eta \in X^* : (\eta \bullet d, \xi) \neq 0\}$. Clearly,

$$I_d(\xi) \subseteq \{\eta \in X^* : \mathrm{ord}(\eta \bullet d) \leq |\xi|\},$$

and using Lemma 1 the latter set has at most cardinality $2^{|\xi| - \mathrm{ord}(d)}$ when $|\xi| \geq \mathrm{ord}(d)$ and zero otherwise. Thus, the family of series in question is locally finite, and the theorem is proved. ■

Example 1: It follows from the definition that $x_0^i x_1 \bullet x_0^j x_1 = x_0^{i+j+1}x_1$ for all $i, j \geq 0$. Therefore, the generating series for two LTI systems $c = \sum_{k\geq 0}(c, x_0^k x_1)x_0^k x_1$ and $d = \sum_{k\geq 0}(d, x_0^k x_1)x_0^k x_1$ satisfies $c \bullet d = c \circ d$, where in this special case the composition product on $\mathbb{R}\langle\langle X\rangle\rangle$ reduces to series convolution, that is, the only possible nonzero coefficients of $c \circ d$ are

$$(c \circ d, x_0^i x_1) = \sum_{j=0}^{i-1} (c, x_0^j x_1)(d, x_0^{i-j-1}x_1),\ i \geq 1.$$

□

Example 2: In general, a word is said to be *drift-free* if it is nonempty and contains no x_0 letters. A straightforward induction using the identity

$$x_0x_1^i \sqcup\!\sqcup\, x_1^j = \sum_{k=0}^{j} \binom{i+j-k}{i} x_1^k x_0 x_1^{i+j-k}$$

gives the pre-Lie product of two drift-free words in the SISO case to be

$$x_1^i \bullet x_1^j = (x_0x_1^j) \sqcup\!\sqcup\, x_1^{i-1}$$

for $i, j \geq 1$. The resulting polynomial has no drift-free words in its support. □

The next lemma will be useful in subsequent sections. It exploits the fact that every $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$.

Lemma 2: For all $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$, $(c \bullet d_F)_N = 0$.

Proof: Consider an arbitrary word written without loss of generality in the form

$$\xi_k = x_0^{n_k}x_1 \cdots x_0^{n_1}x_1x_0^{n_0}, \quad k \geq 0.$$

Trivially, $(\xi_0 \bullet d_F)_N = (x_0^{n_0}(\emptyset \bullet d_F))_N = 0$. Now assume the lemma holds up to some fixed $k \geq 0$. Then

$$\begin{aligned}(\xi_{k+1} \bullet d_F, x_0^i) &= (x_0^{n_{k+1}}x_1(\xi_k \bullet d_F), x_0^i) \\ &\quad + (x_0^{n_{k+1}+1}(\xi_k \sqcup\!\sqcup\, d_F), x_0^i) \\ &= (x_0^{n_{k+1}+1}(\xi_k \sqcup\!\sqcup\, d_F), x_0^i)\end{aligned}$$

using the induction hypothesis. Clearly, this last term is always zero when $i < n_{k+1}+1$. Otherwise, $(\xi_{k+1}\bullet d_F, x_0^i) = (\xi_k \sqcup\!\sqcup\, d_F, x_0^{i-n_{k+1}-1})$. But since every word in the support of d_F contains at least one x_1 letter, these coefficients must also be zero. Thus, $(\xi_{k+1}\bullet d_F)_N = \sum_{i\geq 0}(\xi_{k+1}\bullet d_F, x_0^i) = 0$. Therefore, by induction and the left linearity of the pre-Lie product, $(c \bullet d_F)_N = 0$. ■

IV. The Lie Group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$

To understand how the pre-Lie product in Definition 1 can be used in the context of feedback systems, it is best to step back and first view the problem in the context of Lie groups. Consider the group $\mathscr{F}_\delta$ as an infinite dimensional Lie group. Here

$$F_{c_\delta}[u](t) = u(t) + \sum_{\eta\in X^*}(c,\eta)E_\eta[u](t),$$

where u and t are arbitrary but fixed, and thus, they will be suppressed in the notation. In which case, $F : \mathbb{R}\langle\langle X_\delta\rangle\rangle \to \mathbb{R} : c_\delta \mapsto F_{c_\delta}$. The identity element of the group is $F_{0_\delta} = I$. The first goal is to describe the left-invariant vector field on $\mathscr{F}_\delta$, which for a Lie group uniquely identifies the Lie bracket [19]. The left translation of F_{d_δ} by F_{c_δ} is given by

$$F_{c_\delta} \circ F_{d_\delta} = I + F_d + F_{c\,\tilde{\circ}\,d} = F_{d_\delta} + F_{c\,\tilde{\circ}\,d}.$$

Since the modified composition product is left linear, there is no loss of generality in setting $c = \xi \in X^*$. The differential of $F_{\xi_\delta}\circ : \mathscr{F}_\delta \to \mathscr{F}_\delta$ at I is the linear map $(F_{\xi_\delta}\circ)_* : T_I\mathscr{F}_\delta \to T_{F_{\xi_\delta}}\mathscr{F}_\delta$. Consider for some $\epsilon > 0$ a differentiable path $\gamma : (-\epsilon,\epsilon) \to \mathscr{F}_\delta : t \mapsto F_{d(t)}$ such that $d(0) = 0$. Define the velocity vector at $t = 0$ as F_v, where

$$v = \dot{d}(0) = \sum_{\nu\in X^*}(\dot{d}(0),\nu)\nu.$$

Then specifically the differential of $F_{\xi_\delta}\circ$ at I in the direction of F_v is

$$\begin{aligned}(F_{\xi_\delta}\circ)_*(F_v) &= \frac{d}{dt}F_{d(t)} + F_{\xi\,\tilde{\circ}\,d(t)}\bigg|_{t=0} \\ &= F_v + \sum_{\eta\in X^*}\left(\frac{d}{dt}\xi\,\tilde{\circ}\,d(t)\bigg|_{t=0},\eta\right)E_\eta.\end{aligned}$$

The time derivative in the last expression above is zero when $\xi = \emptyset$, otherwise observe that

$$\begin{aligned}&\frac{d}{dt}(x_i\xi)\,\tilde{\circ}\,d(t)\bigg|_{t=0} \\ &= \frac{d}{dt}\left[x_i\phi_{d(t)}(\xi)(1) + x_0(\phi_{d(t)}(\xi)(1) \sqcup\!\sqcup\, d_i(t))\right]\bigg|_{t=0} \\ &= x_i\frac{d}{dt}\phi_{d(t)}(\xi)(1)\bigg|_{t=0} + x_0(\xi \sqcup\!\sqcup\, \dot{d}_i(0))\end{aligned}$$

using the fact that $d_i(0) = 0$, $i = 0, 1$. Since $\dot{d}_0(0) = 0$ and $\dot{d}_1(0) = v$, it follows in general that

$$\frac{d}{dt}\xi\,\tilde{\circ}\,d(t)\bigg|_{t=0} = \xi \bullet v.$$

In which case,

$$(F_{\xi_\delta}\circ)_*(F_v) = F_{v+\xi\bullet v}. \tag{4}$$

Given the group isomorphism between $\mathscr{F}_\delta$ and $\mathbb{R}\langle\langle X_\delta\rangle\rangle$, the entire setup can be mapped to $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. In this context, the left-invariant vector field on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ is

$$\chi^v : \mathbb{R}\langle\langle X_\delta\rangle\rangle \to T\mathbb{R}\langle\langle X_\delta\rangle\rangle : c_\delta \mapsto v + c \bullet v.$$

The corresponding Lie bracket is then

$$\begin{aligned}[v_1, v_2] &= [\chi^{v_1}, \chi^{v_2}]|_I \\ &= \partial\chi^{v_1}(v_2 + c\bullet v_2) - \partial\chi^{v_2}(v_1 + d\bullet v_1)|_{c=d=0} \\ &= v_2 \bullet v_1 - v_1 \bullet v_2,\end{aligned}$$

where $\partial\chi^v : e \mapsto e \bullet v$. This Lie bracket is clearly induced by a *right* pre-Lie product.

Theorem 3: The Lie algebra of the Lie group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ is the smallest $\mathbb{R}$-vector subspace of $\mathbb{R}\langle\langle X\rangle\rangle$ closed under the bracket $[v_1, v_2] = v_2 \bullet v_1 - v_1 \bullet v_2$.

Example 3: In light of Example 1 and the commutativity of the composition product in the LTI case,

$$[x_0^ix_1, x_0^jx_1] = x_0^jx_1 \circ x_0^ix_1 - x_0^ix_1 \circ x_0^jx_1 = 0.$$

□

Example 4: Continuing Example 2, the Lie bracket of two drift-free words is

$$[x_1^i, x_1^j] = (x_0x_1^i) \sqcup\!\sqcup\, x_1^{j-1} - (x_0x_1^j) \sqcup\!\sqcup\, x_1^{i-1}$$

$$= (x_0x_1^i) \sqcup\!\sqcup\, x_1^{-1}(x_1^j) - (x_0x_1^j) \sqcup\!\sqcup\, x_1^{-1}(x_1^i).$$

It is interesting to note that the linear operators $\tilde{e}_i := (x_0x_1^i) \sqcup\!\sqcup\, x_1^{-1}$, $i \geq 0$, or equivalently, $F_{\tilde{e}_i} = F_{x_0x_1^i}\frac{d}{dt}$ (when $u = 1$), are reminiscent of the basis elements of the Witt Lie algebra $\mathscr{W}_+(\mathbb{R})$ for the group of formal diffeomorphisms of the form $f(t) = t + \sum_{n\geq 2} f_n t^n$ under function composition via the classical Faà di Bruno formula. Namely, the set $\{e_i := t^{i+1}\frac{d}{dt},\ i \geq -1\}$, where $[e_i, e_j] = (j-i)e_{i+j}$. [11]. □

In light of (4), the pre-Lie product can be viewed as a linearization of the modified composition product about zero in its right argument (recall $c\,\tilde{\circ}\,0 = c$).

Theorem 4: For any $c, e \in \mathbb{R}\langle\langle X\rangle\rangle$ there always exist the decomposition

$$c\,\tilde{\circ}\,e = c + c \bullet e + \mathcal{O}(e^2),$$

where $\mathcal{O}(e^2)$ denotes all the nonlinear terms in e, and its linear dependence on c is suppressed in this notation.

Example 5: Consider an LTI system with generating series $c = \sum_{k\geq r} h_k x_0^{k-1}x_1$, where $h_r \neq 0$. So r is the relative degree of the system, and

$$c\,\tilde{\circ}\,e = c + \sum_{k\geq r} x_0^k e.$$

As discussed in the introduction, the group action in this case is linear in e. □

Example 6: Suppose $c = x_1^2$. Then

$$x_1^2\,\tilde{\circ}\,e = \underbrace{x_1^2}_{c} + \underbrace{x_1x_0e + x_0(e \sqcup\!\sqcup\, x_1)}_{c\bullet e} + \underbrace{x_0(e \sqcup\!\sqcup\, (x_0e))}_{\mathcal{O}(e^2)}.$$

Setting $e = x_1 - x_1x_0$ gives

$$\begin{aligned} c \bullet e &= 2x_0x_1^2 + x_1x_0x_1 - \underline{x_0x_1x_0x_1} - 2x_0x_1^2x_0 \\ &\quad - x_1x_0x_1x_0 \\ \mathcal{O}(e^2) &= 2x_0^2x_1^2 + \underline{x_0x_1x_0x_1} - 2x_0^2x_1x_0x_1 - 4x_0^2x_1^2x_0 \\ &\quad - 2x_0x_1x_0^2x_1 - 2x_0x_1x_0x_1x_0 + 2x_0^2x_1x_0x_1x_0 \\ &\quad + 4x_0^2x_1^2x_0^2 + 2x_0x_1x_0^2x_1x_0 + 2x_0x_1x_0x_1x_0^2. \end{aligned}$$

Observe that the linear and nonlinear terms in e share a common word in their supports. So it is possible that $c \bullet e$ might introduce a term that is also shared with an invariant part of c, and then this same term is removed by $\mathcal{O}(e^2)$ resulting in no effect on the invariant series, which by definition must be the case. This suggests that studying the linear term in isolation might reveal nothing about the feedback invariants. But the next section will demonstrate this is *not* the case for one particular class of invariants. □

V. Feedback Group Invariants

It was shown in [13] that $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ acts as a right transformation group on the semigroup $(\mathbb{R}\langle\langle X\rangle\rangle, \circ)$ where

$$\varphi : \mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X_\delta\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : (c, e_\delta) \mapsto c \circ e_\delta = c\,\tilde{\circ}\,e.$$

In which case, feedback product $c@d = c\,\tilde{\circ}\,(-d\circ c)^{\circ -1}$ can be interpreted as the right action of the group element $(-d\circ c)^{\circ-1}$ acting on the generating series c for the system in the forward path. The following theorem describes an invariant subseries of this action.

Theorem 5: [13] Every nonzero series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ which is not equivalent to c_N can be decomposed into the form $c = c_i + \tilde{c}$, where $\mathrm{supp}(c_i) \cap \mathrm{supp}(\tilde{c}) = \emptyset$,

$$c_i = c_{N_0} + x_0^{r_1-1}x_1c_{N_1} + x_0^{r_1-1}x_1x_0^{r_2-1}x_1c_{N_2} + \cdots,$$

$c_{N_\ell} \in \mathbb{R}[X_0]$ (a polynomial in x_0), $r_\ell \geq 1$, $\deg(c_{N_\ell}) \leq r_{\ell+1} - 1$, and c_i is a nonzero invariant series under the transformation group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$. That is, $c\,\tilde{\circ}\,e = c_i + c_v(e)$ with $\mathrm{supp}(c_i) \cap \mathrm{supp}(c_v(e)) = \emptyset$ for all $e \in \mathbb{R}\langle\langle X\rangle\rangle$.

The proof of the theorem is constructive. It boils down to identifying pairs $(c_{N_\ell}, r_{\ell+1})$, $\ell \geq 0$ to extract the invariant c_i from c via the following algorithm:

Step 1: Set $\ell = 0$.

Step 2: Write c in the canonical form

$$c = c_N + x_0^{r-1}c_1 + x_0^r c_2 + \cdots, \tag{5}$$

where $r \geq 1$, c_k are proper series with $x_0^{-1}(c_k) = 0$ for all $k \geq 1$, and $c_1 \neq 0$.

Step 3: Define $c_{N_\ell} = \sum_{k=0}^{r-1}(c_N, x_0^k)x_0^k$ and $r_{\ell+1} = r$.

Step 4: Redefine $c = x_1^{-1}(c_1)$ and set $\ell = \ell + 1$.

Step 5: If $|c|_{x_1} = 0$ set $c_{N_\ell} = c$, $c_{N_k} = 0$, $k > \ell$ and stop. Otherwise, return to Step 2.

The algorithm will only terminate when c is *input-limited*, that is, when $\max_{\eta\in\mathrm{supp}(c)} |\eta|_{x_1}$ is finite. There is no claim that this series is maximal in following sense.

Definition 2: An invariant series c_i^* of $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is called *maximal* if its support contains the support of any other series which is also invariant under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$.

If, however, c has the following property then much more analysis is possible.

Definition 3: A series c has *extended relative degree* r if

$$c = c_N + Kx_0^{r-1}x_1 + x_0^r c'$$

for some $K \neq 0$ and $c' \in \mathbb{R}\langle\langle X\rangle\rangle$.

In particular, it was shown in [13] that c has the maximal invariant polynomial $c_i^* = c_i = c_{N_1} + Kx_0^{r-1}x_1$ with $\deg(c_{N_1}) \leq r-1$, and the group acts freely on the subset of series having extended relative degree. It should be noted that this property is stronger than the classical notion of relative degree defined in either a state space setting [21] or, equivalently, in an input-output setting [15]. It is possessed, for example, by all LTI systems. For the general case, however, no characterization of c_i^* is known at present. The following examples illustrate all the possibilities.

Example 7: The algorithm above applied to the series $c = x_1 + x_1^2$ has the following iterations:

$\ell = 0$: $c = x_1 + x_1^2 \to c_N = 0$, $r = 1$, $c_1 = x_1 + x_1^2 \to c_{N_0} = 0$, $r_1 = 1$,

$\ell = 1$: $c = 1 + x_1 \to c_N = 1$, $r = 1$, $c_1 = x_1 \to c_{N_1} = 1$, $r_2 = 1$

$\ell = 2$: $c = 1 \to c_{N_2} = 1$, $c_{N_k} = 0$, $k > 2$.

Therefore, $c_i = c_{N_0} + x_0^{r_1-1} x_1 c_{N_1} + x_0^{r_1-1} x_1 x_0^{r_2-1} x_1 c_{N_2} = c$. In which case, $c_i^* = c_i = c$ even though c does not have extended relative degree. □

Example 8: Now consider the algorithm applied to the series $c = x_1 x_0 + x_1^2$:

$\ell = 0$: $c = x_1x_0 + x_1^2 \to c_N = 0$, $r = 1$, $c_1 = x_1x_0 + x_1^2 \to c_{N_0} = 0$, $r_1 = 1$

$\ell = 1$: $c = x_0 + x_1 \to c_N = x_0$, $r = 1$, $c_1 = x_1 \to c_{N_1} = 0$, $r_2 = 1$

$\ell = 2$: $c = 1 \to c_{N_2} = 1$, $c_{N_k} = 0$, $k > 2$.

Therefore, $c_i = x_1^2$ since here $N_1 = 0$. To see that $c_i^* = c_i \neq c$, observe that for any $\beta \in \mathbb{R}$

$$c \,\tilde{\circ}\, \beta = x_1x_0 + x_1^2 + \beta(x_0^2 + x_0x_1 + x_1x_0) + \beta^2 x_0^2.$$

In particular, the term x_1x_0 can be removed by setting $\beta = -1$, but no value of β can remove the term x_1^2. Similar experiments with higher order e show that this second term can never be removed as expected. □

Example 9: Next consider the algorithm applied to the series $c = x_1x_0 + x_1^2x_0$:

$\ell = 0$: $c = x_1x_0 + x_1^2x_0 \to c_N = 0$, $r = 1$, $c_1 = x_1x_0 + x_1^2x_0 \to c_{N_0} = 0$, $r_1 = 1$

$\ell = 1$: $c = x_0 + x_1x_0 \to c_N = x_0$, $r = 1$, $c_1 = x_1x_0 \to c_{N_1} = 0$, $r_2 = 1$

$\ell = 2$: $c = x_0 \to c_{N_2} = x_0$, $c_{N_k} = 0$, $k > 2$.

Therefore, $c_i = x_1^2x_0$. But computing $c \,\tilde{\circ}\, e$ explicitly for $e = \beta, \beta x_0, \beta x_1, \ldots$, it quickly becomes apparent that $c_i^* = c \neq c_i$. That is, c_i is not maximal. □

It will be shown next that the pre-Lie product of Foissy gives some additional insight into the nature invariant series c_i. The following subset is useful for a fixed $c \in \mathbb{R}\langle\langle X\rangle\rangle$:

$$R_c = \{\eta \in X^* : \eta \in \mathrm{supp}(c \bullet \nu),\ \nu \in X^*\}.$$

Clearly, R_c is the union of the supports of all the series in the range of $e \mapsto c \bullet e$.

Theorem 6: The invariant series c_i of any $c \in \mathbb{R}\langle\langle X\rangle\rangle$ under the right transformation group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ as described in Theorem 5 satisfies the property $\mathrm{supp}(c_i) \cap R_c = \emptyset$.

Proof: In the first iteration of the algorithm, c is written in the form (5) with $r = r_1$. For any $e \in \mathbb{R}\langle\langle X\rangle\rangle$ it then follows that

$$\begin{aligned} c \bullet e &= x_0^{r_1-1}(c_1 \bullet e) + x_0^{r_1}(c_2 \bullet e) + \cdots \\ &= x_0^{r_1-1}x_1(x_1^{-1}(c_1) \bullet e) + x_0^{r_1}(x_1^{-1}(c_1) \sqcup\!\sqcup\, e) \\ &\quad + x_0^{r_1}x_1(x_1^{-1}(c_2) \bullet e) + x_0^{r_1+1}(x_1^{-1}(c_2) \sqcup\!\sqcup\, e) + \cdots . \end{aligned}$$

Only the first term on the right-hand side above has the potential to share common support with c_i, specifically with the support of $x_0^{r_1-1}x_1c_{N_1}$. This can happen only if $\mathrm{supp}((x_1^{-1}(c_1) \bullet e)_N) \cap \mathrm{supp}(c_{N_1}) \neq \emptyset$. Noting that $(x_1^{-1}(c_1) \bullet e)_N = (x_1^{-1}(c_1) \bullet e_N)_N + (x_1^{-1}(c_1) \bullet e_F)_N$ and applying Lemma 2, this condition reduces to $\mathrm{supp}((x_1^{-1}(c_1) \bullet e_N)_N) \cap \mathrm{supp}(c_{N_1}) \neq \emptyset$. Now by design $\deg(c_{N_1}) \leq r_2 - 1$ and $\mathrm{ord}(x_1^{-1}(c_1)) \geq r_2$. Applying Lemma 1, $\mathrm{ord}((x_1^{-1}(c_1) \bullet e_N)_N) \geq \mathrm{ord}(x_1^{-1}(c_1) \bullet e_N) \geq \mathrm{ord}(x_1^{-1}(c_1)) + \mathrm{ord}(e_N) \geq r_2 + \mathrm{ord}(e_N)$. Thus, the two sets in question must be disjoint, and this proves the theorem. ■

In the context of the previous theorem, the following corollaries are immediate.

Corollary 1: The series c_i is invariant under the linearized version of the action $\varphi : (c, e_\delta) \mapsto c \,\tilde{\circ}\, e$.

Corollary 2: The scalar product $(c_i, c \bullet e) := \sum_{\eta \in X^*} (c_i, \eta)(c \bullet e, \eta) = 0$ for every $e \in \mathbb{R}\langle\langle X\rangle\rangle$.

Corollary 3: In general, $c = c_i$ only if $\mathrm{supp}(c) \cap R_c = \emptyset$.

Example 10: Suppose c has extended relative degree r. Then

$$c \bullet e = (c_N + Kx_0^{r-1}x_1 + x_0^r c') \bullet e = Kx_0^r(e + c' \bullet e).$$

Therefore, $\mathrm{supp}(c_i) \cap R_c = \emptyset$ since for all $e \in \mathbb{R}\langle\langle X\rangle\rangle$

$$\mathrm{supp}(c_{N_1} + Kx_0^{r-1}x_1) \cap \mathrm{supp}(Kx_0^r(e + c' \bullet e)) = \emptyset.$$

As discussed earlier, it is known theoretically that c_i is maximal in this case, but in general $c \neq c_i$. □

Example 11: Reconsider the series in Example 7. Observe

$$c \bullet e = x_0 e + x_1 x_0 e + x_0(x_1 \sqcup\!\sqcup\, e).$$

Since $\mathrm{supp}(c) \cap R_c = \emptyset$, it is possible that $c = c_i$, as was shown earlier to be the case. □

Example 12: For the series in Example 8,

$$c \bullet e = x_0(x_0 \sqcup\!\sqcup\, e) + x_1x_0e + x_0(x_1 \sqcup\!\sqcup\, e).$$

Here $\mathrm{supp}(c) \cap R_c = \{x_0x_1\}$, so the term x_0x_1 can not be part of c_i as was concluded earlier by other means. □

Example 13: For the series in Example 9,

$$c \bullet e = x_0(x_0 \sqcup\!\sqcup\, e) + x_1x_0(x_0 \sqcup\!\sqcup\, e) + x_0(x_1x_0 \sqcup\!\sqcup\, e).$$

Here also $\mathrm{supp}(c) \cap R_c = \emptyset$, but as noted earlier $c \neq c_i$, instead $c = c_i^*$. □

VI. Output Feedback Invariants

One application of the theory presented in the previous sections is describing invariant series under dynamic output feedback. This would correspond to restricting the group transformation to exactly the form given by the feedback product, namely, $c@d = c \,\tilde{\circ}\, (-d \circ c)^{\circ -1}$ with c fixed and d arbitrary. As discussed in the introduction, the feedback product can be linearized with respect to the group action and also with respect to the generating series d in the feedback loop. In the previous section, the former was done. Here the latter topic is pursued. The following lemma describes how to linearize the composition inverse.

Lemma 3: For any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, $e^{\circ -1} = -e + \mathcal{O}(e^2)$.

Proof: By definition $e^{\circ -1}$ must satisfy $(\delta + e) \circ (\delta + e^{\circ -1}) = \delta$. In which case,

$$e^{\circ -1} = -(e \,\tilde{\circ}\, e^{\circ -1}) = -e - e \bullet e^{\circ -1} + \mathcal{O}((e^{\circ -1})^2).$$

The claim is now proved by repeated substitution of the left-hand side of the equation into the right-hand side wherever the series $e^{\circ-1}$ appears. ■

A direct application of this lemma gives the following result.

Theorem 7: For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that

$$c@d = c\,\tilde{\circ}\,(-d\circ c)^{\circ-1} = c + c \bullet (d\circ c) + \mathcal{O}((d\circ c)^2).$$

Note that the second term in the expansion above is linear in d because the composition product on $\mathbb{R}\langle\langle X\rangle\rangle$ is left-linear in general. Now since output feedback is viewed as a restriction of the feedback action to a certain class of elements in the group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$, the maximal output feedback invariant series, say c_i^o, is usually distinct from c_i^*, where the full feedback group is allowed to act on c. Applying the previous theorem, the orbit of c_i^o under this restricted action is described as follows.

Theorem 8: The maximal invariant series under output feedback, c_i^o, satisfies

$$c_i^o@d = c_i^o + c_i^o \bullet (d\circ c_i^o) + \mathcal{O}((d\circ c_i^o)^2).$$

This last identity is a nonlinear generalization of (2).

Example 14: Reconsider the LTI system in Example 5 in the context of output feedback. In light of Example 1,

$$\begin{aligned} c_i^o@(d\circ c_i^o) &= c_i^o + c_i^o \circ (d\circ c_i^o) + c_i^o \circ (d\circ c_i^o)^2 + \cdots \\ &= c_i^o \circ \sum_{k=0}^{\infty} (d\circ c_i^o)^k, \end{aligned}$$

where $(d\circ c_i^o)^0 := \delta$. This is a generating series version of (2). □

VII. Conclusions

The Lie group, Lie algebra and pre-Lie algebra associated with a SISO feedback transformation group for a nonlinear input-output system represented as a Fliess operator were described. Then the linear part of the transformation was identified with the pre-Lie product and used to give a description of a class of invariant series of an arbitrary generating series representing the system in the forward path. Finally, dynamic output feedback, as described by the feedback product was considered. An algebraic description of the orbits of a maximally invariant series under output feedback was given.

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