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Feedback Transformation Group for Nonlinear Input-Output Systems

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***Abstract*— The goal of this paper is to describe a feedback transformation group for the class of nonlinear input-output systems that can be represented in terms of Chen-Fliess functional expansions. There is no a priori requirement that these input-output systems have a state space realization, so the results presented here are independent of any particular state space coordinate system or state space embedding when a realization is available. Using an explicit formula for the generating series of the closed-loop system, an invariant series of this transformation group is described. The order of this invariant series corresponds to the relative degree of the system (defined purely from an input-output point of view) when it is well defined. But in general Brunovsky forms are not on all the orbits of this transformation group unless the system has an *extended* notion of relative degree.**

I. Introduction

Let G be a group and S a given set. G is said to act as a *transformation group* on the right of S if there exists a mapping $\phi : S \times G \rightarrow S : (h, g) \mapsto hg$ such that:

i. $h1 = h$, 1 is the identity element of G;
ii. $h(g_1 g_2) = (hg_1)g_2$ for all $g_1, g_2 \in G$.

The action is said to be *free* if $hg = h$ implies that $g = 1$. Transformation groups have been used extensively in system theory since its inception. The early work of Brockett, Krener and others in the case of linear systems [2], [3] and nonlinear state space systems [1], [19] has been very important in understanding the role of invariance under feedback and coordinate transformations. This has led to a proliferation of normal forms that are quite useful in design and analysis problems [18], [20]. As a specific example, consider the field of (irreducible) rational functions in s with real coefficients denoted by $\mathbb{R}(s)$. Let $G = \mathbb{R}_{p\bar{0}}(s)$ be the subfield of proper elements g of $\mathbb{R}(s)$ with the defining property that $g(+\infty)$ exists and is not zero. Let $S = \mathbb{R}_{p0}(s)$ denote the ring of strictly proper elements h of $\mathbb{R}(s)$. Therefore, $h(+\infty) = 0$. Observe that $g \in \mathbb{R}_{p\bar{0}}(s)$ if and only if $g = K + h$ for some $K \neq 0$ and $h \in \mathbb{R}_{p0}(s)$. In this situation, $\mathbb{R}_{p\bar{0}}(s)$ acts freely on $\mathbb{R}_{p0}(s)$ from the right where the product hg is defined in the usual fashion when $h, g \in \mathbb{R}(s)$. Such a transformation group naturally appears in the context of linear time-invariant (LTI) system theory when feedback is applied. For example, when a single-input, single-output (SISO) plant is modeled by a transfer function $h \in \mathbb{R}_{p0}(s)$ and some feedback $h' \in \mathbb{R}_{p0}(s)$ is applied, the closed-loop system has the transfer function hg where $g = (1+h'h)^{-1}$. In a state space setting, if h has the realization (A, b, c) then $h = b/a$ where $b(s) = c\,\mathrm{adj}(sI - A)b$ and $a(s) = \det(sI - A)$. If static state feedback with gain k is applied so that the closed-loop characteristic polynomial is $a_k(s) = \det(sI - A + bk)$, it is well known from the derivation of the Bass-Gura formula that the closed-loop transfer function has the form $b/a_k = (b/a)(1 + h)$, where $h(s) = -k(sI - A + kb)^{-1}b \in \mathbb{R}_{p0}(s)$ [17]. In which case, both dynamic output feedback and static state feedback can be viewed as special cases of the group $\mathbb{R}_{p\bar{0}}(s)$ acting on $\mathbb{R}_{p0}(s)$. This group action can also represent other types of system transformations such as arbitrary zero placement if one notes that $(b/a)(b'/b) = b'/a$ with $b'/b \in \mathbb{R}_{p\bar{0}}(s)$. It is easy to see then that the Brunovsky forms $1/s^r$ are invariant under this transformation group and induce an equivalence relation with the canonical projection $\pi : \mathbb{R}_{p0}(s) \rightarrow \mathbb{R}_{p0}(s)/\mathbb{R}_{p\bar{0}}(s) : h \mapsto r$, where $r \geq 1$ denotes the relative degree of h.

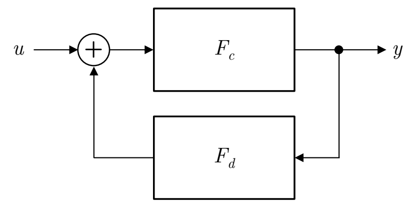

Fig. 1. Feedback connection of Fliess operators

The objective of this paper is to describe a feedback transformation group for the class of nonlinear input-output systems that can be represented in terms of Chen-Fliess functional expansions, so called *Fliess operators* [7], [8]. Such functional series are indexed by words over a noncommutative alphabet. Therefore, their generating series are specified in terms of noncommutative formal power series. There is no a priori requirement that these input-output systems have a state space realization, so the results presented here are independent of any particular state space coordinate system or state space embedding when a realization is available. It was shown in [13], [15] that the feedback interconnection of two Fliess operators F_c and F_d as shown in Fig. 1 always defines another Fliess operator, $F_{c@d}$. An explicit formula for the generating series of the closed-loop system, $c@d$, was derived in [10] via Hopf algebra techniques. Using these results, it will be shown here that this feedback interconnection naturally induces a noncommutative transformation group which leaves a certain subseries of c invariant. The order, r, of this invariant series corresponds to the notion of relative degree (defined purely from an input-output point of view) when it is well defined. But in general r by itself does *not* produce the finest equivalence relation on $\mathbb{R}\langle\langle X \rangle\rangle$ except in certain special cases. One such case is the set of formal power series having an *extended* notion of relative degree as will be discussed. These classes coincide with Brunovsky forms. It is also shown that when the input-output system

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has a state space realization, the transformation group action does *not* include all possible static state feedback laws as in the LTI case. This hints at the possibility of a larger feedback transformation group for the nonlinear setting, but the topic will not be pursued here.

The paper is organized as follows. In the next section, the notation and a short summary of the analysis tools are presented. A more comprehensive introduction can be found in [10], [13], [23]. The main results describing the feedback transformation group and its invariants are presented in Section III. The SISO case is assumed for brevity. In the subsequent section, the relationship between the group invariants and the notion of relative degree and related concepts is addressed. The final section summarizes the paper's conclusions.

II. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. If $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. The *order* of a series is the length of the shortest word in its support. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by $\sqcup\!\sqcup$ [7].

A. Fliess Operators

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0)$$

[7], [8]. Assuming certain growth conditions on the coefficients of c, F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0,\, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [14]. Otherwise, F_c can only be interpreted in a formal sense [15]. Finally, $F_c : u \mapsto y$ is realizable by a state space realization

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z) u_i, \quad z(0) = z_0 \tag{1a}$$

$$y = h(z), \tag{1b}$$

if and only if the Lie rank of c is finite [8], [16]. In this case, $(c, \eta) = L_{g_\eta} h(z_0)$, where

$$L_{g_\eta} h := L_{g_{i_1}} \cdots L_{g_{i_k}} h, \quad \eta = x_{i_k} \cdots x_{i_1},$$

and $L_{g_i} h$ is the Lie derivative of h with respect to g_i.

B. Feedback Connection of Fliess Operators

When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d$$

[5], [6]. Here

$$\eta \circ d = D_\eta(1) = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1),$$

where $\eta = x_{i_k} x_{i_{k-1}} \cdots x_{i_1}$, and the family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup e)$$

indexed by the letters of X generates an $\mathbb{R}$-algebra under composition. (Define $d_0 = 1$, and let $D_\emptyset$ be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$.) The composition product is associative and $\mathbb{R}$-linear in its left argument c. It is linear in its right argument if and only if its left argument is a *linear series*, that is, $\mathrm{supp}(c) \subseteq L$, where

$$L := \{\eta \in X^* : \eta = x_0^{n_1} x_i x_0^{n_0},\ i \in \{1, 2, \ldots, m\},\ n_j \geq 0\}$$

is the set of *linear words*.

In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 1, the closed-loop system has a Fliess operator representation whose generating series is the *feedback product* of c and d, denoted by $c@d$ [13], [15]. Consider, for example, the SISO case where $X = \{x_0, x_1\}$. Define the set of operators

$$\mathscr{F}_\delta = \{I + F_c \ :\ c \in \mathbb{R}\langle\langle X\rangle\rangle\},$$

where I denotes the identity map. It is convenient to introduce the Dirac symbol δ and the definition $F_\delta = I$ such that $I + F_c = F_{\delta + c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. $\mathscr{F}_\delta$ forms a group under composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c\,\tilde{\circ}\, d$, and $\tilde{\circ}$ denotes the *modified* composition product. That is, the product

$$c\,\tilde{\circ}\, d = \sum_{\eta \in X^*} (c, \eta)\, \eta\,\tilde{\circ}\, d,$$

where $\eta\,\tilde{\circ}\,d = \tilde{D}_\eta(1)$ and for each letter $x_i \in X$

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup\, e) \qquad (2)$$

with $d_0 := 0$. It is of central importance that the corresponding group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ has a dual that forms a Faà di Bruno Hopf algebra with antipode, α, satisfying

$$c_\delta^{\circ -1} = \delta + c^{\circ -1} = \delta + \sum_{\eta \in X^*} (\alpha\, a_\eta)(c)\,\eta,$$

where $c^{\circ -1}$ denotes the composition inverse of c,

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta)$$

is the coordinate function for $\eta \in X^*$, and $a_\delta(c_\delta) := 1$ [10]. The antipode can be computed directly by either a series expansion or via a matrix representation of the group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ [11]. An alternative description of this Hopf algebra in terms of pre-Lie algebras is given in [9]. In either setting, the following theorem holds.

Theorem 1: [10] For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that

$$c@d = c\,\tilde{\circ}\,(-d \circ c)^{\circ -1} = c \circ (\delta - d \circ c)^{\circ -1}.$$

This result suggests that $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ acts as a transformation group on the right of $\mathbb{R}\langle\langle X\rangle\rangle$ in a manner analogous to the way that $\mathbb{R}_{p1}(s)$ (the subfield of $\mathbb{R}_{p\bar{0}}(s)$ with elements of the form $1 + h$, $h \in \mathbb{R}_{p0}(s)$) acts on $\mathbb{R}_{p0}(s)$. This topic is addressed in the next section.

III. Transformation Group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$

Define the right action of the transformation group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ on the semigroup $(\mathbb{R}\langle\langle X\rangle\rangle, \circ)$ as

$$\phi : \mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X_\delta\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : (c, e_\delta) \mapsto c \circ e_\delta = c\,\tilde{\circ}\,e.$$

It is well known that the modified composition product is not associative, in particular, $(c\,\tilde{\circ}\,d)\,\tilde{\circ}\,e = c\,\tilde{\circ}\,(e + d\,\tilde{\circ}\,e)$ [21]. In which case,

$$\phi(\phi(c, e_\delta), f_\delta) = (c\,\tilde{\circ}\,e)\,\tilde{\circ}\,f = c\,\tilde{\circ}\,(f + e\,\tilde{\circ}\,f) = \phi(c, (e_\delta \circ f_\delta))$$

as required by definition.

A central result of the paper is an explicit description of a set of invariant series under this transformation group. It follows directly from (2) that if $c = c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ then the entire series is invariant under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. The following theorem addresses the more general case.

Theorem 2: Every nonzero series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ which is not equivalent to c_N can be decomposed into the form $c = c_i + \tilde{c}$, where $\mathrm{supp}(c_i) \cap \mathrm{supp}(\tilde{c}) = \emptyset$,

$$c_i = c_{N_0} + x_0^{r_1 - 1}x_1 c_{N_1} + x_0^{r_1 - 1}x_1 x_0^{r_2 - 1}x_1 c_{N_2} + \cdots,$$

$c_{N_i} \in \mathbb{R}[X_0]$ (a polynomial in x_0), $r_k \geq 1$, and c_i is a nonzero invariant series under the transformation group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$. That is, $c\,\tilde{\circ}\,e = c_i + c_v(e)$ with $\mathrm{supp}(c_i) \cap \mathrm{supp}(c_v(e)) = \emptyset$ for all $e \in \mathbb{R}\langle\langle X\rangle\rangle$.

Proof: First observe that any series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ not equal to c_N can be written canonically in the form

$$c = c_N + x_0^{r-1}c_1 + x_0^r c_2 + \cdots, \qquad (3)$$

where $r \geq 1$, c_k are proper series with $x_0^{-1}(c_k) = 0$ for all $k \geq 1$, and $c_1 \neq 0$. (The left-shift operator $x_i^{-1}(\cdot)$ is the $\mathbb{R}$-linear operator uniquely specified by $x_i^{-1}(\eta) = \eta'$ when $\eta = x_i\eta'$ for $\eta' \in X^*$ and zero otherwise.) Therefore, for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that

$$c\,\tilde{\circ}\,e = c_N + x_0^{r-1}(c_1\,\tilde{\circ}\,e) + x_0^r(c_2\,\tilde{\circ}\,e) + \cdots.$$

Now consider for any fixed $k \geq 1$:

$$\begin{aligned} c_k\,\tilde{\circ}\,e &= \sum_{\eta \in X^*}(c_k, x_1\eta)\tilde{D}_{x_1\eta}(1) \\ &= \sum_{\eta \in X^*}(x_1^{-1}(c_k), \eta)\tilde{D}_{x_1}\tilde{D}_\eta(1) \\ &= \sum_{\eta \in X^*}(x_1^{-1}(c_k), \eta)[x_1\tilde{D}_\eta(1) + x_0(e \sqcup\!\sqcup\, \tilde{D}_\eta(1)] \\ &= x_1(\tilde{c}_k\,\tilde{\circ}\,e) + x_0(e \sqcup\!\sqcup\, (\tilde{c}_k\,\tilde{\circ}\,e)), \end{aligned}$$

where $\tilde{c}_k := x_1^{-1}(c_k)$. Thus,

$$\begin{aligned} c\,\tilde{\circ}\,e =& c_N + x_0^{r-1}x_1(\tilde{c}_1\,\tilde{\circ}\,e) + x_0^r(e \sqcup\!\sqcup\, (\tilde{c}_1\,\tilde{\circ}\,e)) + \\ & x_0^r x_1(\tilde{c}_2\,\tilde{\circ}\,e) + x_0^{r+1}(e \sqcup\!\sqcup\, (\tilde{c}_2\,\tilde{\circ}\,e)) + \cdots. \end{aligned} \qquad (4)$$

Clearly, c_N is independent of e, and the support of the polynomial $c_{N_0} := \sum_{k=0}^{r-1}(c, x_0^k)x_0^k$ is always disjoint from the support of all the other terms in this series. The second term in (4) does depend on e, but it too has a support that is always disjoint from the support of all other terms. On the other hand, terms three and four, for example, may have some inter-term cancelation depending on the specific e selected. The key step now is to extract that part of the second term which is independent of e. First rename r as r_1 and collect all the terms in (4) beyond the second term, as well as series $c_N - c_{N_0}$, into $c_v(e)$. The series $\tilde{c}_1$ is completely arbitrary. Once it is put into form (3), the process above can be repeated for $\tilde{c}_1 \circ e$ to identify a new pair (c_{N_1}, r_2). The additional higher order terms from this iteration are again added to $c_v(e)$. Thus, it follows that

$$\begin{aligned} c\,\tilde{\circ}\,e =& c_{N_0} + x_0^{r_1 - 1}x_1(c_{N_1} + x_0^{r_2 - 1}x_1(x_1^{-1}(\tilde{\tilde{c}}_1) \circ e) + \cdots) \\ & + c_v(e), \end{aligned}$$

where $\tilde{\tilde{c}}_1 = x_1^{-1}((\tilde{c}_1)_1)$. If this process is repeated indefinitely, then the desired c_i is constructed. It is nonzero since c_1 is nonzero, and therefore, $x_0^{r_1 - 1}x_1 c_{N_1}$ is nonzero. By design, c_i is independent of e and disjoint from $c_v(e)$ for all $e \in \mathbb{R}\langle\langle X\rangle\rangle$. In particular, when $e = 0$ then $c = c_i + \tilde{c}$, where $\tilde{c} := c_v(0)$. ■

It should be stressed that c_i can usually not be identified by an inspection of c, but rather via the systematic construction process outlined in the proof. Furthermore, no claim is being made that c_i is the *largest* invariant series under this transformation group, so henceforth this symbol will be reserved for this specific invariant series. In essence, this analysis has simply extracted some lower order invariant pieces of c. Thus, the equivalence relation induced by mapping all series with the same c_i to an equivalence class will not necessarily produce the finest partition of $\mathbb{R}\langle\langle X\rangle\rangle$. Nonetheless, this invariant series still provides significant insight into what is possible and not possible for a variety of feedback systems. In the next section, a necessary and sufficient condition on c is given under which a particular c_i is maximal.

Example 1: Consider an LTI system with transfer function $h(s) = \sum_{k\geq r} h_k s^{-k}$, where $h_r \neq 0$ and $r \geq 1$. The corresponding generating series is the linear series $c = \sum_{k\geq r} h_k x_0^{k-1} x_1$. In which case, $r_1 = r$, $c_{N_1} = h_r$, and $c_{N_i} = 0$ for $i > 1$. Thus, $c_i = x_0^{r-1} x_1 h_r$, which corresponds to a Brunovsky form (modulo the constant h_r) with relative degree r. Observe that $c \,\tilde{\circ}\, e = c_i + \sum_{k\geq r} h_{k+1} x_0^k x_1 + h_k x_0^k e$. □

Example 2: Consider the linear series $c = x_1 + x_1 x_0$. This does not correspond to an LTI system. Observe that $r_1 = 1$, $c_{N_1} = 1 + x_0$, and $c_{N_i} = 0$ for $i > 1$. Therefore, $c_i = c$. A direct calculation gives $c \,\tilde{\circ}\, e = c_i + x_0(e + (e \sqcup\!\sqcup x_0))$. □

Example 3: Consider the linear series $c = x_1 + x_0 x_1 x_0$, which also does not correspond to an LTI system. But similar to the first example, $r_1 = 1$, $c_{N_1} = 1$, and $c_{N_i} = 0$ for $i > 1$. So $c_i = x_1$ and $c \,\tilde{\circ}\, e = c_i + x_0(x_1 x_0 + e + x_0(e \sqcup\!\sqcup x_0))$. □

Example 4: The rational series $c = \sum_{k\geq 0} x_1^k$ plays an important role in showing that the composition product is not a rational operation [5], [6]. With assistance from the Mathematica package NCFPS (Noncommutative Formal Power Series) and NCAlgebra [22], it follows that $c_i = c$. Take as an example the transformation

$$\begin{aligned} c \,\tilde{\circ}\, x_1 = 1 + x_1 + x_0 x_1 + x_1^2 + 2x_0 x_1^2 + x_1 x_0 x_1 + x_1^3 + \\ 2x_0^2 x_1^2 + (x_0 x_1)^2 + 3x_0 x_1^3 + 2x_1 x_0 x_1^2 + x_1^2 x_0 x_1 + \\ 6x_0^2 x_1^3 + 4(x_0 x_1)^2 x_1 + 2x_0 x_1^2 x_0 x_1 + \cdots . \end{aligned}$$

□

To show that the transformation group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ acts freely on $\mathbb{R}\langle\langle X\rangle\rangle$, it is necessary to show that $c \,\tilde{\circ}\, e = c$ for a given c implies that $e = 0$, or equivalently, that the mapping $c \mapsto c \,\tilde{\circ}\, e$ has a fixed point only if $e = 0$. This is in sharp contrast to the situation where $c \mapsto e \,\tilde{\circ}\, c$ is known to have a fixed point for every e [13], [21]. It is obvious for the case where $c = c_N$ that $c \,\tilde{\circ}\, e = c$ for every e. So the action is not free in general. When $c \neq c_N$ it is not difficult to show from the definition of the modified composition product that

$$c \,\tilde{\circ}\, e = c + \sum_{k=1}^{\infty} \sum_{\eta\in X^k} (c,\eta) M_\eta(e_1,\ldots,e_k)|_{e_i=e},$$

where M_η denotes a $\mathbb{R}$-linear k-form. Thus, determining freeness is equivalent to determining whether a linear combination of k-forms is singular. This is quite a difficult problem in general, but it will be shown in the next section that if c has a certain kind of relative degree then the problem is tractable.

IV. Series with Relative Degree

The notion of relative degree is normally associated with state space realizations of the form (1) [16]. It is a central object of study in the theory of feedback linearization. But it was shown in [12] that the concept can also be defined in a purely input-output setting and corresponds to the usual definition when the operator F_c has a realization with finite dimension.

Definition 1: [12] Given $c \in \mathbb{R}\langle\langle X\rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1} X^*$, where $c_F := c - c_N$. Then c has *relative degree* r if the linear word $x_0^{r-1} x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

It is well known that relative degree is invariant under static nonlinear state feedback. The following theorem is of a similar nature but strictly from an input-output point of view.

Theorem 3: A series c has relative degree r if and only if its invariant series c_i has relative degree r. In which case, relative degree is invariant under the transformation group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$.

Proof: Observe that c having relative degree r is equivalent to saying that

$$c = c_N + c_F = c_N + K x_0^{r-1} x_1 + x_0^{r-1} e \tag{5}$$

for some $K \neq 0$ and $x_1 \notin \mathrm{supp}(e)$. Without lost of generality let $e = x_0 e_0 + x_1 e_1$ with e_1 proper. Then

$$c = c_N + x_0^{r-1} x_1 (K + e_1) + x_0^r e_0,$$

and this implies that $c_1 = x_1(K + e_1)$ as defined in (3). In the proof of Theorem 2 then $\tilde{c}_1 = K + e_1$, so that by the usual construction process

$$c_i = c_{N_0} + x_0^{r-1} x_1 K + x_0^{r-1} x_1 x_0^{r_2-1} x_1 c_{N_2} + \cdots .$$

Clearly then c_i has relative degree r.

Conversely, if c_i has relative degree r then

$$c_i = c_{N_0} + x_0^{r-1} x_1 (K + c'_{N_1}) + x_0^{r-1} x_1 e$$

with $K \neq 0$, $c'_{N_1} \in \mathbb{R}[[X_0]]$ proper and

$$e = x_0^{r_2-1} x_1 c_{N_2} + x_0^{r_2-1} x_1 x_0^{r_3-1} x_1 c_{N_3} + \cdots .$$

Since $c = c_i + \tilde{c}$ and the supports of c_i and $\tilde{c}$ are disjoint, it follows that $x_0^{r-1} x_1 \in \mathrm{supp}(c)$. In light of (3), which is the starting point for the construction of c_i, it is immediate that $\mathrm{supp}(c_F) \subseteq x_0^{r-1} X^*$. Thus, c has relative degree r. ■

For a state space system, having a well defined relative degree is a sufficient condition for the existence of a static state feedback law to put the closed-loop input-output system into Brunovsky form. But in Example 2 it is clear that c_i does not correspond to such a form even though F_c is realizable and c has relative degree $r = 1$. Therefore, the only possible conclusion is that unlike the LTI case, the transformation group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ does *not* completely characterize all static state feedback laws. This counterexample is analyzed in more detail below.

Example 5: The series c in Example 2 has Lie rank two. Therefore, F_c has a minimal state space realization of dimension two. For example, letting $z_1 = y = F_c[u]$, it follows that F_c is realized by

$$\dot{z}_1 = (z_2 + 1)u, \quad z_1(0) = 0 \tag{6a}$$
$$\dot{z}_2 = 1, \quad z_2(0) = 0 \tag{6b}$$
$$y = z_1. \tag{6c}$$

The feedback linearizing control law $u = v/(z_2+1)$ gives the closed-loop input-output equation $\dot{y} = v$. A corresponding group element e_δ would necessarily have to satisfy $u = (I + F_e)[v]$, or equivalently, $(I + F_{e^{\circ -1}})[u] = v$. In light of Fig. 2

$$w = u - v = F_{-e^{\circ -1}}[u] =: F_{\tilde{e}}[u].$$

The operator $w = F_{\tilde{e}}[u]$ can be realized by an open-loop state observer followed by a static state feedback law k. Namely, (g_0, g_1, k, z_0) as given in (6a)-(6b) with $w = k(z,u) = -z_2 u$. If an additional state $z_3 = -z_2 u$ is defined then

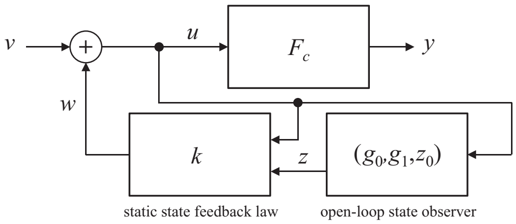

Fig. 2. Feedback law via an open-loop state observer for a system with relative degree

$w = z_3$ and $\dot{z}_3 = -z_2\dot{u}$. The dependence of k on u, or equivalently, the dependence of $\dot{z}_3$ on $\dot{u}$ implies that no suitable $\tilde{e}$ exists with finite Lie rank since otherwise $F_{\tilde{e}}$ would have a realization of the form (1). For the LTI case, this can always be accomplished by simply setting $w = -kz$ for some suitable constant gain vector k (see, for example, Fig. 4.5-1 in [17]). □

Next, a stronger notion of relative degree is given which ensures that a Brunovsky form is always on any corresponding orbit of the transformation group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$.

Definition 2: A series c has *extended relative degree* r if it has relative degree r and

$$c_F = Kx_0^{r-1}x_1 + x_0^r e \tag{7}$$

for some $e \in \mathbb{R}\langle\langle X\rangle\rangle$.

The difference between (5) and (7) appears to be minor, but the consequences are significant. Note, for example, that all LTI systems have a generating series with a well defined extended relative degree. Hence, this concept is most significant in the nonlinear setting. In light of the identity

$$\dot{y} = F_{x_0^{-1}(c)}[u] + uF_{x_1^{-1}(c)}[u]$$

and (5), it follows for a system with relative degree r that

$$y^{(r)} = F_{(x_0^r)^{-1}(c_N)+x_0^{-1}(e)}[u] + (K + F_{x_1^{-1}(e)}[u])u.$$

Since $K+F_{x_1^{-1}(e)}[u](0) = K \neq 0$, the r-th output derivative is the first output derivative that explicitly depends on u, which is the classical interpretation of relative degree. For a system with extended relative degree, this expression further reduces to

$$y^{(r)} = F_{(x_0^r)^{-1}(c_N)+e}[u] + Ku.$$

In a state space setting, this would imply that the realization has a globally defined feedback linearization law. Thus, it has *strong* relative degree in the sense described in [4]. (Strong relative degree, however, does not imply extended relative degree.) In the event that $K = 1$, the feedback linearizing law is $u = v + F_d[v]$, where $d = ((x_0^r)^{-1}(c_N) + e)^{\circ -1}$. The presence of the group inverse in this calculation indicates that a Brunovsky form (modulo the constant K) is on the orbit of the transformation group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ containing c. This fact is established in the following theorem.

Theorem 4: A series c has extended relative degree r if and only if it is on an orbit of $c_N + Kx_0^{r-1}x_1$ under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. In which case, $c_i = c_{N_0} + Kx_0^{r-1}x_1$ as defined in Theorem 2.

Proof: From (7) it follows directly that if c has well defined extended relative degree r then

$$c = c_N + Kx_0^{r-1}x_1 + x_0^r e = (c_N + Kx_0^{r-1}x_1)\,\tilde{\circ}\, e_K,$$

where $e_K = e/K$. Thus,

$$c\,\tilde{\circ}\, e_K^{\circ -1} = c_N + Kx_0^{r-1}x_1. \tag{8}$$

Therefore, c is on an orbit of $c_N + Kx_0^{r-1}x_1$ under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. The converse holds since all the steps above are reversible. Finally, $c = c_i + c_v$, where $c_i = c_{N_0} + Kx_0^{r-1}x_1$ and $c_v = c_N - c_{N_0}$ by virtue of the fact that $c_1 = 1$ in (3). ■

The following corollary is immediate.

Corollary 1: A series c has extended relative degree r if and only if its invariant series c_i has extended relative degree r. In which case, extended relative degree is invariant under the transformation group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$.

The maximality of invariant series in the present context is addressed next.

Definition 3: An invariant series c' of $c \in \mathbb{R}\langle\langle X\rangle\rangle$ will be called *maximal* if its support contains the support of any other series which is also invariant under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$.

It is possible in the current setup to have two invariant series that differ by only a scalar multiple. This distinction could be easy eliminated by slightly expanding the transformation group to include elements like $K\delta + F_c$, where $K \neq 0$. But this minor issue will not be pursued here.

Theorem 5: Suppose c has relative degree r. Then $c_{N_0} + Kx_0^{r-1}x_1$ is its maximal invariant polynomial under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ if and only if c has extended relative degree.

Proof: Suppose c has extended relative degree r but $c_{N_0} + Kx_0^{r-1}x_1$ is not maximal. Then there exists another c' which is also invariant under the group action but contains more terms in its support. However, since c has extended relative degree it follows that (8) holds. That is, the group action has been applied to c, but this c' is clearly not present on the right-hand side of (8) as it must be if it is invariant. Hence, no such c' exists.

The converse claim is addressed next assuming without loss of generality that $c_N = 0$. Suppose that c has relative degree r, but not extended relative degree. In addition, assume that $Kx_0^{r-1}x_1$ is a maximal invariant series. Then it follows that

$$c = Kx_0^{r-1}x_1 + x_0^{r-1}x_1e_1 + x_0^r e_0,$$

where e_1 is proper, so as not to alter the term $Kx_0^{r-1}x_1$, and nonzero, otherwise c has extended relative degree. The series e_0 is completely arbitrary. The main idea is to show that $e_1 \neq 0$ will always render $Kx_0^{r-1}x_1$ non-maximal, and thus produces the needed contradiction. Observe that the middle term $x_0^{r-1}x_1e_1$ above clearly has a support which is disjoint from the other two terms. While it does not have relative degree as a series by itself, it does by Theorem 2 have a nonzero invariant polynomial under $\mathbb{R}\langle\langle X_\delta\rangle\rangle$, say c_1', provided $e_1 \neq 0$. Hence, the polynomial $Kx_0^{r-1}x_1 + c_1'$ defines a larger invariant polynomial if $e_1 \neq 0$, which completes the proof. ■

Example 6: The series $c = x_1 + x_1x_0$ in Examples 2 and 5 has relative degree $r = 1$ but no extended relative degree. Therefore, consistent with Theorems 4 and 5, no transformation from the group $\mathbb{R}\langle\langle X_\delta\rangle\rangle$ will put the closed-loop system into the maximal Brunovsky form x_1. In fact, as pointed out in Example 2, $c_i = c$. □

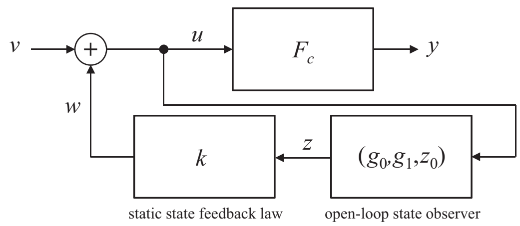

Fig. 3. Feedback law via an open-loop state observer for a system with extended relative degree

Example 7: In Example 3 the series $c = x_1 + x_0 x_1 x_0$ has extended relative degree $r = 1$. It follows directly from the method of proof for Theorem 4 that the feedback law $u = (I + F_e)[v]$ with $e = (x_1 x_0)^{\circ -1}$ will render a closed-loop system with input-output equation $\dot{y} = v$. In which case, the corresponding feedback law as shown in Fig. 3 is $w = F_{\tilde{e}}[u]$ with $\tilde{e} = -e^{\circ -1} = -x_1 x_0$. But since c also has finite Lie rank, a construction of the feedback law based on state space realizations is also possible. Specifically, setting $z_1 = y = F_c[u]$ and $z_2 = F_{x_1 x_0}[u]$, it is readily verified that F_c is realized by

$$g_0(z) = \begin{bmatrix} z_2 \\ 0 \\ 1 \end{bmatrix}, \quad g_1(z) = \begin{bmatrix} 1 \\ z_3 \\ 0 \end{bmatrix}, \quad z_0 = 0, \quad h(z) = z_1.$$

Therefore, $F_{\tilde{e}}$ is realized by $(g_0, g_1, k, 0)$ with $k(z) = -z_2$ since $w = u - v$ and $\dot{y} = \dot{z}_1 = z_2 + u = v$. This can be confirmed independently by showing that $L_{g_\eta} k(z_0) = -1$ when $\eta = x_1 x_0$ and $L_{g_\eta} k(z_0) = 0$ otherwise. Thus, $\tilde{e} = -x_1 x_0$. □

Another consequence of extended relative degree is given below.

Theorem 6: The transformation group $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ acts freely on the subset of $\mathbb{R}\langle\langle X \rangle\rangle$ having extended relative degree.

Proof: Assume c has extended relative degree r. Without loss of generality let $c_N = 0$. Then there exists an e_c such that $c \circ (\delta + e_c)^{-1} = x_0^{r-1} x_1$. So if $c \circ (\delta + e) = c$ for some e, then it follows immediately that

$$(c \circ (\delta + e)) \circ (\delta + e_c)^{-1} = c \circ (\delta + e_c)^{-1}$$
$$(c \circ (\delta + e_c)^{-1}) \circ (\delta + e^{e_c}) = c \circ (\delta + e_c)^{-1},$$

where e^{e_c} corresponds to the conjugate action $\delta + e^{e_c} = (\delta + e_c) \circ (\delta + e) \circ (\delta + e_c)^{-1}$. In which case,

$$x_0^{r-1} x_1 \circ (\delta + e^{e_c}) = x_0^{r-1} x_1$$
$$x_0^{r-1} x_1 + x_0^r e^{e_c} = x_0^{r-1} x_1,$$

and therefore, $e^{e_c} = 0$, or equivalently, $(\delta + e_c) \circ (\delta + e) \circ (\delta + e_c)^{-1} = \delta$. The only possible conclusion is that $e = 0$. ■

V. CONCLUSIONS

A feedback transformation group for the class of nonlinear input-output systems that can be represented in terms of Chen-Fliess functional expansions was presented. In particular, a class of invariant series for this transformation group was described explicitly and related to the notion of relative degree, when it is well defined. In general, Brunovsky forms are not on all the orbits of this transformation group unless the series describing the plant has extended relative degree. In which case, the group action is free, and the invariant series, in this case invariant polynomials, are maximal.

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