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Preservation of Commutation Relations and Physical Realizability of Open Two-Level Quantum Systems*

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***Abstract*— Coherent feedback control considers purely quantum controllers in order to overcome disadvantages such as the acquisition of suitable quantum information, quantum error correction, etc. These approaches lack a systematic characterization of quantum realizability. Recently, a condition characterizing when a system described as a linear stochastic differential equation is quantum was developed. Such condition was named physical realizability, and it was developed for linear quantum systems satisfying the quantum harmonic oscillator canonical commutation relations. In this context, open two-level quantum systems escape the realm of the current known condition. When compared to linear quantum system, the challenges in obtaining such condition for such systems radicate in that the evolution equation is now a bilinear quantum stochastic differential equation and that the commutation relations for such systems are dependent on the system variables. The goal of this paper is to provide a necessary and sufficient condition for the preservation of the Pauli commutation relations, as well as to make explicit the relationship between this condition and physical realizability.**

I. Introduction

In the last twenty years, the use of quantum feedback control systems have became critical for the development of quantum and nano technologies [1], [4], [5]. However, the majority of approaches consider a classical controller in the feedback loop. In this context, coherent feedback control considers purely quantum controllers in order to overcome disadvantages such as the acquisition of suitable quantum information, quantum error correction, etc [3], [10], [15]. Unfortunately, these approaches lack a systematic characterization of quantum realizability. In [9], a condition characterizing when a system described as a linear stochastic differential equation is quantum was developed. Such condition was named *physical realizability*, and it was developed specifically for linear systems satisfying the quantum harmonic oscillator canonical commutation relations. The class of systems for which this condition is known to be satisfied is still too limited for applications. In this paper, the focus is on systems describing the dynamics of open two-level quantum systems. Compared to a linear quantum system, the problem is more complicated and requires extra machinery for two basic reasons. The first is that the system being analyzed is bilinear, and the second is that the commutation relations that the system has to obey are now dependent on the system variables, which was not the case for linear quantum systems related to the quantum harmonic oscillator [9], [12]. In [6], a characterization of the physical realizability for open two-level quantum systems was provided. However, it is not clear whether or not such condition imply the preservation in time of the commutation relations for the system variables of the bilinear quantum stochastic differential equation (QSDE) describing the system. Thus, the main contribution of this paper, given in Section V, is to provide a necessary and sufficient condition for the preservation of Pauli commutation relations, as well as to make explicit the relationship between this condition and physical realizability. Furthermore, in Section IV, the physical realizability condition of open two-level quantum systems is reformulated in terms of the quadrature of the interacting Boson field, which yields a more natural self-adjoint (all component matrices are real) representation of the system and the physical realizability condition.

The paper is organized as follows. Section II presents the basic preliminaries on open quantum systems. In Section III, the necessary algebraic machinery to study open two-level quantum systems is given. This is followed by Section IV, in which the definition of physical realizability is provided as well as a condition for a bilinear QSDE to be physically realizable. In Section V, it is shown that a physically realizable system preserves the commutation relations established for spin operators. Finally, Section VI gives the conclusions. The proofs of all results are included in the full version of the paper; see [7].

II. Open Two-Level Quantum Systems

Systems governed by the laws of quantum mechanics that interact with an external environment (e.g., electromagnetic field) are known as *open quantum systems*. In order to study such systems, one has to give a quantum description of both the system and the interacting environment. The quantum

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mechanical behavior of the system is based on the notions of *observables* and *states*. Observables represent physical quantities that can be measured, as self-adjoint operators on a complex separable Hilbert space $\mathfrak{H}$, while states give the current status of the system, as elements of $\mathfrak{H}$, allowing the computation of expected values of observables. The set of operators in $\mathfrak{H}$ is denoted by $\mathfrak{T}(\mathfrak{H})$, and the set of $n \times m$ dimensional arrays of operators in $\mathfrak{T}(\mathfrak{H})$ is denoted by $\mathfrak{T}(\mathfrak{H})^{n\times m}$. Here open quantum systems are treated in the context of quantum stochastic processes (see [2], [14] for more information). For this purpose, observables may be thought as quantum random variables that do *not* in general commute. A measure of the non commutativity between observables is usually given by the *commutator* between operators. The commutator of x and y in $\mathfrak{T}(\mathfrak{H})$ is an antisymmetric bilinear operation defined as

$$[x, y] = xy - yx.$$

Also, for $x \in \mathfrak{T}(\mathfrak{H})^n$ and $y \in \mathfrak{T}(\mathfrak{H})^m$, the commutator of x and y is

$$[x, y^T] \triangleq xy^T - (yx^T)^T \in \mathfrak{T}(\mathfrak{H})^{n\times m}.$$

In particular, the commutator of x and its adjoint $x^\dagger$ is the $n \times n$ matrix of operators

$$[x, x^\dagger] \triangleq xx^\dagger - (x^{\#}x^T)^T \in \mathfrak{T}(\mathfrak{H})^{n\times n},$$

where

$$x^{\#} \triangleq \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$$

and * denotes the operator adjoint. In the case of complex vectors (matrices) * denotes the complex conjugate while † denotes the conjugate transpose. The non-commutativity of observables is a fundamental difference between quantum systems and classical systems in which the former must satisfy certain commutation relations originating from *Heisenberg uncertainty principle*. The environment consists of a collection of oscillator systems each with annihilation field operator $w(t)$ and creation field operator $w^*(t)$ used for the annihilation and creation of quanta at point t, and commonly known as the *boson quantum field* (with parameter t). Here it is assumed that t is a real time parameter. The field operators $w(t)$ and $w^*(t)$ satisfy commutation relations as well. That is,

$$[w(t), w^*(t')] = \delta(t - t'),$$

for all $t, t' \in \mathbb{R}$, where $\delta(t)$ denotes the Dirac delta. Its mathematical description is given in terms of a Hilbert space called a *Fock space*. When the boson quantum field is in the vacuum state, i.e., no physical particles are present, it then represents a natural quantum extension of white noise, and may be described using the quantum Itô calculus [2], [14]. This amounts to have three interacting signals (inputs) in the evolution of the system: the annihilation processes $W(t)$, the creation process $W^\dagger(t)$, and the counting process $\Lambda(t)$. The evolution of an open quantum system (i.e., the system together with the environment) is unitary. That is, if ψ is a state then $\psi(t) = U(t)\psi$, where $U(t)$ is unitary for all t, and is the solution of

$$dU(t) = \Big((S - \hat{I})\, d\Lambda(t) + L\, dW^\dagger(t) - L^\dagger S\, dW(t) - \frac{1}{2}(L^\dagger L + \boldsymbol{i}\mathcal{H})\, dt \Big) U(t),$$

with initial condition $U(0) = \hat{I}$, $\hat{I}$ denoting the identity operator in $\mathfrak{T}(\mathfrak{H})$ and $\boldsymbol{i}$ being the imaginary unit. Here, $\mathcal{H}$ is a fixed self-adjoint operator representing the *Hamiltonian* of the system, and L and S are operators determining the *coupling* of the system to the field, with S unitary. The evolution of ψ is equivalent to the evolution of the observable X given by

$$X(t) = U^\dagger(t)(X \otimes \hat{I})\, U(t),$$

whose evolution is referred as the *Heisenberg picture* while the one for ψ is known as the *Schrödinger picture*. This paper exclusively takes the point of view of the Heisenberg picture. Quantum stochastic calculus allows then to express the Heisenberg picture evolution of X as

$$dX = (S^\dagger X S - X)\, d\Lambda + \mathcal{L}(X)\, dt + S^\dagger[X, L]\, dW^\dagger + [L^\dagger, X]S\, dW, \tag{1}$$

where $\mathcal{L}(X)$ is the Lindblad operator defined as

$$\mathcal{L}(X) = -\boldsymbol{i}[X, \mathcal{H}] + \frac{1}{2}\left(L^\dagger[X, L] + [L^\dagger, X]L\right). \tag{2}$$

The output field is given by

$$Y(t) = U(t)^\dagger W(t) U(t),$$

which amount to

$$dY = Ldt + SdW.$$

In summary, one can say from the discussion above that the dynamics of an open quantum systems is uniquely determined by the triple of operators $(S, L, \mathcal{H})$. Hereafter, the operator S is assumed to be the identity operator ($S = \hat{I}$).

The main focus of this paper is on the dynamics of open two-level quantum systems interacting with one boson quantum field. The Hilbert space for this system is $\mathfrak{H} = \mathbb{C}^2$, the two dimensional complex vector space. The vector of system variables for a two level system is then

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathfrak{T}(\mathfrak{H})^3,$$

Given that these operators are self-adjoint, the vector of system variables x satisfies $x = x^{\#}$. In particular, a self-adjoint operator $\hat{\sigma}$ in $\mathfrak{T}(\mathfrak{H})$ is spanned by the Pauli matrices [13], i.e.,

$$\hat{\sigma} = \frac{1}{2}\sum_{i=0}^{3} \alpha_i \sigma_i,$$

where $\alpha_0 = \mathrm{Tr}(\hat{\sigma})$, $\alpha_i = \mathrm{Tr}(\hat{\sigma}\sigma_i)$, and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -\boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli matrices. Thus, $\alpha_0, \alpha_1, \alpha_2$ and α_3 determine uniquely the operator $\hat{\sigma}$. The initial value of the system variables can be set to $x(0) = (\sigma_1, \sigma_2, \sigma_3)$. The product of Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij} + \boldsymbol{i} \sum_k \epsilon_{ijk} \sigma_k$$

for $i, j, k \in \{1, 2, 3\}$. It is then clear that the commutation relations for Pauli matrices are

$$[\sigma_i, \sigma_j] = 2\boldsymbol{i} \sum_k \epsilon_{ijk} \sigma_k, \tag{3}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} denotes the Levi-Civita tensor defined as

$$\epsilon_{ijk} = \begin{cases} +1, \ \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ -1, \ \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ \ \ 0, \ \text{otherwise.} \end{cases}$$

Due to the fact that the Pauli matrices form a complete orthogonal set, any Hamiltonian and coupling operators of polynomial type are representable as linear functions of x. Therefore, assuming linearity captures a large class of Hamiltonian and coupling operators without much loss of generality, i.e.,

$$\mathcal{H} = \alpha x \ \text{ and } \ L = \Gamma x,$$

where $\alpha^T \in \mathbb{R}^3$ and $\Gamma^T \in \mathbb{C}^3$. As mentioned before, the coupling operator specifies how the interacting field acts on x. In general, the dimensionality of the coupling matrix Γ depends proportionally on the number of interacting fields.

It is customary to express QSDEs in terms of its interaction with quadrature fields. The quadrature fields are given by the transformation

$$\begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\boldsymbol{i} & \boldsymbol{i} \end{pmatrix} \begin{pmatrix} W \\ W^\dagger \end{pmatrix}, \tag{4}$$

where the operators $\bar{W}_1$ and $\bar{W}_2$ are now self-adjoint. In [8], the Itô table for W and $W^\dagger$ is

$$\begin{pmatrix} dW \\ dW^\dagger \end{pmatrix} \begin{pmatrix} dW & dW^\dagger \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dt,$$

which in terms of the quadrature fields is

$$\begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix} \begin{pmatrix} d\bar{W}_1 & d\bar{W}_2 \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{i} \\ -\boldsymbol{i} & 1 \end{pmatrix} dt.$$

Observe that, in general, the evolution of x (*standard form*) falls into a class of bilinear QSDEs expressed as

$$dx = F_0\, dt + Fx\, dt + G_1 x\, d\bar{W}_1 + G_2 x\, d\bar{W}_2, \tag{5}$$

where $F_0 \in \mathbb{R}^3$ and $F, G_1, G_2 \in \mathbb{R}^{3\times3}$. The fact that all matrices in (5) are real is due to the quadrature transformation (4). The output field is

$$dY = Hx\, dt + \frac{1}{2}\left(d\bar{W}_1 + \boldsymbol{i} d\bar{W}_2\right)$$

with $H^T \in \mathbb{C}^3$. Similarly, the quadrature form of the output fields can be obtained from the transformation

$$\begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\boldsymbol{i} & \boldsymbol{i} \end{pmatrix} \begin{pmatrix} Y \\ Y^\dagger \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} x\, dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \tag{6}$$

where

$$H_1 = H + H^\# \ \text{ and } \ H_2 = \boldsymbol{i}(H^\# - H)$$

are obviously real matrices.

In this context, the goal of the paper can now be stated more specifically. Given a bilinear QSDE as in (5), under what condition there exist $\mathcal{H}$ and L such that (5) can be written as in (1). Such condition is given in Section IV.

III. Notation and Algebraic Relations

In order to continue the description of open two-level quantum systems, some linear algebra identities are needed. Let $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)^T$ be a vector of operators in $\mathfrak{T}(\mathfrak{H})$, and define the linear mapping $\Theta : \mathfrak{T}(\mathfrak{H})^3 \to \mathfrak{T}(\mathfrak{H})^{3\times3}$ as

$$\Theta(\hat{\sigma}) = \begin{pmatrix} 0 & \hat{\sigma}_3 & -\hat{\sigma}_2 \\ -\hat{\sigma}_3 & 0 & \hat{\sigma}_1 \\ \hat{\sigma}_2 & -\hat{\sigma}_1 & 0 \end{pmatrix}.$$

With any vector $\beta = (\beta_1, \beta_2, \beta_3)^T \in \mathbb{C}^3$ one can associate the vector of operators $\hat{\beta} = (\beta_1 \hat{I}, \beta_2 \hat{I}, \beta_3 \hat{I})^T \in \mathfrak{T}(\mathfrak{H})^3$. Abusing the notation slightly, for a vector $\beta \in \mathbb{C}^3$ the mapping $\Theta(\beta)$ simply means that

$$\Theta(\beta) = \begin{pmatrix} 0 & \beta_3 \hat{I} & -\beta_2 \hat{I} \\ -\beta_3 \hat{I} & 0 & \beta_1 \hat{I} \\ \beta_2 \hat{I} & -\beta_1 \hat{I} & 0 \end{pmatrix},$$

In a similar manner, any complex matrix is associated to a matrix of operators by considering the identity operator $\hat{I}$ in each component. For convenience the identity operator $\hat{I}$ is usually suppressed. The definition of Θ can further be understood for row vectors in the sense that $\Theta(\beta) = \Theta(\beta^T)$. The fact that β is either a column or a row vector will be clear from the context. It will also be convenient to rewrite $\Theta(\beta)$ in terms of its columns. That is,

$$\Theta(\beta) = (\Theta_1(\beta), \Theta_2(\beta), \Theta_3(\beta)).$$

The product of Pauli operators can be expressed in a compact matrix form thanks to the mapping Θ. That is,

$$xx^T = I + \boldsymbol{i}\Theta(x),$$

where I denotes the identity matrix. Similarly, the commutation relations for Pauli operators are written as

$$[x, x^T] = 2\boldsymbol{i}\Theta(x).$$

Consider now the *stacking operator* $\mathrm{vec} : \mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ whose action on a matrix creates a column vector by stacking its columns below one another. With the help of vec, the matrix $\Theta(\beta)$ can be reorganized so that it gives

$$\mathrm{vec}(\Theta(\beta)) = \begin{pmatrix} \Theta_1(\beta) \\ \Theta_2(\beta) \\ \Theta_3(\beta) \end{pmatrix} = E\beta,$$

where $\beta \in \mathbb{C}^3$, $\Theta_i(\beta) = \bar{e}_i^T\beta$, $E \triangleq (\bar{e}_1, \bar{e}_2, \bar{e}_3)^T$, and

$$\bar{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\bar{e}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The set $\{-\boldsymbol{i}\bar{e}_1, -\boldsymbol{i}\bar{e}_2, -\boldsymbol{i}\bar{e}_3\}$ can be identified to be the adjoint representation of $SU(2)$, which has as generators the Pauli matrices. It is thus that one can rewrite the matrix $\bar{e}_k$ as

$$\bar{e}_i = \begin{pmatrix} \epsilon_{i11} & \epsilon_{i12} & \epsilon_{i13} \\ \epsilon_{i21} & \epsilon_{i22} & \epsilon_{i23} \\ \epsilon_{i31} & \epsilon_{i32} & \epsilon_{i33} \end{pmatrix},$$

where the Levi-Civita tensor is known as the completely antisymmetric *structure constant* of $SU(2)$. Observe also that

$$\bar{e}_k = \epsilon_{ijk}(\mathbb{1}_{ji} - \mathbb{1}_{ij})$$

with $i \neq j \neq k$ and $\mathbb{1}_{ij} \in \mathbb{R}^{3\times 3}$ being an elementary matrix (i.e., matrix consisting of 1 in the (i, j) position and 0 everywhere else). In addition, the matrix E satisfies

$$E^T E = 2I.$$

If one defines the block matrix $\mathbb{1}_E = \{\mathbb{1}_{ji}\}_{i,j=1}^3 \in \mathbb{R}^{9\times 9}$, then E also satisfies

$$EE^T = I - \mathbb{1}_E, \ \text{ and } \ \mathbb{1}_E E = -E.$$

The matrix $\mathbb{1}_E$ can be identified as a tensor permutation matrix, which comes from the fact that the Levi-Civita tensor satisfies the contraction epsilon identity

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

The properties of $\Theta(\beta)$ are summarized in the next lemma.

Lemma 1: Let $\beta, \gamma \in \mathbb{C}^3$ be column vectors. The mapping Θ satisfies

i. $\Theta(\beta)\gamma = -\Theta(\gamma)\beta$,

ii. $\Theta(\beta)\beta = 0$,

iii. $\bar{e}_i\Theta(\beta) = \beta e_i^T - \beta_i I$,

iv. $\Theta(\beta)\Theta(\gamma) = \gamma\beta^T - \beta^T\gamma I$,

v. $\Theta(\Theta(\beta)\gamma) = [\Theta(\beta), \Theta(\gamma)]$,

where e_i is an elementary vector with i indicating the position of the nonzero element.

The explicit computation of the vector fields in (1) is given by the next lemma.

Lemma 2: The component coefficients of equations (1) and (2) are

$$[x, \mathcal{H}] = -2\boldsymbol{i}\Theta(\alpha)x, \quad (7a)$$

$$[x, L] = -2\boldsymbol{i}\Theta(\Gamma)x, \quad (7b)$$

$$[x, L^\dagger] = -2\boldsymbol{i}\Theta(\Gamma^\#)x, \quad (7c)$$

$$L^\dagger[x, L] = -2\boldsymbol{i}\Theta(\Gamma)\Gamma^\dagger - 2(\Gamma\Gamma^\dagger I - \Gamma^\dagger\Gamma)x, \quad (7d)$$

$$[x, L^\dagger]L = 2\boldsymbol{i}\Theta(\Gamma)\Gamma^\dagger + 2(\Gamma\Gamma^\dagger I - \Gamma^T\Gamma^\#)x. \quad (7e)$$

From (7a)-(7e), one can now write equation (5) as the following bilinear QSDE

$$\begin{aligned} dx = & -2\boldsymbol{i}\Theta(\Gamma)\Gamma^\dagger\, dt - 2\Theta(\alpha)x\, dt \\ & + \left(-2\Gamma\Gamma^\dagger I + \Gamma^\dagger\Gamma + \Gamma^T\Gamma^\#\right) x\, dt \\ & + \boldsymbol{i}\Theta(\Gamma^\# - \Gamma)x\, d\bar{W}_1 - \Theta(\Gamma + \Gamma^\#)x\, d\bar{W}_2. \end{aligned} \quad (8)$$

Note that $(\Theta(\Gamma)\Gamma^\dagger)^* = -\Theta(\Gamma)\Gamma^\dagger$, which assures that

$$Re\{\Theta(\Gamma)\Gamma^\dagger\} = \frac{1}{2}\left(\Theta(\Gamma)\Gamma^\dagger + (\Theta(\Gamma)\Gamma^\dagger)^*\right) = 0.$$

Also, observe that $\Gamma^\# - \Gamma$ is purely imaginary and $\Gamma + \Gamma^\#$ is purely real. Therefore, all matrices in (8) are real.

As mentioned in Section II, the output fields Y_1 and Y_2 depend linearly on L, $L^\dagger$ and the fields $\bar{W}_1$ and $\bar{W}_2$, i.e.,

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} \Gamma + \Gamma^\# \\ i(\Gamma^\# - \Gamma) \end{pmatrix} x\, dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}.$$

IV. Physical Realizability

In an environment where the classical laws of physics apply, standard control techniques such as optimization or a Lyapunov procedures do not worry in general of the nature of the controller they synthesized. In other words, their implementation is always possible since the physics behind them still holds. However, if one desires to implement a controller that obeys the laws imposed by quantum mechanics (quantum coherent control), then such a task is not so easily achieved unless an explicit characterization of those laws is given in terms of the control system vector fields. This is exactly the purpose for introducing the concept of a *physically realizable* system in the next definition.

Definition 1: System (5) with output equation (6) is said to be physically realizable if there exist $\mathcal{H} = \alpha x$, with $\alpha^T \in \mathbb{R}^3$, and $L = \Gamma x$, with $\Gamma^T \in \mathbb{C}^3$ such that

$$\begin{aligned} F_0 &= -2\boldsymbol{i}\Theta(\Gamma)\Gamma^\dagger, \\ F &= -2\Theta(\alpha) + \Gamma^\dagger\Gamma + \Gamma^T\Gamma^\# - 2\Gamma\Gamma^\dagger I, \\ G_1 &= \Theta\left(\boldsymbol{i}(\Gamma^\# - \Gamma)\right), \\ G_2 &= -\Theta(\Gamma + \Gamma^\#), \\ H_1 &= \Gamma + \Gamma^\#, \\ H_2 &= \boldsymbol{i}\left(\Gamma^\# - \Gamma\right). \end{aligned}$$

Note by direct inspection that for a physically realizable system ${G_i}^T = -G_i$ for $i = 1, 2$.

From a control perspective, it is necessary to characterize when a bilinear QSDE posses underlying Hamiltonian and coupling operators which allows to express the matrices comprising (5) and (6) as in Definition 1. Thus, the main result of the paper is given in the next theorem, which establishes necessary and sufficient conditions for the physical realizability of a bilinear QSDE.

Theorem 1: System (5) with output equation (6) is physically realizable if and only if

i. $F_0 = \frac{1}{2}(G_1 - \boldsymbol{i}G_2)(H_1 + \boldsymbol{i}H_2)^\dagger$,

ii. $G_1 = \Theta(H_2)$,

iii. $G_2 = -\Theta(H_1)$,

iv. $F + F^T + G_1{G_1}^T + G_2{G_2}^T = 0$.

In which case, one can identify the matrix α defining the system Hamiltonian as

$$\alpha = \frac{1}{8}\mathrm{vec}(F - F^T)^T E,$$

and the coupling matrix can be identified to be

$$\Gamma = \frac{1}{2}(H_1 + \boldsymbol{i}H_2).$$

V. Preservation of Canonical Commutation Relations

The goal of this section is to show that the conditions presented in Theorem 1 are necessary and sufficient for preserving the Pauli commutation relations (3) by the system (5). To achieve this task, a property of the stacking operator and a lemma are needed. Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times l}$ and $C \in \mathbb{R}^{l\times r}$ for $n, m, l, r \in \mathbb{N}$. It is well-known that the stacking operator used at the end of Section III satisfies

$$\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B).$$

Lemma 3: Let $E = (\bar{e}_1, \bar{e}_2, \bar{e}_3)^T$, and $A, B \in \mathbb{R}^{3\times 3}$. Then

i. $E^T(A \otimes B)E = E^T(B \otimes A)E$.

ii. $E^T(I \otimes A)E = \mathrm{Tr}(A)I - A^T$.

iii. $E^T(A \otimes B)E = A^TB^T + B^TA^T + \mathrm{Tr}(A)\mathrm{Tr}(B)I - \mathrm{Tr}(B)A^T - \mathrm{Tr}(A)B^T - \mathrm{Tr}(AB)I$.

iv. $EE^T(A \otimes B)E = (A \otimes B)E + (B \otimes A)E$.

In order to be considered a quantum system, the system variables of (5) must preserve (3) for all times. The condition that (5) has to satisfy is

$$d[x, x^T] - 2\boldsymbol{i}\Theta(dx) = 0. \tag{9}$$

Note by the linearity of the map Θ that

$$\begin{aligned}\Theta(dx) = \ & \Theta(F_0)dt + \Theta(Fx)dt \\ & + \Theta(G_1x)d\bar{W}_1 + \Theta(G_2x)d\bar{W}_2.\end{aligned}$$

A condition for system (5) to satisfy (9) is given in the next theorem.

Theorem 2: Let $[x_i(0), x_j(0)] = 2\boldsymbol{i}\sum_{k=1}^{3}\epsilon_{ijk}x_k(0)$ for $i, j \in \{1, 2, 3\}$. System (5) implies

$$[x_i(t), x_j(t)] = 2\boldsymbol{i}\sum_{k=0}^{3}\epsilon_{ijk}x_k(t)$$

for all $t \geq 0$ if and only if

$$\begin{aligned}G_1 + G_1^T = G_2 + G_2^T = 0 \\ G_1G_2^T - G_2G_1^T - \Theta(F_0) = 0 \\ F^T + F + G_1{G_1}^T + G_2{G_2}^T = 0.\end{aligned}$$

Theorem 3: A physically realizable system satisfies the conditions of Theorem 2.

VI. Conclusions

A condition for physical realizability was given for open two-level quantum systems. Under this condition it was shown that there exist operators $\mathcal{H}$ and L such that the bilinear QSDE (5) with output equation (6) can be written as in (1). Also, it was shown that physical realizability implies preservation of the Pauli commutation relations for all times. Future work includes extending the formalism for the case of multi-particle spin systems.

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