49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

On the Local Convergence of Fliess Operators Driven by L_2-Itô Random Processes

Luis A. Duffaut Espinosa W. Steven Gray Oscar R. González

***Abstract*— Fliess operators, which are a type of functional series expansion, have been used to describe a broad class of nonlinear input-output maps driven by deterministic inputs. But in most applications, a system's inputs have noise components. It has been shown that the notion of a Fliess operator can be generalized to admit a class of L_2-Itô stochastic input processes, and that they converge absolutely over an arbitrarily large but finite time interval when a certain coefficient growth condition is met. However, a significant number of systems fail to meet this condition. In this paper, the methodology is extended by considering instead an interval of convergence with a random length. It results in a less restrictive sufficient condition for convergence, and thus, is applicable to a larger class of systems.**

I. Introduction

Fliess operators provide a general framework under which analytic nonlinear input-output systems can be studied [6]–[11], [13], [14], [21]. When the inputs are deterministic, these operators are described by an infinite summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell : \eta \mapsto (c, \eta)$. The set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For a measurable function $u : [a, b] \to \mathbb{R}^m$ define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \leq i \leq m\}$, where $\|u_i\|_{L_p}$ is the usual L_p-norm for a measurable real-valued component function u_i. Define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\eta'}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$. The input-output operator corresponding to c is then

$$F_c[u](t) \triangleq \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t),$$

which is called a *Fliess operator*. The most general results regarding the convergence of Fliess operators were presented in [14]. There it was shown that if the generating series c is *globally convergent*, i.e., satisfies the growth condition

$$|(c, \eta)| \leq KM^{|\eta|}$$

for all $\eta \in X^*$, where $|\eta|$ denotes the number of symbols in η and $K, M > 0$, then $F_c[u]$ converges absolutely on $[t_0, \infty)$ for $u \in L_{p,e}[t_0, \infty)$. On the other hand, if the generating series c is *locally convergent*, i.e., satisfies the growth condition

$$|(c, \eta)| \leq KM^{|\eta|}|\eta|!$$

for all $\eta \in X^*$, then $F_c[u]$ converges absolutely on $[t_0, t_0+T]$ if T and $\|u\|_{L_p}$ are sufficiently small.

When noisy inputs are introduced, similar approaches to defining Fliess operators have been presented in [1], [6], [8], [11], [17], however, only Wiener processes are admissible inputs. More recently in [3]–[5], it was shown that the notion of a Fliess operator can be generalized to admit a class of L_2-Itô stochastic processes. This class of processes was chosen since a large number of important phenomena can be described by Itô processes. Specifically, such operators were defined as an infinite summation of Lebesgue and Stratonovich iterated integrals, and it was shown that for any suitable L_2-Itô process w, $F_c[w]$ is absolutely mean square convergent over an arbitrarily large but finite interval of time provided the generating series c is globally convergent. This class of input-output systems, however, is still fairly limited for engineering applications. For example, consider the stochastic differential equation in Stratonovich form

$$\begin{aligned} dz(t) &= Mz^2(t)\, dW(t), \quad z(0) = 1, \\ y(t) &= Kz(t), \end{aligned} \tag{1}$$

where W is a Wiener process and $K, M > 0$ are fixed. By the procedure described in [4], the generating series for (1) is specified using the alphabet $XY = \{x_0, y_0\}$ as $(c, \eta) = KM^{|\eta|}\,|\eta|!$ if $\eta = y_0^k$, $k \geq 0$ and 0 otherwise. This series is clearly not globally convergent and its corresponding output is

$$\begin{aligned} y(t) &= F_c[0](t) \\ &= \sum_{k=0}^{\infty} KM^k k! \oint_0^t \cdots \oint_0^{t_2} dW(t_1) \cdots dW(t_k). \end{aligned}$$

Since Stratonovich integrals follow the rules of standard integral calculus,

$$y(t) = F_c[0](t) = \sum_{k=0}^{\infty} KM^k W^k(t).$$

At first glance, y appears not to be convergent. However, if for a fixed $0 < R < 1$ the *first passage time*

$$\tau_R \triangleq \inf\{t > 0 : |MW(t)| = R\}$$

is positive, then $y(t)$ will be a well-defined random variable $Kz(t) = \frac{K}{1-MW(t)}$ for any $t \in [0, \tau_R]$, where $[0, \tau_R]$ is a nonzero interval of time. This motivates the idea that a broader class of stochastic input-output systems can be

This research was supported by the NASA Langley Research Center under grant NNX07AD52A.

The authors are affiliated with the Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, 23529-0246, USA. lduff004@odu.edu, {gray,gonzalez}@ece.odu.edu

978-1-4244-7744-9/10/$26.00 ©2010 IEEE

described by Fliess operators having only locally convergent generating series. Clearly, the interval of convergence now has a random nature, i.e.,

$$[0,\tau_R] = \{0 \leq t \leq \tau_R(\omega) : (\tau_R(\omega),\omega) \in [0,\infty) \times \Omega\},$$

where Ω is the sample space. To reinforce this observation, the solution of (1) can also be obtained directly by separation of variables as

$$\int_0^t \frac{dz(s)}{z^2(s)} = \frac{z(t)-1}{z(t)} = \oint_0^t M\, dW(s) = MW(t).$$

Thus,

$$y(t) = Kz(t) = \frac{K}{1-MW(t)} \tag{2}$$

for any t such that $MW(t) < 1$, which implies that (2) is valid for all $t \in [0,\tau_R]$, $R < 1$. It should be noted that Arous studied the series expansions of the solutions of stochastic differential equations over stochastic time intervals when the coefficients of the series grow at a locally convergent rate [1]. But his approach was based upon a priori knowledge of a state equation and certain conditions on the corresponding vector fields. The goal of this paper is to describe a theoretical framework under which a Fliess operator driven by an L_2-Itô random process and having a locally convergent generating series converges over a time interval of random length, independent of the existence of any state space model. The approach utilizes only the input-output description of the system.

The paper is organized as follows. Section II introduces the stochastic tools and definitions used throughout the paper. In Section III, the definition of a Fliess operator driven by an L_2-Itô random process input is presented. Then its local convergence (in the mean square sense) over a stochastic time interval is analyzed. Finally, Section IV provides the conclusions and suggestions for future work.

II. The Stochastic Setting

Consider a Wiener process, $W(t)$, defined over a complete probability space $(\Omega,\mathcal{F},P)$. For a predictable function $u : \Omega \times [t_0,t_0+T] \to \mathbb{R}^m$ let $\|u\|_p = \max\{\|u_i\|_{L_p} : 1 \leq i \leq m\}$, where $\|\cdot\|_{L_p}$ is the usual norm on $L_p(\Omega \times [t_0,t_0+T],\mathcal{P},P\otimes\lambda)$, the set of all predictable functions defined on $[t_0,t_0+T]$ having finite $\|\cdot\|_{L_p}$-norm, $\mathcal{P}$ is the predictable algebra, and λ is the Lebesgue measure. The Stratonovich integral is defined in terms of an Itô integral.

Definition 1: [15], [16] For a stochastic process $v \in L_2(\Omega \times [t_0,t_0+T],\mathcal{P},P\otimes\lambda)$, the *Stratonovich integral* of v is defined by

$$\oint_{t_0}^t v(s)\, dW(s) \triangleq \int_{t_0}^t v(s)\, dW(s) + \frac{1}{2}\langle v,W\rangle_{[t_0,t]},$$

where the quadratic covariation is

$$\langle v,W\rangle_{[t_0,t]} \triangleq \lim_{\|\Pi\|\to 0} \sum_{k=0}^{n-1} \left(v(t_{k+1})+v(t_k)\right)\left(W(t_{k+1})-W(t_k)\right),$$

where $\|\Pi\| = \max_{k=0,\ldots,n-1}(t_{k+1}-t_k)$ is the measure of the partition Π.

A well known property of Stratonovich integrals is that they obey the usual integration by parts formula.

Definition 2: [2], [16] Let $T > 0$ and t_0 be fixed. An m-dimensional stochastic process w over $[t_0,t_0+T]$ is called an L_2-*Itô process* if it can be written as

$$w(t) = \int_{t_0}^t a(s)\, ds + \int_{t_0}^t b(s)\, dW(s),$$

where $a,b \in L_2^m(\Omega \times [t_0,t_0+T],\mathcal{P},P\otimes\lambda)$.
The set of all L_2-Itô processes will be denoted by $\mathscr{I}$. It is known that $\mathscr{I}$ is a subset of $L_2^m(\Omega\times[t_0,t_0+T],\mathcal{P},P\otimes\lambda)$.

Definition 3: Let $0 \leq t_0 < T$. Consider the set of all m-dimensional stochastic processes over $[t_0,t_0+T]$, denoted by $\widetilde{\mathcal{UV}}^m[t_0,t_0+T]$, which can be written as

$$w(t) = \int_{t_0}^t u(s)\, ds + \oint_{t_0}^t v(s)\, dW(s)$$

for some $u,v \in \mathscr{I}$. The latter are called the *drift* and *diffusion* inputs, respectively. Moreover, the subset $\mathcal{UV}^m[t_0,t_0+T] \subset \widetilde{\mathcal{UV}}^m[t_0,t_0+T]$ will refer to all processes satisfying:

a) Each integrand consists of m components such that $\mathbf{E}[u_i(t)] < \infty$, $\mathbf{E}[v_i(t)] < \infty$, $t \in [t_0,t_0+T]$.

b) The integrands u and v are such that

$$\|u\|_{L_2}, \|v\|_{L_2}, \|b\|_{L_2}, \|v\|_{L_4} \leq R \in \mathbb{R}^+,$$

where b is the integrand of the Itô integral in v.

c) The random variables $u_i(t_1)$, $u_i(t_2)$, $v_i(t_1)$ and $v_i(t_2)$ are independent for $1 \leq i \leq m$ and $t_1 \neq t_2$.

Observe that since $u,v \in \mathscr{I}$, then by Definition 1 any $w \in \widetilde{\mathcal{UV}}^m[t_0,t_0+T]$ is also an L_2-Itô process. Without loss of generality, it is assumed hereafter that $t_0 = 0$.

Definition 4: [18] Let $\mathbf{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ be a filtration with $\mathcal{T} = [0,\infty]$. A random variable $\tau : \Omega \to \mathcal{T}$ is a *stopping time* with respect to $\mathbf{F}$ if the event $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \mathcal{T}$.

Definition 5: [18] Let X be a stochastic process, and let D be a Borel set in $\mathbb{R}$. Define the *hitting time* of D for X as $\tau = \inf\{t > 0 : X(t) \in D\}$ or $\tau = +\infty$ if $X(t) \notin D$ for all $t \in \mathcal{T}$.

Theorem 1: [18] Let X be an adapted càdlàg stochastic process, and let D be either an open or closed Borel set. Then the hitting time, τ, of D is a stopping time.

Example 1: Consider an almost sure (a.s.) continuous time stochastic process X. A special case of a hitting time is $\tau_R = \inf\{t > 0 : X(t) = R\}$, where $R \in \mathbb{R}^+$. It is usually called the *first passage time* for the barrier R. When X is a Wiener process, the probability density function of τ_R is

$$P(t) = \frac{|R|}{\sqrt{2\pi}}\, t^{-\frac{3}{2}} e^{-\frac{R^2}{2t}},$$

which is an inverse gamma density function with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{R^2}{2}$ [15]. In Fig. 1, a Monte Carlo generated estimate of the probability density function of τ_R is shown for $R = 1$.

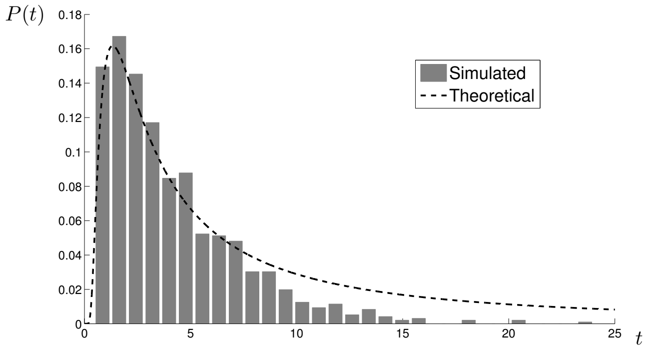

Fig. 1: Simulated probability density function for τ_1. $\square$

Definition 6: Let X be a stochastic process on $\mathcal{T}$, and let τ be a stopping time. A *stopped* or *truncated process* is any process of the form

$$\begin{aligned} X^\tau(t,\omega) &\triangleq X(t\wedge\tau(\omega),\omega) \\ &\triangleq X(t,\omega)\mathbb{1}_{[0,\tau(\omega))}(t) + X(\tau(\omega),\omega)\mathbb{1}_{[\tau(\omega),\infty)}(t), \end{aligned}$$

where $t\wedge\tau \triangleq \min(t,\tau)$ for $t\geq 0$, and $\mathbb{1}_A(t)$ indicates whether or not $t\in A$. The random variable $X_\tau(\omega) \triangleq X(\tau(\omega),\omega)$ is called a *stopped random variable*.
Observe that if the stopped process $X^\tau(t,\omega)$ is restricted to the stochastic interval

$$[0,\tau] \triangleq \{(t,\omega)\in[0,\infty)\times\omega : 0\leq t\leq\tau(\omega)\},$$

then $X^\tau(t,\omega) = X(t,\omega)\mathbb{1}_{[0,\tau(\omega)]}(t)$. Usually, a path of this process is denoted simply by $X(t\wedge\tau)$. Also, if $X(t) = \int_0^t v(s)\,dW(s)$ then the stopped random variable X_τ satisfies $X_\tau = \int_0^\tau v(s)\,dW(s) = \int_{\mathcal{T}} v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s)$.

Theorem 2: [19], [20] Let τ be a stopping time and $v\in L_2(\Omega\times[0,T],\mathcal{P},P\otimes\lambda)$. Define $X(t) = \int_0^t v(s)\,dW(s)$. Then X stopped at τ satisfies

$$\begin{aligned} X^\tau(t) &= X(t\wedge\tau) \\ &= \int_0^{t\wedge\tau} v(s)\,dW(s) \\ &= \int_0^t v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s). \end{aligned}$$

When v is an L_2-Itô process, a similar result for Stratonovich integrals follows directly from Theorem 2.

Corollary 1: Let τ be a stopping time and $v\in\mathscr{I}$. If $X(t) = \fint_0^t v(s)\,dW(s)$ then the stopped process $X^\tau(t) = \fint_0^{t\wedge\tau} v(s)\,dW(s)$ satisfies

$$X^\tau(t) = \fint_0^t v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s).$$

Proof: Since v can be written as

$$v(t) = \int_0^t a(s)\,ds + \int_0^t v(s)\,dW(s),$$

for $a,b\in L_2(\Omega\times[0,T],\mathcal{P},P\otimes\lambda)$, then X can be written as

$$X(t) = \int_0^t v(s)\,dW(s) + \int_0^t \frac{b(s)}{2}\,ds.$$

It follows from Theorem 2 that

$$\begin{aligned} &X^\tau(t) \\ &= X(t)\mathbb{1}_{[0,\tau]}(t) \\ &= \int_0^{t\wedge\tau} v(s)\,dW(s) + \int_0^{t\wedge\tau}\frac{b(s)}{2}\,ds \\ &= \int_0^t v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s) + \int_0^t \frac{b(s)}{2}\mathbb{1}_{[0,\tau]}(s)\,ds \\ &= \int_0^t v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s) + \frac{1}{2}\left\langle \int_0^\cdot a(s)\mathbb{1}_{[0,\tau]}(s)\,ds \right. \\ &\quad \left. + \int_0^\cdot b(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s), W\right\rangle_{[0,t]} \\ &= \int_0^t v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s) + \frac{1}{2}\left\langle v\mathbb{1}_{[0,\tau]}, W\right\rangle_{[0,t]} \\ &= \fint_0^t v(s)\mathbb{1}_{[0,\tau]}(s)\,dW(s). \end{aligned}$$

This completes the proof. $\blacksquare$

Theorem 3: Let $X(t) = \fint_0^t v(s)\,dW(s)$, where $v\in\mathscr{I}$ and $t\in[0,T]$. Then:

a) For any finite $R>0$, there exists a positive stopping time $\tau_R = \inf\{t>0 : |X(t)| = R\}\wedge T$.
b) $X^{\tau_R}(t,\omega)$ restricted to $[0,\tau_R]$ is a well-defined L_2-bounded, a.s. continuous and adapted L_2-Itô process.

Proof: For part a), since $v\in\mathscr{I}$ with diffusion b, X can be written as

$$X(t) = \int_0^t \frac{b(s)}{2}\,ds + \int_0^t v(s)\,dW(s).$$

Therefore, τ_R is a positive well-defined stopping time since the Lebesgue and Itô integrals produce a.s. continuous processes, and the absolute value function is continuous. Moreover, since v is defined over $[0,T]$, it is clear that if $|X(t)|<R$ for all $t\in[0,T']$ then $\tau_R = T$ when $T'>T$. Regarding part b), it is well known that since X is adapted and a.s. continuous, the stopped process is also adapted and a.s. continuous [18]. Now evaluate $\int_0^T \mathbf{E}\left[X^2(s)\right]ds$. By Itô's formula,

$$\begin{aligned} X^2(t) &= 2\int_0^t X(s)\bar{b}(s)\,ds + 2\int_0^t X(s)v(s)\,dW(s) \\ &\quad + \int_0^t v^2(s)\,ds, \end{aligned}$$

where $\bar{b}(s) = b(s)/2$. Since $2k_1k_2 \leq k_1^2 + k_2^2$ for all $k_1,k_2\in\mathbb{R}$, it follows from the properties of the Itô integral that

$$\mathbf{E}[X^2(t)] \leq \mathbf{E}\left[\int_0^t \left(X^2(s)+\bar{b}^2(s)\right)\,ds + \int_0^t v^2(s)\,ds\right].$$

The L_2 bound for X restricted to $[0,\tau_R]$ can be calculated from the previous expression as

$$\begin{aligned} \mathbf{E}[X^2(t\wedge\tau_R)] \leq\; & \mathbf{E}\left[\int_0^t \left(X^2(s\wedge\tau_R)+\bar{b}^2(s\wedge\tau_R)\right)\,ds\right. \\ & \left. + \int_0^t v^2(s\wedge\tau_R)\,ds\right]. \end{aligned}$$

Given that $\bar{b}, v \in L_2(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda)$, define a real number $M \geq \mathbf{E}\left[\int_0^t \bar{b}^2(s)\,ds\right] + \mathbf{E}\left[\int_0^t v^2(s)\,ds\right]$. Then by Fubini's theorem

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq M + \int_0^t \mathbf{E}\left[X^2(s \wedge \tau_R)\right]\,ds. \quad (3)$$

If (3) is applied recursively to the integrand of (3) then

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq M + Mt + \int_0^t \int_0^s \mathbf{E}\left[X^2(r \wedge \tau_R)\right]\,drds.$$

Repeating this procedure infinitely many times and noting that $\mathbf{E}\left[X^2(r \wedge \tau_R)\right] \leq R^2$, it follows that

$$\mathbf{E}[X^2(t \wedge \tau_R)] \leq \lim_{p\to\infty} M \sum_{n=0}^{p} \frac{t^n}{n!} + R^2 \frac{t^p}{p!} = Me^t$$

for any $t \in [0,T]$. This implies that $\int_0^T \mathbf{E}\left[X^2(t \wedge \tau_R)\right] dt < \infty$, and thus $X^{\tau_R}(t,\omega) \in L_2(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda)$. This completes the proof. ■

III. Convergence of Fliess Operators driven by L_2-Itô Stochastic Inputs

To describe an iterated integral over $\mathcal{UV}^m[0,T]$, consider the following alphabets: $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_m\}$ and $XY = X \cup Y$. For each $\eta \in XY^*$, define recursively the mapping $E_\eta : L_2^m(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda) \to \mathcal{C}_{a.s.}[0,T]$ by first setting $E_\emptyset = 1$ and then letting

$$E_{x_i\eta'}[w](t) \triangleq \int_0^{t-} u_i(s) E_{\eta'}[w](s)\,ds,\ x_i \in X, \quad (4)$$

$$E_{y_i\eta'}[w](t) \triangleq \oint_0^{t-} v_i(s) E_{\eta'}[w](s)\,dW(s),\ y_i \in Y, \quad (5)$$

where $\eta' \in XY^*$, $u_0 = v_0 = 1$, and the notation $t-$ (suppressed in subsequent sections) indicates that the integration is over $[0,t)$. The iterated integral defined in (4) and (5) can be extended linearly to polynomials as

$$E_p[w](t) = \sum_{\eta \in \mathrm{supp}(p)} (p,\eta) E_\eta[w](t),$$

where $p \in \mathbb{R}\langle XY \rangle$, i.e., the set of polynomials in XY. The set of all such integrals forms a vector space denoted as $\mathcal{E}(\mathbb{R}\langle XY \rangle)$. A Fliess operator is defined over $\mathcal{UV}^m[0,T]$ as follows.

Definition 7: [3]–[5] A causal m-input, ℓ-output *Fliess operator* F_c, $c \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$, driven by a random process in $\mathcal{UV}^m[0,T]$ is formally defined as

$$F_c[w](t) = \sum_{\eta \in XY^*} (c,\eta)\, E_\eta[w](t), \quad (6)$$

where each E_η is given in (4)-(5).

For any $A \subset XY$, let $|\eta|_A$ denote the number of letters in η that belongs to A. Consider the following definition.

Definition 8: Let $X^kY^n \triangleq \{\eta \in XY^*, |\eta|_X = k, |\eta|_Y = n\}$ and $\|\cdot\|_2$ be the usual norm on $L_2^m(\Omega, \mathcal{F}, P)$. For a fixed $t \in [0,T]$, the series $F_c[w](t)$ in (6) is said to be a *Cauchy series* if for any $\epsilon > 0$ there exist an $N > 0$ such that

$$\left\| \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta \in X^kY^{j-k}} (c,\eta) E_\eta[w](t) \right\|_2 < \epsilon,$$

when $N_2 > N_1 > N$.

Two m-dimensional random vectors x, y are called *equivalent* if $P(\omega \in \Omega : x(\omega) = y(\omega)) = 1$. It is well known that $L_2^m(\Omega, \mathcal{F}, P)$ with its usual inner product is a Hilbert space modulo this equivalence relation. One can show that a Fliess operator $F_c[w]$ is convergent in the mean square sense by proving that it forms a Cauchy series.

Next a stochastic notion of local convergence is introduced using the concept of a stopping time. Then a corresponding sufficient condition for local convergence is presented.

Definition 9: Let $X_i(t) = \oint_0^t v_i(s)\,dW(s)$, where $v_i \in \mathscr{I}$ and $i = \{0,1,\ldots,m\}$. The set $\mathcal{UV}^m[0,\tau_R]$ is defined as the set of processes $w \in \mathcal{UV}^m[0,T]$ stopped at

$$\tau_R = \min_{i \in \{0,1,\cdots,m\}} \inf\{t > 0 : |X_i(t)| = R\} \wedge T. \quad (7)$$

The theorem below is the main result of the paper.

Theorem 4: Suppose that for a series $c \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$, there exist real numbers $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}\,|\eta|!, \ \ \forall \eta \in XY^*.$$

Then for a sufficiently small $R > 0$ and any random process $w \in \mathcal{UV}^m[0,\tau_R]$ with τ_R defined as in (7), the series

$$F_c[w](t) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{\eta \in X^kY^{j-k}} (c,\eta) E_\eta[w](t) \quad (8)$$

converges in the mean square sense to the random vector $y(t) = F_c[w](t)$ for all $t \in [0,\tau_R]$. Here R is called the *radius of convergence* and $[0,\tau_R]$ the *interval of convergence*. Note that in (8) there is an implied order of summation over XY^*. Thus, the current proof for the convergence of F_c is strictly speaking addressing *conditional* local convergence.

The following terminology is used in the proof of Theorem 4. Let $\mathbb{N}^{m+1}$ be the set of all vectors with its $m+1$ components in $\mathbb{N} = \{0,1,\ldots\}$. For a fixed word $\eta \in XY^*$, define the vectors $\boldsymbol{\alpha} = (\alpha_0,\cdots,\alpha_m) \in \mathbb{N}^{m+1}$ and $\boldsymbol{\beta} = (\beta_0,\cdots,\beta_m) \in \mathbb{N}^{m+1}$, where $\alpha_i = |\eta|_{x_i}$, $\beta_i = |\eta|_{y_i}$, $k = \sum_{i=0}^m \alpha_i$ and $n = \sum_{i=0}^m \beta_i$. Let $\sqcup\!\sqcup : \mathbb{R}\langle\langle XY \rangle\rangle \times \mathbb{R}\langle\langle XY \rangle\rangle \to \mathbb{R}\langle\langle XY \rangle\rangle$ represent the *shuffle* product [3], [12]. As a consequence of the integration by parts formula for the Stratonovich integral, the shuffle product and the scalar product $E_{\eta_1}[w](t)E_{\eta_2}[w](t)$ for $\eta_1, \eta_2 \in XY^*$ are related by

$$E_{\eta_1}[u](t)E_{\eta_2}[u](t) = E_{\eta_1 \sqcup\!\sqcup\, \eta_2}[u](t). \quad (9)$$

For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ define the polynomials $p_{\boldsymbol{\alpha}} = x_0^{\alpha_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, x_m^{\alpha_m}$ and $p_{\boldsymbol{\beta}} = y_0^{\beta_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, y_m^{\beta_m}$, respectively. The summations over all possible α_i's that sum to k and all possible β_i's that sum to n are denoted, respectively, by $\sum_{\|\alpha\|=k}$ and $\sum_{\|\beta\|=n}$.

Lemma 1: [3] The characteristic series, $\boldsymbol{X^kY^n}$, of the language X^kY^n can be written in terms of the shuffle product as

$$\boldsymbol{X^kY^n} \triangleq \sum_{\eta \in X^kY^n} \eta \ = \sum_{\|\alpha\|=k, \|\beta\|=n} p_{\boldsymbol{\alpha}} \sqcup\!\sqcup\, p_{\boldsymbol{\beta}}.$$

For fixed $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$, $w \in \mathcal{UV}^m[0,T]$ and $t \in [0,T]$, define the following sum of iterated integrals

$$\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t) \triangleq E_{p_{\boldsymbol{\alpha}} \sqcup\!\sqcup\, p_{\boldsymbol{\beta}}}[w](t) = E_{p_{\boldsymbol{\alpha}}}[w](t)E_{p_{\boldsymbol{\beta}}}[w](t).$$

The importance of $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w]$ comes from the fact that, using the commutativity of the shuffle product and (9), the

Lebesgue integrals and Stratonovich integrals are separable, and thus, an L_2 upper bound for $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)$ can be obtained by calculating individual L_2 upper bounds for the random variables $E_{p_{\boldsymbol{\alpha}}}[w](t)$ and $E_{p_{\boldsymbol{\beta}}}[w](t)$. Then from the independence assumptions in Definition 3,

$$\|\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)\|_2^2 = \|E_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \left\|E_{p_{\boldsymbol{\beta}}}[w](t)\right\|_2^2. \tag{10}$$

Now consider the next lemma.

Lemma 2: [3], [4] Let u be the drift input of $w \in \mathcal{UV}^m[0,T]$. Then for $\boldsymbol{\alpha} = (\alpha_0,\ldots,\alpha_m) \in \mathbb{N}^{m+1}$ and any real numbers $0 \le s < t \le T$

$$|E_{p_{\boldsymbol{\alpha}}}[w](t)| \le \prod_{i=0}^{m} \frac{U_i^{\alpha_i}(t)}{\alpha_i!}$$

and

$$\mathbf{E}\left[\prod_{i=0}^{m}(U_i(t)-U_i(s))^{\alpha_i}\right] \le \prod_{i=0}^{m}\bar{U}_i^{\alpha_i}(t),$$

where $U_i(t) \triangleq \int_0^t |u_i(s)|\, ds$ and $\bar{U}_i(t) \triangleq \int_0^t \mathbf{E}\left[|u_i(s)|\right] ds$.

Using the above lemma, an upper bound for $\|E_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2$ can be easily calculated as

$$\|E_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \le \prod_{i=0}^{m}\frac{\bar{U}_i^{2\alpha_i}(t)}{(\alpha_i!)^2} \le \frac{R^{2k}}{(\alpha!)^2}, \tag{11}$$

where $\alpha! \triangleq \alpha_0!\cdots\alpha_m!$ and $\|u\|_{L_1} \le R$. Since $E_{p_{\boldsymbol{\beta}}}[w](t)$ is comprised exclusively of Stratonovich integrals, a different approach has to be employed to determine a bound for $\left\|E_{p_{\boldsymbol{\beta}}}[w](t)\right\|_2^2$ as described below.

Proof of Theorem 4: Without loss of generality, it is assumed that $\ell = 1$. Select R as in (11) and pick any $w \in \mathcal{UV}^m[0,\tau_R]$ and $t \in [0,\tau_R]$. By definition, $\tau_R \le T$, thus, R also bounds $\|u\|_{L_1}$ when computed over $[0,\tau_R]$. Furthermore, recall that τ_R is the first time in which the fastest $X_i(t) = \oint_0^t v_i(s)\ dW(s)$ hits the barrier $(-R,R)$, and, therefore, $|X_i(t\wedge\tau_R)| \le R$, where $i = 0,\ldots,m$. Since each $X_i(t)$ is an a.s. continuous process, and the absolute value function is a continuous function, one can always choose, without loss of generality, a continuous version of the process X. Then by Theorem 3 the random variable τ_R^i is a positive stopping time. Thus, the stopped process $X^{\tau_R^i}$ is a well-defined L_2-bounded, a.s. continuous and adapted L_2-Itô process. Now, using (9) observe

$$E_{p_{\boldsymbol{\beta}}}[w](t) = E_{y_0^{\beta_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup y_m^{\beta_m}}[w](t) = \prod_{i=0}^{m} E_{y_i^{\beta_i}}[w](t)$$
$$= \prod_{i=0}^{m}\frac{E_{y_i^{\sqcup\!\sqcup\, \beta_i}}[w](t)}{\beta_i!} = \prod_{i=0}^{m}\frac{(E_{y_i}[w](t))^{\beta_i}}{\beta_i!},$$

where $y_i^{\sqcup\!\sqcup\, \beta_i} \triangleq y_i \sqcup\!\sqcup\, y_i^{\sqcup\!\sqcup\, (\beta_i-1)}$ and $y_i^{\sqcup\!\sqcup\, 0} = 1$. Given that $E_{y_i}[w](t) = \oint_0^t v_i(s)\, dW(s)$, the L_2-norm for $E_{p_{\boldsymbol{\beta}}}[w](t)$ truncated at the stopping time τ_R is

$$\left\|E_{p_{\boldsymbol{\beta}}}[w](t\wedge\tau_R)\right\|_2^2$$
$$= \frac{1}{(\beta!)^2}\mathbf{E}\left[\prod_{i=0}^{m}\left(\left|\oint_0^{t\wedge\tau_R} v_i(s)\, dW(s)\right|\right)^{2\beta_i}\right] \le \frac{R^{2n}}{(\beta!)^2}.$$

Define

$$a_{k,n}(t) = KM^{k+n}(k+n)!\sum_{\|\alpha\|=k,\|\beta\|=n}\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t).$$

Using (10), (11), and the multinomial theorem, the following bound is obtained

$$\|a_{k,n}(t\wedge\tau_R)\|_2$$
$$\le KM^{k+n}(k+n)!\sum_{\|\alpha\|=k,\|\beta\|=n}\frac{R^k}{\alpha!}\frac{R^n}{\beta!}$$
$$= K(MR)^{k+n}(k+n)!\frac{(m+1)^k}{k!}\frac{(m+1)^n}{n!}$$
$$= K\left(MR\,(m+1)\right)^{k+n}\binom{k+n}{n}.$$

To show that (8) is mean square convergent, it is sufficient to show that it is a Cauchy series. In light of Lemma 1 and the triangle inequality,

$$\left\|\sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\sum_{\eta\in X^kY^{j-k}}(c,\eta)E_\eta[w](t)\right\|_2 \le \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\|a_{k,j-k}(t)\|_2$$

for $N_2 > N_1$. For any $\epsilon > 0$ there exists an $N > 0$ such that

$$\sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\|a_{k,j-k}(t\wedge\tau_R)\|_2$$
$$\le \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}K(MR\,(m+1))^j\frac{j!}{k!(j-k)!}$$
$$= \sum_{j=N_1}^{N_2}K(MR\,(m+1))^j\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}$$
$$= \sum_{j=N_1}^{N_2}K(2MR\,(m+1))^j < \epsilon,$$

provided that $2MR(m+1) < 1$ and $N_2 > N_1 > N$. Note that if

$$R < \frac{1}{2M(m+1)} \tag{12}$$

then the series (8) is Cauchy on $[0,\tau_R]$, and thus, the theorem is proved. ■

One can remove the conditionality in Theorem 4 if an extra condition on the generating series of a Fliess operator is introduced.

Definition 10: [7] Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^{m+1}$ and define

$$\boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \left\{\eta \in XY^* : |\eta|_{x_i} = \alpha_i, |\eta|_{y_i} = \beta_i, i = 0,\ldots,m\right\}.$$

A series $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ is called *exchangeable* if for every $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^{m+1}$ the words in $\boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ have the same coefficient.

Corollary 2: Let $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ be an exchangeable and locally convergent series. Then for a sufficiently small $R > 0$ and any random process $w \in \mathcal{UV}^m[0,\tau_R]$ with τ_R defined as in (7), the series

$$F_c[w](t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\eta\in X^kY^{j-k}}(c,\eta)\,E_\eta[w](t)$$

converges absolutely in the mean square sense for all $t \in [0,\tau_R]$.

Proof: Since c is exchangeable, one can group all the iterated integrals associated with words having the same $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, i.e.,

$$\begin{aligned} F_c[w](t) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\eta\in X^kY^{j-k}}(c,\eta)\,E_\eta[w](t) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,S_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t), \quad (13)\end{aligned}$$

where $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} = (c,\eta)$ for all $\eta \in \boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Here the series (13) is Cauchy if for any $\epsilon > 0$ there exist an $N > 0$ such that

$$\left\| \sum_{j=N_1}^{N_2}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,S_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)\right\|_2 < \epsilon,$$

when $N_2 > N_1 > N$. Using (9), $F_c[w]$ can be written uniquely as

$$\begin{aligned} &F_c[w](t) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\prod_{i=0}^{m+1}\frac{E_{x_i}^{\alpha_i}[w](t)}{\alpha_i!}\frac{E_{y_i}^{\beta_i}[w](t)}{\beta_i!}. \quad (14)\end{aligned}$$

Therefore, following the procedure in the proof of Theorem 4, (14) is Cauchy, and thus,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\substack{\|\alpha\|=k\\ \|\beta\|=j-k}} |c_{\boldsymbol{\alpha},\boldsymbol{\beta}}|\prod_{i=0}^{m+1}\left\|\frac{E_{x_i}^{\alpha_i}[w](t)}{\alpha_i!}\frac{E_{y_i}^{\beta_i}[w](t)}{\beta_i!}\right\|_2 < \infty$$

holds for any $t \in [0, \tau_R]$. This completes the proof. ■

Example 2: Consider the formal power series

$$c = \sum_{k=0}^{\infty} k!\,(x_1+y_1)^k = \sum_{k=0}^{\infty}(x_1+y_1)^{\sqcup\!\sqcup\, k}.$$

This series is both locally convergent and exchangeable. Observe

$$\begin{aligned} y(t) = F_c[w](t) &= \sum_{k=0}^{\infty} E_{(x_1+y_1)^{\sqcup\!\sqcup\, k}}[w](t) \\ = \sum_{k=0}^{\infty} E_{(x_1+y_1)}^{k}[w](t) &= (1-E_{(x_1+y_1)}[w](t))^{-1} \quad (15)\end{aligned}$$

for all $t \in [0, \tau_R]$, where τ_R is defined in (7) and $R < 1/4$ from (12). Note that

$$\frac{d}{dt}E_{(x_1+y_1)}[w](t) = \frac{d}{dt}w(t) = \dot{w}(t).$$

Here $\dot{w}$ is the formal derivative of $w \in \mathcal{UV}[0,T]$, i.e., if

$$w(t) = \int_0^t u(s)\,ds + \oint_0^t v(s)\,dW(s),$$

then $\dot{w}(t) = u(t) + v(t)\bar{w}(t)$, where $\bar{w}$ denotes white Gaussian noise. Given that the Stratonovich integral obeys the usual differential chain rule, it follows that

$$\begin{aligned} \frac{d}{dt}F_c[w](t) &= \frac{d}{dt}(1-E_{(x_1+y_1)}[w])^{-1} \\ &= (1-E_{(x_1+y_1)}[w])^{-2}\dot{w} \\ &= (F_c[w](t))^2\dot{w}.\end{aligned}$$

Thus, $y = F_c[w]$ has a state space realization

$$\frac{d}{dt}z(t) = z^2(t)\,\dot{w}, \quad y(t) = z(t), \quad z(0) = 1.$$

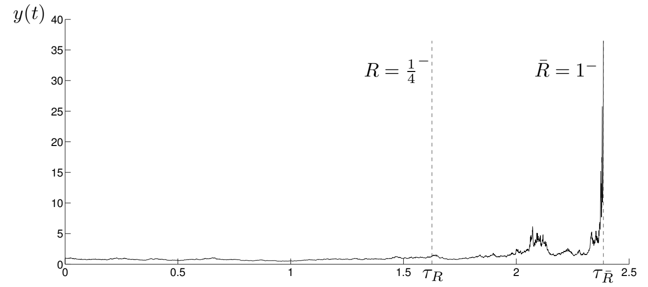

Fig. 2: Sample path of the output in Example 2.

A sample path of the output for this system when the input $\dot{w} = \bar{w}$ is shown in Fig. 2. The corresponding sample value of the random variable $\tau_R = 1.626$. However, from (15) it also follows directly that $F_c[\bar{w}](t)$ converges on $[0, \tau_{\bar{R}}]$ with $\tau_{\bar{R}} = \inf\{t > 0 : |W(t)| = \bar{R}\} \wedge T$ and $\bar{R} < 1$. Recall that R as defined in the proof of Theorem 4 is the maximum of certain norm bounds for u and v individually. On the other hand, $\bar{R}$ involves a bound on the sum of the integrals of u and v. Here the sample value of the random variable $\tau_{\bar{R}} = 2.386$. Observe that $\bar{R}$ can always be chosen to be larger than R since $R < 1/4$, and therefore, the interval of convergence $[0, \tau_{\bar{R}}]$ is almost surely longer than $[0, \tau_R]$. In other words, the generic interval of convergence found in Theorem 4 can often be improved upon in specific cases. □

Example 3: Consider the following state space system

$$\frac{d}{dt}z(t) = z^3(t)\,\dot{w}, \quad y(t) = z(t), \quad z(0) = 1, \qquad (16)$$

where $w \in \mathcal{UV}[0, \tau_R]$ for a sufficiently small R. Strictly speaking, (16) is only meaningful in its integral form

$$z(t) = \int_0^t z^3(s)u(s)\,ds + \oint_0^t z^3(s)v(s)\,dW(s). \qquad (17)$$

Given that the shuffle product represents the product of iterated integrals, the n-th power of $z(t)$ can be uniquely associated with the series $c^{\sqcup\!\sqcup\, n}$ such that (17) can be written algebraically as

$$c = (x_1+y_1)\,c^{\sqcup\!\sqcup\, 3}.$$

Thus, c can be calculated as $c = \sum_{k=0}^{\infty} c_k$, where

$$\begin{aligned} c_0 &= 1 \\ c_k &= (x_1+y_1)\sum_{\substack{i_1+i_2+i_3=k-1\\ i_1,i_2,i_3<k}} c_{i_1}\sqcup\!\sqcup\, c_{i_2}\sqcup\!\sqcup\, c_{i_3}\end{aligned}$$

[11], [17], [19]. The next three c_k's are given below:

$$\begin{aligned} c_1 &= (x_1+y_1)\,(c_0\sqcup\!\sqcup\, c_0\sqcup\!\sqcup\, c_0) = (x_1+y_1) \\ c_2 &= (x_1+y_1)\,(c_0\sqcup\!\sqcup\, c_0\sqcup\!\sqcup\, c_1 + c_0\sqcup\!\sqcup\, c_1\sqcup\!\sqcup\, c_0 + c_1\sqcup\!\sqcup\, c_0\sqcup\!\sqcup\, c_0) \\ &= (x_1+y_1)\,((x_1+y_1)+(x_1+y_1)+(x_1+y_1))\end{aligned}$$

$$
\begin{aligned}
&= 3\,(x_1+y_1)^2 \\
c_3 &= (x_1+y_1)\,(c_0 \sqcup\!\sqcup c_0 \sqcup\!\sqcup c_2 + c_0 \sqcup\!\sqcup c_2 \sqcup\!\sqcup c_0 + c_2 \sqcup\!\sqcup c_0 \sqcup\!\sqcup c_0 \\
&\quad + c_0 \sqcup\!\sqcup c_1 \sqcup\!\sqcup c_1 + c_1 \sqcup\!\sqcup c_0 \sqcup\!\sqcup c_1 + c_1 \sqcup\!\sqcup c_1 \sqcup\!\sqcup c_0) \\
&= (x_1+y_1)\left(9(x_1+y_1)^2 + 3(x_1+y_1)^{\sqcup\!\sqcup\, 2}\right) \\
&= 15\,(x_1+y_1)^3.
\end{aligned}
$$

Inductively, one can show that $c_k = (2k-1)!!(x_1+y_1)^k$, $k \geq 0$. (The double factorial is defined as $n!! = n(n-2)\cdots 1$ when n is odd and likewise when n is even.) It is easy to verify that

$$
(2k-1)!! = \frac{(2k)!}{2^k k!} = \frac{(2k)!}{2^k k! k!}\, k! = \frac{C_k^{2k}}{2^k}\, k! \leq 2^k k!.
$$

Therefore, c is locally convergent and exchangeable, and Theorem 4 and Corollary 2 are applicable. This implies from (12) that $y(t) = F_c[w](t)$ converges for all $t \in [0, \tau_R]$ for any $R < 1/(2M(m+1)) = 1/8$. However, applying properties of the shuffle product gives

$$
c = \sum_{k=0}^{\infty} (2k+1)!!\,(x_1+y_1)^k = \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k} (x_1+y_1)^{\sqcup\!\sqcup\, k}.
$$

Observe that the series expansion of

$$
\frac{1}{\sqrt{1-2z}} = 1 + z + \frac{3}{2}z^2 + \frac{5}{2}z^3 + \frac{35}{8}z^4 + O(z^5) = \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k}\, z^k.
$$

Hence, c is the generating series of the input-output operator

$$
F_c[w](t) = \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k} \underbrace{E^k_{(x_1+y_1)}[w](t)}_{w^k(t)} = \frac{1}{\sqrt{1-2\,w(t)}}, \quad (18)
$$

where $t \in [0, \tau_{\bar{R}}]$ with $\tau_{\bar{R}} = \inf\{t > 0 : |2\,w(t)| = \bar{R}\} \wedge T$ and $\bar{R} < 1$. Let t' be a sample value of τ_R. Then here the radii of convergence R and $\bar{R}$ can be related as follows:

$$
\begin{aligned}
\bar{R} &= |2\,w(t')| \leq 2\left(\left|\int_0^{t'} u(s)\,ds\right| + \left|\oint_0^{t'} v(s)\,dW(s)\right|\right) \\
&\leq 2\,(R+R) \leq \frac{1}{2}.
\end{aligned}
$$

Hence, it is evident that one can always choose the radius of convergence $\bar{R}$ to be larger than R. This implies again that Theorem 4 almost surely gives a more conservative interval of convergence than the one found directly from (18). A sample path of the output for this system confirms this when $\dot{w} = \bar{w}$ as shown in Fig. 3. □

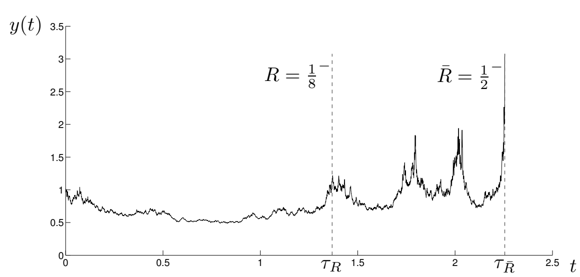

Fig. 3: Sample path of the output in Example 3.

IV. Conclusions

A Fliess operator $F_c[w]$ with $w \in \mathcal{UV}^m[0,T]$, $T > 0$ has been shown to be conditionally convergent in the mean square sense when c is locally convergent. Moreover, if c is also exchangeable then $F_c[w]$ is absolutely convergent in the mean square sense over a finite interval of time having a random length. Future work includes relaxing the exchangeability condition for the local convergence, and extending the method for Poisson input processes.

References

[1] G. B. Arous, "Flots et séries de Taylor stochastiques," *Prob. Theory Related Fields*, vol. 81, pp. 29–77, 1989.

[2] K. L. Chung and R. J. Williams, *Introduction to Stochastic Integration*. Birkhäuser, Boston, 1990.

[3] L. A. Duffaut Espinosa, *Interconnections of Nonlinear Systems Driven by L_2-Itô Stochastic Processes*, Ph.D. dissertation, Old Dominion University, 2009.

[4] L. A. Duffaut Espinosa, W. S. Gray, and O. R. González, "On Fliess operators driven by L_2-Itô random processes," in *Proc. 48th IEEE Conference on Decision and Control*, Shanghai, China, 2009, pp. 7478-7484.

[5] ——, "On the absolute convergence of Fliess operators driven by L_2-Itô random processes," in *Proc. 42nd IEEE Southeastern Symposium on System Theory*, Tyler, Texas, 2010, pp. 280-285.

[6] M. Fliess, "Intégrales itérées de K. T. Chen, bruit blanc Gaussien et filtrage non linéaire," *C. R. Acad. Sc. Paris*, vol. A-284, pp. 459–462, 1977.

[7] ——, "Fonctionnelles causales non linéaires et indéterminées non commutatives," *Bull. Soc. Math. France*, vol. 109, pp. 3–40, 1981.

[8] ——, "Stability of an elementary type of stochastic differential equation driven by multidimensional noises," *Stochastics*, vol. 14, pp. 205–213, 1981.

[9] ——, "Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives," *Invent. Math.*, vol. 71, pp. 521–537, 1983.

[10] M. Fliess, M. Lamnabhi, and F. Lamnabhi-Lagarrigue, "An algebraic approach to nonlinear functional expansions," *IEEE Trans. Circuits Systems*, vol. CAS-30, pp. 554–570, 1983.

[11] M. Fliess and F. Lamnabhi-Lagarrigue, "Applications of a new functional expansion to the cubic anharmonic oscillator," *J. Math Phys.*, vol. 23, pp. 495–502, 1981.

[12] J. Gaines, "The algebra of iterated stochastic integrals," *Stoch. Stoch. Rep.*, vol. 49, pp. 169–179, 1994.

[13] W. S. Gray and Y. Li, "Generating series for interconnected analytic nonlinear systems," *SIAM J. Contr. Optim.*, vol. 44, pp. 646–672, 2005.

[14] W. S. Gray and Y. Wang, "Fliess operators on L_p spaces: Convergence and continuity," *Systems Control Lett.*, vol. 46, pp. 67–74, 2002.

[15] F. C. Klebaner, *Introduction to Stochastic Calculus with Applications*. Imperial College Press, London, 1998.

[16] P. E. Kloeden and E. Platen, *Numerical Solutions of Stochastic Differential Equations*. Springer-Verlag, Heidelberg, 1992.

[17] F. Lamnabhi-Lagarrigue and M. Lamnabhi, "Algebraic computation of the statistics of the solutions of some nonlinear stochastic differential equations," in *Computer Algebra*, ser. Lectures Notes in Computer Sci., G. Goos and J. Hartmanis, Eds. Springer-Verlag, Berlin, 1983, vol. 162, pp. 55–67.

[18] P. E. Protter, *Stochastic Integration and Differential Equations*, 2nd ed. Springer, Berlin, 2005.

[19] E. Riccomagno, "Calcolo simbolico per processi stocastici," Masters thesis, Università di Genova, 1993.

[20] M. J. Steele, *Stochastic Calculus and Financial Applications*. Springer, New York, 2003.

[21] Y. Wang, *Algebraic Differential Equations and Nonlinear Control Systems*, Ph.D. dissertation, Rutgers University, 1990.