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On Fliess Operators Driven by L_2-Itô Random Processes

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Abstract— **Fliess operators with deterministic inputs have been studied since the late 1970's and are well understood. When the inputs are stochastic processes the theory is less developed. There have been several interesting approaches for Wiener process inputs. But the interconnection of systems is not well-posed in this context, and this limits their use in applications. This paper has two specific goals. The first goal is to describe the theoretical framework under which a Fliess operator can be driven by a class of L_2-Itô random processes. The second goal is to derive a sufficient condition for the stochastic convergence of the series which defines the corresponding output process.**

I. Introduction

Functional series expansions of nonlinear input-output operators have been utilized since the early 1900's in engineering, mathematics and physics. Among the more representative approaches are those of M. Fliess [6]–[11], V. Volterra [25] and N. Wiener [26]. From a deterministic point of view, a broad class of nonlinear systems can be described by Fliess operators, which are input-output maps constructed using the Chen-Fliess formalism [3], [7]. Such an operator is comprised of a summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate an m-input, ℓ-output operator F_c in the following manner. Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p \triangleq \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$-norm and $B_p^m(R)[a, b] \triangleq \{u \in L_p^m[a, b] : \|u\|_p \leq R\}$. With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset = 1$, and

$$E_{x_i\eta'}[u](t) \triangleq \int_{t_0}^{t} u_i(s)E_{\eta'}[u](s)\, ds, \quad (1)$$

where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$. The input-output operator corresponding to c is then

$$F_c[u](t) \triangleq \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t),$$

which is called a *Fliess operator*. All Volterra operators with analytic kernels, for example, are Fliess operators. In the classical literature where these operators first appeared, it was normally assumed that there exist real numbers $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!$, $\forall\, \eta \in X^*$, where $|z| = \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of symbols in η [7], [9], [10], [23]. This growth condition on the coefficients of c ensures that there exist positive real numbers R and T such that for all measurable u with $\|u\|_\infty \leq R$ the series defining F_c converges uniformly and absolutely on $[t_0, t_0+T]$. Such a power series c is said to be *locally convergent*. More recently, Gray and Wang showed in [14] that

$$|E_\eta[u](t)| \leq \prod_{i=0}^{m} \frac{U_i^{\alpha_i}(t)}{\alpha_i!}, \quad (2)$$

where for each x_i, $U_i(t) = \int_{t_0}^{t} |u_i(s)|\, ds$, and $\alpha_i = |\eta|_{x_i}$ is the number of times the letter x_i appears in η. This bound can be used to show that F_c constitutes a well-defined operator from $B_p^m(R)[t_0,\, t_0 + T]$ into $B_q^\ell(S)[t_0,\, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $p, q \in [1, \infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ with $(1, \infty)$ being a conjugate pair by convention. It also allows one to characterize well-posed interconnections of analytic nonlinear input-output systems [13].

In most applications, a system's inputs usually have *noise* components. In such circumstances, additional mathematical machinery is needed to properly describe an input-output map in the sense of Fliess. Several authors have formulated approaches under which Wiener processes are admissible inputs. One example is the series expansion of the solution of a stochastic differential equation, where iterated Itô and Stratonovich integrals play a central role [12], [16]. Sussmann gave a detailed description of the situation using Lie series and showed that a particularly suitable mathematical formulation involves the use of Stratonovich integrals because they obey the rules of ordinary differential calculus [22], [24]. On the other hand, Itô integrals are still useful for computing estimates of process moments [1], [19]–[21]. Similar approaches have been presented in [1], [6], [8], [11], [12], [18]. It is easily verified, however, that the corresponding output process of a nonlinear input-output system is in general not a Wiener process. Hence, these approaches are not suitable for modeling interconnected systems. In this paper, a broader class of stochastic processes

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known as L_2-Itô processes are considered as inputs [4], [16]. It is argued that this input class is more appropriate for practical applications. Then stochastic versions of (1) and (2) are defined using Lebesgue and Stratonovich integrals. In this context, the paper has two specific goals. The first goal is to describe the theoretical framework under which a Fliess operator can be driven by a subset of L_2-Itô random processes. The second goal is to derive a sufficient condition for the stochastic convergence of the series which defines the corresponding output process.

The paper is organized as follows. Section II will introduce the basic stochastic tools and definitions used throughout the paper. In Section III, bounds for stochastic iterated integrals are introduced. Section IV extends the definition of a Fliess operator to admit L_2-Itô random processes and provides the main convergence result for these operators.

II. Preliminaries

Consider a Wiener process, $W(t)$, defined over a probability space $(\Omega, \mathcal{F}, P)$. For a predictable function $u : \Omega \times [t_0, t_0+T] \to \mathbb{R}^m$ let $\|u\|_p = \max\{\|u_i\|_{L_p} : \ 1 \le i \le m\}$, where $\|\cdot\|_{L_p}$ is the usual norm on $L_p(\Omega\times[t_0,t_0+T],\mathcal{P},P\otimes\lambda)$, the set of all predictable functions defined on $[t_0,t_0+T]$ having finite $\|\cdot\|_{L_p}$-norm, $\mathcal{P}$ is the predictable algebra, and λ is the Lebesgue measure. The Stratonovich integral is defined in terms of an Itô integral.

Definition 1: For a stochastic process $v \in L_2(\Omega\times[t_0,t_0+T],\mathcal{P},P\otimes\lambda)$, the *Stratonovich integral* of v is defined by

$$\oint_{t_0}^{t} v(s)\,dW(s) \triangleq \int_{t_0}^{t} v(s)\,dW(s) + \frac{1}{2}\langle v, W\rangle_{[t_0,t]},$$

where the quadratic covariation is

$$\langle v, W\rangle_{[t_0,t]} \triangleq \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(v(t_{k+1})+v(t_k)\right)\left(W(t_{k+1})-W(t_k)\right).$$

A well known property of Stratonovich integrals is that they obey the usual integration by parts formula.

Definition 2: [4], [16] Let $T>0$ and t_0 be fixed. An m-dimensional stochastic process w over $[t_0,t_0+T]$ is called an *L_2-Itô process* if it can be written as

$$w(t) = \int_{t_0}^{t} a(s)\,ds + \int_{t_0}^{t} b(s)\,dW(s),$$

where $a, b \in L_2^m(\Omega\times[t_0,t_0+T],\mathcal{P},P\otimes\lambda)$.
The set of all L_2-Itô processes will be denoted by $\mathscr{I}$. It is known that $\mathscr{I}$ is a subset of $L_2^m(\Omega\times[t_0,t_0+T],\mathcal{P},P\otimes\lambda)$.

Definition 3: Consider the set of all m-dimensional stochastic processes over $[t_0,t_0+T]$, denoted by $\widetilde{\mathcal{UV}}^m[t_0,t_0+T]$, which can be written as

$$w(t) = \int_{t_0}^{t} u(s)\,ds + \oint_{t_0}^{t} v(s)\,dW(s)$$

for some $u, v \in \mathscr{I}$. The latter are called the *drift* and *diffusion* inputs, respectively. Moreover, the subset $\mathcal{UV}^m[t_0,t_0+T] \subset \widetilde{\mathcal{UV}}^m[t_0,t_0+T]$ will refer to all processes satisfying:

i. The integrands u and v are such that

$$\|u\|_1, \|v\|_2, \|v\|_4 \le R \in \mathbb{R}^+.$$

ii. Each integrand consists of m components such that $\mathbf{E}[u_i(t)] < \infty$, $\mathbf{E}[v_i(t)] < \infty$, $t \in [t_0,t_0+T]$, and the random variables $u_i(t_1)$, $u_i(t_2)$, $v_i(t_1)$ and $v_i(t_2)$ are independent for $1 \le i \le m$ and $t_1 \ne t_2$.

Observe that since $u, v \in \mathscr{I}$, then by Definition 1 any $w \in \widetilde{\mathcal{UV}}^m[t_0,t_0+T]$ is also an L_2-Itô process.

To describe an iterated integral over $\mathcal{UV}^m[t_0,t_0+T]$, consider the following alphabets: $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_m\}$ and $XY = X \cup Y$. For each $\eta \in XY^*$, define recursively the mapping $E_\eta : L_2^m(\Omega\times[t_0,t_0+T],\mathcal{P},P\otimes\lambda) \to \mathcal{C}_{a.s.}[t_0,t_0+T]$ by first setting $E_\emptyset = 1$ and then letting

$$E_{x_i\eta'}[w](t) \triangleq \int_{t_0}^{t-} u_i(s)E_{\eta'}[w](s)\,ds,\ x_i \in X, \tag{3}$$

$$E_{y_i\eta'}[w](t) \triangleq \oint_{t_0}^{t-} v_i(s)E_{\eta'}[w](s)\,dW(s),\ y_i \in Y, \tag{4}$$

where $\eta' \in XY^*$, $u_0 = v_0 = 1$, and the notation $t-$ (suppressed in subsequent sections) indicates that the integration is over $[t_0, t)$.

The following terminology is used throughout. Let $\mathbb{N}^{m+1}$ be the set of all vectors with its $m+1$ components in $\mathbb{N} = \{0,1,\ldots\}$. For a fixed word $\eta \in XY^*$, define the vectors $\boldsymbol{\alpha} = (\alpha_0,\cdots,\alpha_m) \in \mathbb{N}^{m+1}$ and $\boldsymbol{\beta} = (\beta_0,\cdots,\beta_m) \in \mathbb{N}^{m+1}$, where $\alpha_i = |\eta|_{x_i}$, $\beta_i = |\eta|_{y_i}$, $k = \sum_{i=0}^m \alpha_i$ and $n = \sum_{i=0}^m \beta_i$. Let $\sqcup\!\sqcup : \mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle$ represent the *shuffle* product [2, p. 20]. As a consequence of the integration by parts formula, the shuffle product and the scalar product $E_{\eta_1}[u](t)E_{\eta_2}[u](t)$ for $\eta_1, \eta_2 \in X^*$ are intimately related, i.e.,

$$E_{\eta_1}[u](t)E_{\eta_2}[u](t) = F_{\eta_1 \sqcup\!\sqcup \eta_2}[u](t). \tag{5}$$

A straightforward extension of (5) from X^* to XY^* is possible since Stratonovich integration also obeys the same integration by parts rule. For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ define the polynomials $p_{\boldsymbol{\alpha}} = x_0^{\alpha_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{\alpha_m}$ and $p_{\boldsymbol{\beta}} = y_0^{\beta_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup y_m^{\beta_m}$, respectively. The summations over all possible α_i's that sum to k and all possible β_i's that sum to n are denoted, respectively, by $\sum_{\|\alpha\|=k}$ and $\sum_{\|\beta\|=n}$.

Lemma 1: [5] Let $X^kY^n = \{\eta \in XY^*, |\eta|_X = k, |\eta|_Y = n\}$. The characteristic series, $\boldsymbol{X^kY^n}$, of the language X^kY^n can be written in terms of the shuffle product as

$$\boldsymbol{X^kY^n} \triangleq \sum_{\eta\in X^kY^n} \eta = \sum_{\|\alpha\|=k,\|\beta\|=n} p_{\boldsymbol{\alpha}} \sqcup\!\sqcup p_{\boldsymbol{\beta}}.$$

Example 1: In the deterministic case, i.e., when the alphabet Y is empty and the drift inputs are deterministic, one can show that

$$|F_{p_{\boldsymbol{\alpha}}}[u](t)| \leq \prod_{i=0}^{m} \frac{U_i^{\alpha_i}(t)}{\alpha_i!} \leq \frac{R^k}{\alpha_0! \cdots \alpha_m!}, \tag{6}$$

when $\max\{\|u\|_1, T\} \leq R$ on $[0,T]$. Furthermore, it is easily verified that for any $u \in L_1^m[0,T]$ and $\eta \in X^*$

$$|E_\eta[u](t)| \leq E_\eta[\bar{u}](t), \quad 0 \leq t \leq T,$$

where $\bar{u} \in L_1^m[0,T]$ has components $\bar{u}_j = |u_j|$, $j = 0, 1 \ldots, m$. Now fix $T > 0$. Pick any $u \in L_1^m[0,T]$ and let $R = \max\{\|u\|_1, T\}$. From Lemma 1, observe that

$$\begin{aligned}
\sum_{\eta \in X^*} |(c,\eta) E_\eta[u](t)| &\leq \sum_{k=0}^{\infty} \sum_{\eta \in X^k} |(c,\eta)|\, E_\eta[\bar{u}](t) \\
&\leq \sum_{k=0}^{\infty} K M^k k! \sum_{\|\alpha\|=k} F_{p_{\boldsymbol{\alpha}}}[\bar{u}](t) \\
&\leq \sum_{k=0}^{\infty} K (MR)^k \sum_{\|\alpha\|=k} \frac{k!}{\alpha_0! \cdots \alpha_m!} \\
&= \sum_{k=0}^{\infty} K(MR(m+1))^k.
\end{aligned}$$

Therefore, the series defining F_c converges absolutely and uniformly on an open ball in $L_1[0,T]$ of radius $R < 1/M(m+1)$. In [14], the more conservative radius of convergence $1/M(m+1)^2$ was proved. □

III. Iterated Stochastic Integrals and their L_2 Upper Bounds

For fixed $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$, $w \in \mathcal{UV}^m[0,T]$ and $t \geq 0$, define the following sum of iterated integrals

$$\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t) \triangleq F_{p_{\boldsymbol{\alpha}} \sqcup\!\sqcup\, p_{\boldsymbol{\beta}}}[w](t) = F_{p_{\boldsymbol{\alpha}}}[w](t) F_{p_{\boldsymbol{\beta}}}[w](t).$$

The importance of $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w]$ comes from the fact that, using the commutativity of the shuffle product and relation (5), the Lebesgue integrals and Stratonovich integrals can be completely separated and thus, an L_2 upper bound for $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)$ can be obtained by calculating individual L_2 upper bounds for the random vectors $F_{p_{\boldsymbol{\alpha}}}[w](t)$ and $F_{p_{\boldsymbol{\beta}}}[w](t)$. Then from the independence assumptions in Definition 3,

$$\|\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)\|_2^2 = \|F_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \left\|F_{p_{\boldsymbol{\beta}}}[w](t)\right\|_2^2. \tag{7}$$

The main goal of this section is to compute an upper bound for the right handside of (7). A bound for $\|F_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2$ can be easily calculated from the following lemma.

Lemma 2: [5] Let u be the drift input of $w \in \mathcal{UV}^m[0,T]$. Then for $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_m) \in \mathbb{N}^{m+1}$ and any real numbers $t > s \geq 0$

$$\mathbf{E}\left[\prod_{i=0}^{m} (U_i(t) - U_i(s))^{\alpha_i}\right] \leq \prod_{i=0}^{m} \bar{U}_i^{\alpha_i}(t),$$

where $\bar{U}_i(t) \triangleq \int_0^t \mathbf{E}\left[|u_i(s)|\right] ds$.

Using the above lemma, equation (6), and Definition 3, it follows that

$$\|F_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \leq \prod_{i=0}^{m} \frac{\bar{U}_i^{2\alpha_i}(t)}{(\alpha_i!)^2} \leq \frac{R^{2k}}{(\alpha!)^2}, \tag{8}$$

where $\alpha! \triangleq \alpha_0! \cdots \alpha_m!$.

Now, given that $F_{p_{\boldsymbol{\beta}}}[w](t)$ is comprised exclusively of Stratonovich integrals, a different approach has to be employed to determine a bound for $\left\|F_{p_{\boldsymbol{\beta}}}[w](t)\right\|_2^2$. It is known that Stratonovich integrals lack certain important properties such as isometry [16]. But if a Stratonovich integral is written in terms of Itô integrals, then all the properties associated with Itô integrals are available. There exist several formulas for writing iterated Stratonovich integrals as sums of iterated Itô integrals [15], [17]. An analogous formula for $E_\xi[w]$, $\xi \in Y^*$, is obtained directly by succesive applications of Definition 1.

Theorem 1: [5] Let $\xi \in Y^n$ and $w \in \mathcal{UV}^m[0,T]$ be arbitrary. Then

$$E_\xi[w](t) = \sum_{r_1=0, r_2=0}^{n, \lfloor \frac{n}{2} \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}} \in \bar{A}_{nr_2} \\ \boldsymbol{s}_{\boldsymbol{r_1}} \in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}} \mathbf{I}_\xi^{\substack{\boldsymbol{s}_{\boldsymbol{r_1}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t)\ , \tag{9}$$

where

$$\begin{aligned}
\bar{A}_{nr_2} = \{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}} = (\bar{s}_{r_2}, \ldots, \bar{s}_1) \in \mathbb{N}^{r_2} : \bar{s}_{l_2} + 1 < \bar{s}_{l_2+1}, \\
1 \leq l_2 \leq r_2 - 1,\ 1 \leq \bar{s}_{l_2} \leq n-1\}
\end{aligned}$$

for $1 \leq r_2 \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\bar{A}_{n0} = \emptyset$,

$$\begin{aligned}
A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} = \{\boldsymbol{s}_{\boldsymbol{r_1}} = (s_{r_1}, \ldots, s_1) \in \mathbb{N}^{r_1} : s_{l_1} < s_{l_1+1}, \\
1 \leq l_1 \leq r_1 - 1,\ s_{l_1} \neq \bar{s}_{l_2} \text{ or } \bar{s}_{l_2} + 1,\ \bar{s}_{l_2} \in \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}, \\
1 \leq s_{l_1} \leq n\}
\end{aligned}$$

for $1 \leq r_1 \leq n$, $A_{n0}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}} = \emptyset$, and $\lfloor \cdot \rfloor$ is the floor function. In addition, if $\xi = y_{i_n} \xi'$ then

$$\mathbf{I}_\xi[w](t) \triangleq \int_0^t v_{i_n}(t_n) \mathbf{I}_{\xi'}[w](t_n)\, dW(t_n) \tag{10}$$

$$\mathbf{I}_\xi^{\substack{\boldsymbol{s}_{\boldsymbol{r_1}} \\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t) \tag{11}$$

$$\triangleq \quad \mathbf{I}_\xi[w](t) \Bigg|_{\substack{\int v_{i_{\bar{s}_{l_2}+1}} \int v_{i_{\bar{s}_{l_2}}} dW(t') dW(t) \longrightarrow \int v_{i_{\bar{s}_{l_2}+1}} v_{i_{\bar{s}_{l_2}}} dt \\ \int v_{i_{s_{l_1}}} dW(t) \longrightarrow \int b_{i_{s_{l_1}}} dt}}$$

with $b_{i_{s_l}} \in L_2(\Omega \times [0,T], \mathcal{P}, P \otimes \lambda)$, $1 \leq l_1 \leq r_1$, $1 \leq l_2 \leq r_2$, and $i_{s_{l_1}}, i_{\bar{s}_{l_2}} \in \{0, \ldots, m\}$ are the indices of the s_{l_1}-th and $\bar{s}_{l_2}$-th letters of ξ.

The next two theorems present L_2 upper bounds for the iterated Itô integrals (10) and (11).

Theorem 2: [5] Let $\xi \in Y^n$ and $w \in \mathcal{UV}^m[0,T]$ be arbitrary. An L_2 upper bound for the iterated Itô integral (10) at a fixed $t \in [0,T]$ is

$$\|\mathbf{I}_\xi[w](t)\|_2^2 \leq \prod_{i=0}^{m} \frac{V_i^{\beta_i}(t)}{\beta_i!}, \tag{12}$$

where $V_i(t)=\int_0^t \mathbf{E}\left[v_i^2(s)\right] ds$.

Theorem 3: Let $\xi\in Y^n$ and $w\in\mathcal{UV}^m[0,T]$ be arbitrary. An L_2 upper bound for the iterated Itô integral (11) at a fixed $t\in[0,T]$ is

$$\left\|\mathbf{I}_\xi^{\substack{\boldsymbol{s}_{\boldsymbol{r_1}}\\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t)\right\|_2^2 \le 2^{r_2}t^{r_1+r_2}\prod_{i=0}^m \frac{V_i^{\bar\beta_i}(t)}{\bar\beta_i!}\frac{\bar V_i^{\bar\gamma_i}(t)}{\sqrt{\bar\gamma_i!}}\frac{B_i^{\gamma_i}(t)}{\gamma_i!}, \quad (13)$$

where $\bar\gamma_i=\sum_{l_2=1}^{r_2}(\delta_{ii_{\bar s_{l_2}}}+\delta_{i(i_{\bar s_{l_2}}+1)})$, $\bar s_{l_2}\in\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}$; $\gamma_i=\sum_{l_1=1}^{r_1}\delta_{ii_{s_{l_1}}}$, $s_{l_1}\in\boldsymbol{s}_{\boldsymbol{r_1}}$; $\bar\beta_i=\beta_i-\bar\gamma_i-\gamma_i$; $\bar V_i^2(t)=\int_0^t\mathbf{E}\left[v_i^4(s)\right]ds$ and $B_i(t)=\int_0^t\mathbf{E}\left[b_i^2(s)\right]ds$. Here δ_{ij} denotes the Kronecker delta function.

Proof: This inequality is proved by induction over $r=r_1+r_2$. If $r_1=0$ and $r_2=0$ then inequality (13) reduces trivially to inequality (12). Now, assume that (13) holds up to $r-1\ge 0$. Set $\xi=y_{i_n}\cdots y_{i_{s_r}}\xi'$, with $\xi'\in Y^{s_r-1}$ and $s_r=s_{r_1}$ or $\bar s_{r_2}+1$. For $s_{r_1}>\bar s_{r_2}+1$, applying the isometry property $n-s_{r_1}$ times and Theorem 2 gives

$$\begin{aligned}
&\left\|\mathbf{I}_\xi^{\substack{\boldsymbol{s}_{\boldsymbol{r_1}}\\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t)\right\|_2^2=\mathbf{E}\Bigg[\Bigg(\int_0^t v_{i_n}(t_n)\cdots\int_0^{t_{s_{r_1}+2}} v_{i_{s_{r_1}+1}}\cdot\\
&\quad\int_0^{t_{s_{r_1}+1}} b_{i_{s_{r_1}}}(t_{s_{r_1}})\mathbf{I}_{\xi'}^{\substack{\boldsymbol{s}_{\boldsymbol{r_1-1}}\\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t_{s_{r_1}})dt_{s_{r_1}}\cdot\\
&\quad dW(t_{s_{r_1}+1})\cdots dW(t_n)\Bigg)^2\Bigg]\\
&\le\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[v_{i_{s_{r_1}+1}}^2(t_{s_{r_1}+1})\right]\cdot\\
&\quad t_{s_{r_1}+1}\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b_{i_{s_{r_1}}}^2(t_{s_{r_1}})\right]\left\|\mathbf{I}_{\xi'}^{\substack{\boldsymbol{s}_{\boldsymbol{r_1-1}}\\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t_{s_r})\right\|_2^2 dt_{s_{r_1}}\cdot\\
&\quad dt_{s_{r_1}+1}\cdots dt_n\\
&\le\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[v_{i_{s_{r_1}+1}}^2(t_{s_{r_1}+1})\right]\cdot\\
&\quad t_{s_{r_1}+1}\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b_{i_{s_{r_1}}}^2(t_{s_{r_1}})\right]2^{r_2}t_{s_r}^{r_1-1+r_2}\cdot\\
&\quad\prod_{i=0}^m\frac{V_i^{\bar\beta}(t_{s_{r_1}})}{\bar\beta_i!}\frac{\bar V_i^{\bar\gamma_i}(t_{s_{r_1}})}{\sqrt{\bar\gamma_i!}}\frac{B_i^{\gamma_i}(t_{s_{r_1}})}{\gamma_i!}\,dt_{s_{r_1}}dt_{s_{r_1}+1}\cdots dt_n\\
&\le\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[v_{i_{s_{r_1}+1}}^2(t_{s_{r_1}+1})\right]\cdot\\
&\quad 2^{r_2}t_{s_r+1}^{r_1+r_2}\prod_{i=0}^m\frac{V_i^{\bar\beta}(t_{s_{r_1}+1})}{\bar\beta_i!}\frac{\bar V_i^{\bar\gamma_i}(t_{s_{r_1}+1})}{\sqrt{\bar\gamma_i!}}\prod_{\substack{l=0\\ l\neq i_{s_{r_1}}}}^m\frac{B_l^{\gamma_l}(t_{s_{r_1}+1})}{\gamma_l!}\cdot\\
&\quad\int_0^{t_{s_{r_1}+1}}\mathbf{E}\left[b_{i_{s_{r_1}}}^2(t_{s_{r_1}})\right]\frac{B_{i_{s_{r_1}}}^{\gamma_{i_{s_{r_1}}}}(t_{s_{r_1}})}{\gamma_{i_{s_{r_1}}}!}\,dt_{s_{r_1}}dt_{s_{r_1}+1}\cdots dt_n\\
&\le 2^{r_2}t^{r_1+r_2}\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{s_{r_1}+2}}\mathbf{E}\left[v_{i_{s_{r_1}+1}}^2(t_{s_{r_1}+1})\right]\cdot\\
&\quad\prod_{i=0}^m\frac{V_i^{\bar\beta}(t_{s_{r_1}+1})}{\bar\beta_i!}\frac{\bar V_i^{\bar\gamma_i}(t_{s_{r_1}+1})}{\sqrt{\bar\gamma_i!}}\frac{B_l^{\gamma_l'}(t_{s_{r_1}+1})}{\gamma_l'!}\,dt_{s_{r_1}+1}\cdots dt_n,
\end{aligned}$$

where $\gamma_i'=\gamma_i+\delta_{ii_{s_{r_1}}}$. Observe that $\sum_{i=0}^m\gamma_i'=r_1$. The remaining $(n-s_{r_1})$ nested integrals are evaluated in a similar manner except that the $\bar\beta_i$'s increase instead of the γ_i's. Therefore,

$$\left\|\mathbf{I}_\xi^{\substack{\boldsymbol{s}_{\boldsymbol{r_1}}\\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t)\right\|_2^2\le 2^{r_2}t^{r_1+r_2}\prod_{i=0}^m\frac{V_i^{\bar{\bar\beta}}(t)}{\bar{\bar\beta}_i!}\frac{\bar V_i^{\bar\gamma_i}(t)}{\sqrt{\bar\gamma_i!}}\frac{B_l^{\gamma_i'}(t)}{\gamma_i'!},$$

where $\sum_{i=0}^m\left(\bar{\bar\beta}_i+\bar\gamma_i+\gamma_i'\right)=n$, $\bar{\bar\beta}_j=\bar\beta_j+\sum_{l=s_{r_1}+1}^n\delta_{ji_l}$ for $i_l\in\{0,\ldots,m\}$, and $\beta_i=\bar{\bar\beta}_i+\bar\gamma_i+\gamma_i'$. Similarly, for $\bar s_{r_2}>s_{r_1}$, one applies instead the isometry property $n-(\bar s_{r_2}-1)$ times. There are two situations. The first is when $i_{\bar s_{r_2}}+1\neq i_{\bar s_{r_2}}$. It then follows by Hölder's inequality that

$$\begin{aligned}
&\left\|\mathbf{I}_\xi^{\substack{\boldsymbol{s}_{\boldsymbol{r_1}}\\ \bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}[w](t)\right\|_2^2\le\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar s_{r_2}+3}}\mathbf{E}\left[v_{i_{\bar s_{r_2}+2}}^2(t_{\bar s_{r_2}+1})\right]\cdot\\
&\quad t_{\bar s_{r_2}+1}\int_0^{t_{\bar s_{r_2}+1}}\mathbf{E}\left[v_{i_{\bar s_{r_2}+1}}^2(t_{\bar s_{r_2}})v_{i_{\bar s_{r_2}}}^2(t_{\bar s_{r_2}})\right]2^{r_2-1}t_{\bar s_{r_2}}^{r_1+r_2-1}\cdot\\
&\quad\prod_{i=0}^m\frac{V_i^{\bar\beta}(t_{\bar s_{r_2}})}{\bar\beta_i!}\frac{\bar V_i^{\bar\gamma_i}(t_{\bar s_{r_2}})}{\sqrt{\bar\gamma_i!}}\frac{B_i^{\gamma_i}(t_{\bar s_{r_2}})}{\gamma_i!}\,dt_{\bar s_{r_2}}dt_{\bar s_{r_2}+1}\cdots dt_n\\
&\le\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar s_{r_2}+3}}\mathbf{E}\left[v_{i_{\bar s_{r_2}+2}}^2(t_{\bar s_{r_2}+1})\right]2^{r_2-1}t_{\bar s_{r_2}+1}^{r_1+r_2}\cdot\\
&\quad\prod_{i=0}^m\frac{V_i^{\bar\beta}(t_{\bar s_{r_2}+1})}{\bar\beta_i!}\frac{B_i^{\gamma_i}(t_{\bar s_{r_2}+1})}{\gamma_i!}\prod_{\substack{l=0\\ l\neq i_{\bar s_{r_2}},i_{\bar s_{r_2}+1}}}^m\frac{\bar V_l^{\bar\gamma_l}(t_{\bar s_{r_2}+1})}{\sqrt{\bar\gamma_l!}}\cdot\\
&\quad\left(\int_0^{t_{\bar s_{r_2}+1}}\mathbf{E}\left[v_{i_{\bar s_{r_2}}+1}^2(t_{\bar s_{r_2}})\right]\frac{\bar V_{i_{\bar s_{r_2}+1}}^{2\bar\gamma_{i_{\bar s_{r_2}+1}}}(t_{\bar s_{r_2}})}{\bar\gamma_{i_{\bar s_{r_2}}}!}\,dt_{\bar s_{r_2}}\right)^{\frac{1}{2}}\cdot\\
&\quad\left(\int_0^{t_{\bar s_{r_2}+1}}\mathbf{E}\left[v_{i_{\bar s_{r_2}}}^2(t_{\bar s_{r_2}})\right]\frac{\bar V_{i_{\bar s_{r_2}}}^{2\bar\gamma_{i_{\bar s_{r_2}}}}(t_{\bar s_{r_2}})}{\bar\gamma_{i_{\bar s_{r_2}}}!}\,dt_{\bar s_{r_2}}\right)^{\frac{1}{2}}\cdot\\
&\quad dt_{\bar s_{r_2}+1}\cdots dt_n\\
&\le\int_0^t\mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar s_{r_2}+3}}\mathbf{E}\left[v_{i_{\bar s_{r_2}+2}}^2(t_{\bar s_{r_2}+1})\right]2^{r_2-1}t_{\bar s_{r_2}+1}^{r_1+r_2}\cdot\\
&\quad\prod_{i=0}^m\frac{V_i^{\bar\beta}(t_{\bar s_{r_2}+1})}{\bar\beta_i!}\frac{B_i^{\gamma_i}(t_{\bar s_{r_2}+1})}{\gamma_i!}\prod_{\substack{l=0\\ l\neq i_{\bar s_{r_2}},i_{\bar s_{r_2}+1}}}^m\frac{\bar V_l^{\bar\gamma_l}(t_{\bar s_{r_2}+1})}{\sqrt{\bar\gamma_l!}}\cdot
\end{aligned}$$

$$\frac{\bar{V}_{i_{\bar{s}_{r_2}+1}}^{\bar{\gamma}_{i_{\bar{s}_{r_2}+1}}+1}(t_{\bar{s}_{r_2}+1})}{\sqrt{(\bar{\gamma}_{i_{\bar{s}_{r_2}+1}}+1)!}}\frac{\bar{V}_{i_{\bar{s}_{r_2}}}^{\bar{\gamma}_{i_{\bar{s}_{r_2}}}+1}(t_{\bar{s}_{r_2}+1})}{\sqrt{(\bar{\gamma}_{i_{\bar{s}_{r_2}}}+1)!}}\,dt_{\bar{s}_{r_2}+1}\cdots dt_n$$

$$\leq \int_0^t \mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[v_{i_{\bar{s}_{r_2}+2}}^2(t_{\bar{s}_{r_2}+1})\right] 2^{r_2-1} t_{\bar{s}_{r_2}+1}^{r_1+r_2}\cdot \prod_{i=0}^m \frac{V_i^{\bar{\beta}}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i'}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_i'!}}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\cdot dt_{\bar{s}_{r_2}+1}\cdots dt_n,$$

where $\bar{\gamma}_i' = \bar{\gamma}_i + \delta_{ii_{\bar{s}_{r_2}}} + \delta_{i(i_{\bar{s}_{r_2}}+1)}$. Observe that $\sum_{i=0}^m \bar{\gamma}_i' = 2r_2$. In the second situation, $i_{\bar{s}_{r_2}}+1 = i_{\bar{s}_{r_2}}$. Since $(\bar{\gamma}_i+2)\geq \sqrt{\bar{\gamma}_i+1}\sqrt{\bar{\gamma}_i+2}$, it follows that

$$\left\|\mathbf{I}_{\xi}^{\boldsymbol{s_{r_1}}\,\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}[w](t)\right\|_2^2 \leq \int_0^t \mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[v_{i_{\bar{s}_{r_2}+2}}^2(t_{\bar{s}_{r_2}+1})\right]\cdot 2^{r_2-1} t_{\bar{s}_{r_2}+1}^{r_1+r_2}\prod_{i=0}^m \frac{V_i^{\bar{\beta}}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\prod_{\substack{l=0\\ l\neq i_{\bar{s}_{r_2}+1}}}^m \frac{\bar{V}_l^{\bar{\gamma}_l}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_l!}}\cdot \int_0^{t_{\bar{s}_{r_2}+1}} \mathbf{E}\left[v_{i_{\bar{s}_{r_2}+1}}^4(t_{\bar{s}_{r_2}})\right]\frac{\bar{V}_{i_{\bar{s}_{r_2}+1}}^{\bar{\gamma}_{i_{\bar{s}_{r_2}+1}}}(t_{\bar{s}_{r_2}})}{\sqrt{\bar{\gamma}_{i_{\bar{s}_{r_2}+1}}!}}\,dt_{\bar{s}_{r_2}}dt_{\bar{s}_{r_2}+1}\cdots dt_n$$

$$\leq \int_0^t \mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[v_{i_{\bar{s}_{r_2}+2}}^2(t_{\bar{s}_{r_2}+1})\right] 2^{r_2-1} t_{\bar{s}_{r_2}+1}^{r_1+r_2}\cdot \prod_{i=0}^m \frac{V_i^{\bar{\beta}}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\prod_{\substack{l=0\\ l\neq i_{\bar{s}_{r_2}+1}}}^m \frac{\bar{V}_l^{\bar{\gamma}_l}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_l!}}\cdot 2\frac{\bar{V}_{i_{\bar{s}_{r_2}+1}}^{\bar{\gamma}_{i_{\bar{s}_{r_2}+1}}+2}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_{i_{\bar{s}_{r_2}}}!}(\bar{\gamma}_{i_{\bar{s}_{r_2}}}+2)}\,dt_{\bar{s}_{r_2}+1}\cdots dt_n$$

$$\leq \int_0^t \mathbf{E}\left[v_{i_n}^2(t_n)\right]\cdots\int_0^{t_{\bar{s}_{r_2}+3}} \mathbf{E}\left[v_{i_{\bar{s}_{r_2}+2}}^2(t_{\bar{s}_{r_2}+1})\right] 2^{r_2} t_{\bar{s}_{r_2}+1}^{r_1+r_2}\cdot \prod_{i=0}^m \frac{V_i^{\bar{\beta}}(t_{\bar{s}_{r_2}+1})}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i'}(t_{\bar{s}_{r_2}+1})}{\sqrt{\bar{\gamma}_i'!}}\frac{B_i^{\gamma_i}(t_{\bar{s}_{r_2}+1})}{\gamma_i!}\,dt_{\bar{s}_{r_2}+1}\cdots dt_n,$$

where $\bar{\gamma}_i' = \bar{\gamma}_i + 2\delta_{ii_{\bar{s}_{r_2}}}$ and $\sum_{i=0}^m \bar{\gamma}_i' = 2r_2$. The remaining $n-(\bar{s}_{r_2}-1)$ nested integrals are evaluated in a similar manner except that the $\bar{\beta}_i$'s increase instead of the $\bar{\gamma}_i$'s. In addition, the above analysis verifies in general that $\sum_{i=0}^m \beta_i = \sum_{i=0}^m(\bar{\beta}_i+\bar{\gamma}_i+\gamma_i) = n$, which completes the proof. ∎

A consequence of the previous theorems is the L_2 upper bound for the random variable $E_\xi[w](t)$, $\xi\in Y^n$.

Theorem 4: Let $\xi \in Y^n$ and $w\in\mathcal{UV}^m[0,T]$ be arbitrary. Then for a fixed $t\in[0,T]$

$$\|E_\xi[w](t)\|_2 < \frac{(\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\beta!)^{\frac{1}{4}}},$$

where $\beta! \triangleq \beta_0!\cdots\beta_m!$ and $\max\{\|v\|_2, \|v_0\|_2, \|v\|_4\}\leq R$.

Proof: From (9) a Stratonovich iterated integral can be written in terms of Itô iterated integrals. Note that $\#\{\bar{A}_{nr_2}\} = \binom{n-r_2}{r_2}\leq\binom{n}{r_2}$ and $\#\{A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}\} = \binom{n-2r_2}{r_1}\leq\binom{n}{r_1}$. Using the triangle inequality, Theorem 1, Theorem 3 and the binomial theorem

$$\|E_\xi[w](t)\|_2 \leq \sum_{r_1=0,r_2=0}^{n,\lfloor\frac{n}{2}\rfloor}\frac{1}{2^{r_1}2^{r_2}}\sum_{\substack{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}\in\bar{A}_{nr_2}\\ \boldsymbol{s_{r_1}}\in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}}\left\|\mathbf{I}_{\xi}^{\boldsymbol{s_{r_1}}\,\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}[w](t)\right\|_2$$

$$\leq \sum_{r_1=0,r_2=0}^{n,\lfloor\frac{n}{2}\rfloor}\frac{t^{\frac{r_1+r_2}{2}}}{2^{r_1}2^{\frac{r_2}{2}}}\sum_{\substack{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}\in\bar{A}_{nr_2}\\ \boldsymbol{s_{r_1}}\in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}}\sqrt{\prod_{i=0}^m\frac{V_i^{\bar{\beta}_i}(t)}{\bar{\beta}_i!}\frac{\bar{V}_i^{\bar{\gamma}_i}(t)}{\sqrt{\bar{\gamma}_i!}}\frac{B_i^{\gamma_i}(t)}{\gamma_i!}}$$

$$\leq R^n\sum_{r_1=0,r_2=0}^{n,\lfloor\frac{n}{2}\rfloor}\frac{t^{\frac{r_1+r_2}{2}}}{2^{r_1}2^{\frac{r_2}{2}}}\sum_{\substack{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}\in\bar{A}_{nr_2}\\ \boldsymbol{s_{r_1}}\in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}}\prod_{i=0}^m\frac{1}{(\bar{\beta}_i!)^{\frac{1}{2}}(\bar{\gamma}_i!)^{\frac{1}{4}}(\gamma_i!)^{\frac{1}{2}}}$$

$$\leq \left(\frac{R^n}{(\beta!)^{\frac{1}{4}}}\right)\sum_{r_1=0,r_2=0}^{n,\lfloor\frac{n}{2}\rfloor}\frac{t^{\frac{r_1+r_2}{2}}}{2^{r_1}2^{\frac{r_2}{2}}}\sum_{\substack{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}\in\bar{A}_{nr_2}\\ \boldsymbol{s_{r_1}}\in A_{nr_1}^{\bar{\boldsymbol{s}}_{\boldsymbol{r_2}}}}}\prod_{i=0}^m\frac{\beta!}{\bar{\beta}_i!\bar{\gamma}_i!\gamma_i!}$$

$$\leq \left(\frac{R^n}{(\beta!)^{\frac{1}{4}}}\right)3^n\sum_{r_1=0}^n\frac{t^{\frac{r_1}{2}}}{2^{r_1}}\binom{n}{r_1}\sum_{r_2=0}^n\frac{t^{\frac{r_2}{2}}}{2^{\frac{r_2}{2}}}\binom{n}{r_2}$$

$$\leq \frac{(3\sqrt{2}R(\sqrt{t}+2)(\sqrt{t}+\sqrt{2}))^n}{4^n(\beta!)^{\frac{1}{4}}}$$

$$< \frac{(\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\beta!)^{\frac{1}{4}}}.$$

∎

IV. Fliess Operators with Stochastic Inputs from $\mathcal{UV}^m[0,T]$

In this section Fliess operators are suitably extended in order to accept random processes from $\mathcal{UV}^m[0,T]$ as inputs.

Definition 4: A causal m-input, ℓ-output *Fliess operator* F_c, $c\in\mathbb{R}^\ell\langle\langle XY\rangle\rangle$, driven by a random process from $\mathcal{UV}^m[0,T]$ is formally defined as

$$F_c[w](t)\triangleq\sum_{j=0}^\infty\sum_{k=0}^j\sum_{\eta\in X^kY^{j-k}}(c,\eta)\,E_\eta[w](t), \qquad (14)$$

where each E_η is given in (3)-(4).

The operator F_c is only formally defined unless its convergence is proved. Since (14) involves stochastic integrals, a mean square notion of convergence is assumed. The procedure used here is motivated by Riccomagno in [20], [21]. In this regard, consider the following definition.

Definition 5: For a fixed $t\in[0,T]$, the series $F_c[w](t)$ in (14) is said to be a *Cauchy series* if for any $\epsilon>0$ there exist an $N>0$ such that

$$\left\|\sum_{j=N_1}^{N_2}\sum_{k=0}^j\sum_{\eta\in X^kY^{j-k}}(c,\eta)E_\eta[w](t)\right\|_2<\epsilon,$$

when $N_2 > N_1 > N$.

Two m-dimensional random vectors x, y are called *equivalent* if $P(\omega \in \Omega : x(\omega) = y(\omega)) = 1$. It is well known that $L_2^m(\Omega, \mathcal{F}, P)$ with its usual norm is a Hilbert space modulo this equivalence relation. The following theorem ensures that a Fliess operator converges in the mean square sense to produce a well defined output process when its coefficients satisfy a *rational* growth condition.

Theorem 5: Suppose for a series $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ there exists real numbers $K > 0$ and $M > 0$ such that

$$|(c, \eta)| \leq KM^{|\eta|}, \ \ \forall \eta \in XY^*.$$

Then for any random process $w \in \mathcal{UV}^m[0, T]$, $T > 0$, the series (14) converges in the mean square sense to a well defined random vector $y(t) = F_c[w](t)$, $t \in [0, T]$.

Proof: Without loss of generality it is assumed that $\ell = 1$. Pick a $t \in [0, T]$ and any $w \in \mathcal{UV}^m[0, T]$. Let $R = \max\{\|u\|_1, \|v\|_2, \|v_0\|_2, \|v\|_4\}$. For a word $\eta \in XY^*$, recall $k = \sum_{i=0}^m \alpha_i$ is the number of Lebesgue integrals in η, while $n = \sum_{i=0}^m \beta_i$ is the number of stochastic integrals in η. Define

$$a_{k,n}(t) = KM^{n+k} \sum_{\|\alpha\|=k, \|\beta\|=n} \mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t).$$

To show that (14) is mean square convergent, it is sufficient to show that it is a Cauchy series. Since $|\eta| = |\eta|_X + |\eta|_Y = k + n = j$, it follows immediately from Lemma 1 that

$$\left\| \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \sum_{\eta \in X^k Y^{j-k}} (c, \eta) E_\eta[w](t) \right\|_2 \leq \sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \|a_{k,j-k}(t)\|_2,$$

where $N_2 > N_1 > N$, provided that $N > 0$ is sufficiently large. Applying the multinomial theorem, equations (7) and (8), and Theorem 4

$$\|a_{k,n}(t)\|_2 \leq KM^{n+k} \sum_{\|\alpha\|=k, \|\beta\|=n} \frac{(\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\beta!)^{\frac{1}{4}}} \frac{R^k n!}{\alpha! \beta!}.$$

Without loss of generality, it is assumed that $R = \max\{\|u\|_1, \|v\|_2, \|v_0\|_2, \|v\|_4\} \geq 1$. If $R' \triangleq 4R(R+4)$, then

$$\begin{aligned}
\|a_{n,k}(t)\|_2 &\leq K(MR')^{k+n} \frac{(m+1)^k}{k!(n!)^{\frac{1}{4}}} \sum_{\|\beta\|=n} \frac{(n!)^{\frac{5}{4}}}{(\beta!)^{\frac{5}{4}}} \\
&\leq K(MR')^{k+n} \frac{(m+1)^k}{k!(n!)^{\frac{1}{4}}} \left(\sum_{\|\beta\|=n} \frac{n!}{\beta!} \right)^2 \\
&= K(MR')^{k+n} \frac{(m+1)^k}{k!(n!)^{\frac{1}{4}}} (m+1)^{2n} \\
&\leq \frac{K(MR'(m+1)^2)^{k+n}}{k!(n!)^{\frac{1}{4}}}.
\end{aligned}$$

Finally, taking the indicated summations,

$$\begin{aligned}
&\sum_{j=N_1}^{N_2} \sum_{k=0}^{j} \|a_{k,j-k}(t)\|_2 \\
&\leq K \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(MR'(m+1)^2)^k (MR'(m+1)^2)^{j-k}}{k!((j-k)!)^{\frac{1}{4}}} \\
&= K \sum_{k=0}^{\infty} \frac{(MR'(m+1)^2)^k}{k!} \sum_{n=0}^{\infty} \frac{(MR'(m+1)^2)^n}{(n!)^{\frac{1}{4}}} \\
&= K' \sum_{n=0}^{\infty} \frac{(MR'(m+1)^2)^n}{(n!)^{\frac{1}{4}}} = K' \sum_{n=0}^{\infty} \frac{(M'')^n}{(n!)^{\frac{1}{4}}},
\end{aligned}$$

where $M'' \triangleq (MR'(m+1)^2)$ and $K' \triangleq Ke^{(MR'(m+1)^2)}$. Note that $\frac{(M'')^n}{(n!)^{\frac{1}{4}}}$ is the n-th term of an absolutely convergent series. By the ratio test

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{M''}{(n+1)^{\frac{1}{4}}} = 0.$$

Hence, the series (14) is Cauchy. ■

Note in (14) that there is an implied order to the summation over XY^*. This is to indicate that the current proof for the convergence of F_c is strictly speaking for *conditional* convergence. That is, the series converges when computed in the order indicated. The authors conjecture that the series is in fact *absolutely* convergent, i.e., $\sum_{\eta \in XY^*} |(c, \eta)| \, \|E_\eta[w]\|_2 < \infty$, but this issue will be addressed elsewhere.

An important observation applicable to the well-posedness of interconnected systems is made in the following corollary.

Corollary 1: [5] In the context of Theorem 5, the operator

$$F_c : \mathcal{UV}^m[0, T] \to \widetilde{\mathcal{UV}}^\ell[0, T]$$

for any $T > 0$.

Example 2: Consider an autonomous system modeled by a stochastic differential equation in integral form

$$z_t = z_0 + \int_0^t \bar{f}(z_s) \, ds + \int_0^t g(z_s) \, dW(s), \qquad (15)$$

where $\bar{f}(z)$ and $g(z)$ are suitably defined functions and $z_s = z(s)$ [16]. For a $\mathcal{C}^2$ function F, the Stratonovich chain rule is

$$dF(z_t) = f(z_t) \frac{\partial}{\partial z} F(z_t) \, dt + g(z_t) \frac{\partial}{\partial z} F(z_t) \, d_s W(t), \ \ (16)$$

where $d_s W(t)$ refers to Stratonovich integration and $f(z) = \bar{f}(z) + \frac{g(z)}{2} \frac{\partial}{\partial z} g(z)$. Using these equations one can identify the Lie differentiation operators $L_f = f(z)\frac{\partial}{\partial z}$ and $L_g = g(z)\frac{\partial}{\partial z}$. Now, let $F(z)$ in (16) be replaced by either f or g, and substitute $f(z_t)$ and $g(z_t)$ into (15). This yields

$$\begin{aligned}
z_t &= z_0 + f(z_0)\int_0^t ds + g(z_0)\oint_0^t dW(s) \\
&+ \int_0^t\int_0^s L_f f(z_r)\, drds + \int_0^t\oint_0^s L_g f(z_r)\, dW(r)ds \\
&+ \oint_0^t\int_0^s L_f g(z_r)\, drdW(s) + \oint_0^t\oint_0^s L_g g(z_r)\, dW(r)dW(s) \\
&= z_0 + f(z_0)\int_0^t ds + g(z_0)\oint_0^t dW(s) + R_1(z_t),
\end{aligned}$$

where $R_1(z_t)$ contains all the integrals whose integrands do not depend on z_0. In light of (3)-(4), define $X = \{x_0\}$, $Y = \{y_0\}$ and the iterated Lie derivatives $L_{g_{x_0\eta}} = L_{g_\eta}L_{g_{x_0}}$ and $L_{g_{y_0\eta}} = L_{g_\eta}L_{g_{y_0}}$, where $g_{x_0} = f$, $g_{y_0} = g$, and $\eta \in XY^*$. Then repeating the procedure once more,

$$\begin{aligned}
z_t = z_0 &+ L_{g_{x_0}}I(z_0)E_{x_0}[0](t) + L_{g_{y_0}}I(z_0)E_{y_0}[0](t) \\
&+ L_{g_{x_0x_0}}I(z_0)E_{x_0x_0}[0](t) + L_{g_{y_0x_0}}I(z_0)E_{y_0x_0}[0](t) \\
&+ L_{g_{x_0y_0}}I(z_0)E_{x_0y_0}[0](t) + L_{g_{y_0y_0}}I(z_0)E_{y_0y_0}[0](t) \\
&+ R_2(z_t),
\end{aligned}$$

where I denotes the identity map. This produces the Peano-Baker formula for the solution of (15)

$$z_t = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\eta\in X^kY^{j-k}} L_{g_\eta}I(z_0)\; E_\eta[0](t).$$

Thus (f, g, I, z_0) realizes the operator F_c driven by only noise when $(c, \eta) = L_{g_\eta}I(z_0)$, $\forall\eta \in XY^*$. This example is analogous to the known deterministic case described in [7]. □

Example 3: Suppose the alphabet X is empty and $Y = \{y_0\}$. The input-output operator associated with the series $(c, \eta) = KM^{|\eta|}$, $\forall\eta \in Y^*$ is

$$\begin{aligned}
y(t) = F_c[0](t) &= \sum_{n=0}^{\infty} KM^n \oint_0^t \cdots \oint_0^{t_2} dW(t_1)\cdots dW(t_k) \\
&= \sum_{n=0}^{\infty} KM^n \frac{W^n(t)}{n!} = Ke^{MW(t)},
\end{aligned}$$

which is a well defined output according to Theorem 5. If instead $(c, \eta) = KM^{|\eta|}\,|\eta|!$, then Theorem 5 does not apply. However, if for a positive $R < 1$ there exist a stopping time $\tau_R \triangleq \inf\{t : |MW(t)| = R\}$, then $y(t)$ will converge almost surely to $\frac{K}{1-MW(t)}$, $\forall t \le \tau_R$. In which case, the convergence condition given in Theorem 5 is only sufficient. □

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