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On the Bilinearity of Cascaded Bilinear Systems

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***Abstract*— When two bilinear state space systems are interconnected in a cascade fashion, the resulting input-output map may not have a bilinear realization. In 1972, Brockett asked under what conditions bilinearity is preserved under composition. In 1979, Ferfera produced the least restrictive sufficient condition which is presently known using formal power series representations of the input-output maps. As this result is not widely known, the first goal of this paper is to present the idea in a broad context and to supply a concise, self-contained proof. The approach taken here follows largely from the original and is based on the theory of rational transductions. A second goal of the paper is to explore generalizations of Ferfera's sufficient condition.**

I. Introduction

Consider a state space system of the form

$$\begin{aligned}\dot{z}(t) &= Az(t)+\sum_{j=1}^{m}N_j z(t)u_j(t), \;\; z(0)=z_0\\ y(t) &= Cz(t),\end{aligned}$$

where $z(t)\in\mathbb{R}^n$; $u_j(t)\in\mathbb{R}$; $y(t)\in\mathbb{R}^\ell$; and A, N_j and C are matrices of appropriate dimensions. Systems of this type are called *bilinear systems*. They have a well developed theoretical foundation (e.g., see [22]) and have been employed in a wide number of applications from engineering to ecology and medicine [15], [21]. From a mathematical perspective, they can be viewed as a *bridge* between the class of linear systems and the class of affine-input nonlinear systems, i.e., systems of the form

$$\begin{aligned}\dot{z}(t) &= f(z(t))+\sum_{j=1}^{m}g_j(z(t))u_j(t), \;\; z(0)=z_0\\ y(t) &= h(z(t)),\end{aligned}$$

where f, g_j and h are vector fields defined in terms of local coordinates on a state space manifold [13], [14], [18], [25]. Such nonlinear systems have a considerable literature, for example, in geometric nonlinear control [15].

It is easily verified that if two linear state space systems are interconnected in a cascade fashion, that is, if $m=\ell$ and one feeds the output of one system into the input of the other, then the resulting input-output system always has a linear realization. The same closure property also holds for the class of affine-input nonlinear systems. Unfortunately, this very convenient property does *not* hold in general for the bilinear case. For example, if $(A_i, N_{\cdot,i}, C_i, z_{i,0})$, $i=1,2$ are two bilinear systems then one possible state space realization for the input-output mapping $u_1\mapsto y_2$ is clearly

$$\begin{aligned}\dot{z}_1(t) &= A_1z_1(t)+\sum_{j=1}^{m}N_{j,1}z_1(t)u_{j,1}(t),\; z_1(0)=z_{1,0}\\ \dot{z}_2(t) &= A_2z_2(t)+\sum_{j=1}^{m}N_{j,2}z_2(t)(C_1z_1(t))_j,\; z_2(0)=z_{2,0}\\ y_2(t) &= C_2z_2(t),\end{aligned}$$

which appears at first inspection to be only in the affine-input nonlinear class. (Here $(v)_j$ denotes the j-th component of $v\in\mathbb{R}^m$). In 1972, Brockett asked under what conditions is bilinearity preserved under composition [2]. One trivial sufficient condition can be identified immediately from the state space system above: when a bilinear system is followed by a linear system, the resulting system is bilinear since in this case $N_{j,2}=0$, $j=1,2,\ldots,m$. But this condition is very restrictive and not necessary. In 1979, Ferfera provided in [4], [5] a much less restrictive sufficient condition using formal power series representations of the input-output mappings, namely, $F_{c_i}: u_i\mapsto y_i$, where c_i is a generating series written in terms of a noncommutative alphabet $X=\{x_0,x_1,\ldots,x_m\}$ [8]–[10]. In this setting, system composition can be described by $F_{c_2}\circ F_{c_1}=F_{c_2\circ c_1}$, where $c_2\circ c_1$ denotes the composition product of two formal power series [4], [5], [11], [20]. Bilinearity, in this context, is equivalent to having a *rational* or *regular* generating series [1]. Ferfera introduced the notion of an *input-limited* rational series (generating series for linear systems being a special case) and showed that rationality is preserved under composition when an arbitrary rational series is followed by an input-limited rational series. It is easily demonstrated, however, that this condition is not necessary. In fact, at present, no necessary condition is available in the literature concerning this property.

As Ferfera's sufficient condition is not widely known, the first goal of this paper is to present the idea in a broad context and to supply a concise, self-contained proof. The approach taken here follows largely from the original and is based on the theory of rational transductions [6], [16], [19], [23]. A second goal of the paper is to explore generalizations of Ferfera's sufficient condition. The paper is organized as follows. In Section II, Ferfera's sufficient condition is described, as well as some basic background for the problem. In Section III, the necessary tools from the theory of rational transductions are provided. In the next section, the main proof is given in detail. The final section is devoted to generalizations of Ferfera's sufficient condition.

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II. Ferfera's Sufficient Condition via the Composition Product

A finite nonempty set of symbols $X=\{x_0,x_1,\ldots,x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite string of letters from X, $\eta=x_{i_k}\cdots x_{i_1}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k will be denoted by X^k. The set of all words including the empty word, $\emptyset$, will be denoted by X^*. It forms a monoid under catenation. A *language* is any subset of X^*. Any mapping $c: X^*\rightarrow\mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta\in X^*$ is written as (c,η). Typically, c is represented as the formal sum $c=\sum_{\eta\in X^*}(c,\eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an $\mathbb{R}$-algebra under the catenation product. Given $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$, the subset of X^* defined by $\mathrm{supp}(c)=\{\eta:(c,\eta)\neq 0\}$ is called the *support* of c. The subset of $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ consisting of all the series with finite support is denoted by $\mathbb{R}^\ell\langle X\rangle$, and its elements are called *polynomials*. c is called *proper* if $\emptyset\notin\mathrm{supp}(c)$ and *invertible* if there exists a series $c^{-1}\in\mathbb{R}\langle\langle X\rangle\rangle$ such that $cc^{-1}=c^{-1}c=1$. In the event that c is not proper, it is always possible to write $c=(c,\emptyset)(1-c')$, where $c'\in\mathbb{R}\langle\langle X\rangle\rangle$ is proper. It then follows that

$$c^{-1}=\frac{1}{(c,\emptyset)}(1-c)^{-1}=\frac{1}{(c,\emptyset)}(c')^*,$$

where $(c')^*:=\sum_{i\geq 0}(c')^i$. It can be shown that c is invertible if and only if c is not proper. Now let S be any subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}\langle\langle X\rangle\rangle$. S is said to be *rationally closed* when every invertible $c\in S$ has $c^{-1}\in S$. The *rational closure* of any set $E\subset\mathbb{R}\langle\langle X\rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ containing E.

Definition 1: [1] A series $c\in\mathbb{R}\langle\langle X\rangle\rangle$ is **rational** if it belongs to the rational closure of $\mathbb{R}\langle X\rangle$.

Thus, a given rational series can be obtained from a finite set of polynomials by performing a finite number of additions, scalar products, catenation products and inversions (or star operations), the so called *rational operations*. The following definitions and theorem provide another characterization of rational series [1], [24].

Definition 2: A **linear representation** of a series $c\in\mathbb{R}\langle\langle X\rangle\rangle$ is any triple (μ,γ,λ), where $\mu: X^*\rightarrow\mathbb{R}^{n\times n}$ is a monoid morphism, $\gamma,\lambda^T\in\mathbb{R}^{n\times 1}$, and $(c,\eta)=\lambda\mu(\eta)\gamma$ for all $\eta\in X^*$.

Definition 3: A series is called **recognizable** if it has a linear representation.

Theorem 1: A formal power series is rational if and only if it is recognizable.

For each $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p}\geq 1$ and $t_0<t_1$ be given. For a measurable function $u:[t_0,t_1]\rightarrow\mathbb{R}^m$, define $\|u\|_\mathfrak{p}=\max\{\|u_i\|_\mathfrak{p}: 1\leq i\leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0,t_1]$. Let $L_\mathfrak{p}^m[t_0,t_1]$ denote the set of all measurable functions defined on $[t_0,t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B_\mathfrak{p}^m(R)[t_0,t_1]:=\{u\in L_\mathfrak{p}^m[t_0,t_1]:\|u\|_\mathfrak{p}\leq R\}$. Define recursively for each $\eta\in X^*$ the mapping $E_\eta: L_1^m[t_0,t_1]\rightarrow\mathcal{C}[t_0,t_1]$ by setting $E_\emptyset[u]\equiv 1$, and

$$E_{x_i\bar\eta}[u](t,t_0)=\int_{t_0}^t u_i(\tau)E_{\bar\eta}[u](\tau,t_0)\,d\tau,$$

where $x_i\in X$, $\bar\eta\in X^*$ and $u_0(t)\equiv 1$. The input-output operator corresponding to c is then

$$F_c[u](t)=\sum_{\eta\in X^*}(c,\eta)\,E_\eta[u](t,t_0),$$

which is referred to as a *Fliess operator* [8]–[12], [20]. When there exist real numbers $K,M>0$ such that $|(c,\eta)|\leq KM^{|\eta|}|\eta|!$ for all $\eta\in X^*$, where $|z|:=\max\{|z_1|,|z_2|,\ldots,|z_\ell|\}$ when $z\in\mathbb{R}^\ell$, then F_c constitutes a well-defined operator from $B_\mathfrak{p}^m(R)[t_0,\,t_0+T]$ into $B_\mathfrak{q}^\ell(S)[t_0,\,t_0+T]$ for sufficiently small $R,S,T>0$, where the numbers $\mathfrak{p},\mathfrak{q}\in[1,\infty]$ are conjugate exponents, i.e. $1/\mathfrak{p}+1/\mathfrak{q}=1$ [12]. Such a power series c is said to be *locally convergent*. It can be easily shown via Theorem 1 that every rational series is locally convergent. Given any linear representation (μ,γ,λ) of a rational c, it follows that

$$c=\sum_{k=0}^{\infty}\sum_{i_1,\ldots,i_k=0}^{m}(\lambda N_{i_k}\cdots N_{i_1}\gamma)\,x_{i_k}\cdots x_{i_1},$$

where $N_i=\mu(x_i)$. The corresponding Fliess operator, F_c, is realized by the bilinear realization

$$\dot z(t)=N_0z(t)+\sum_{i=1}^m N_iz(t)\,u_i(t),\quad z(t_0)=\gamma \qquad (1)$$

$$y(t)=\lambda z(t) \qquad (2)$$

in the sense that (1) has a well-defined solution $\Phi(t,t_0,\gamma,u)$ on some interval $[t_0,t_1]$ for every $u\in B_\mathfrak{p}^m(R)[t_0,t_1]$ with $\mathfrak{p}\geq 1$ and $R>0$ sufficiently small, and

$$F_c[u](t)=\lambda\Phi(t,t_0,\gamma,u),\quad \forall t\in[t_0,t_1].$$

The composition of two Fliess operators F_c and F_d, where $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$, can be described in terms of the composition product given below.

Definition 4: [4], [5] For any $\eta\in X^*$ and series $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$, the **composition** of η with d is defined in a recursive manner by

$$\eta\circ d=\begin{cases}\eta & : \ |\eta|_{x_i}=0,\ \forall\, i\neq 0\\ x_0^{k+1}[d_i\sqcup\!\sqcup(\eta'\circ d)] & : \ \eta=x_0^kx_i\eta',\ k\in\mathbb{N},\\ & \quad\ i\neq 0,\ \eta'\in X^*,\end{cases}$$

where $\sqcup\!\sqcup$ denotes the shuffle product on $\mathbb{R}\langle\langle X\rangle\rangle$, $|\eta|_{x_i}$ is the number of times the letter x_i appears in η, and $d_i:\xi\mapsto(d,\xi)_i$ with $(d,\xi)_i$ being the i-th component of the coefficient (d,ξ). The **composition** of any $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ with d is

$$c\circ d=\sum_{\eta\in X^*}(c,\eta)\eta\circ d.$$

Theorem 2: [4], [11] Let $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$. The composition $F_c\circ F_d$ has generating series $c\circ d$, i.e., $F_c\circ F_d=F_{c\circ d}$. In addition, if c and d are locally convergent then $c\circ d$ is also locally convergent.

The following example (due to Ferfera [4]) shows that the composition product does not preserve rationality.

Example 1: Consider the scalar bilinear realization

$$\dot{z}(t) = z(t)u(t),\ z(0) = 1$$
$$y(t) = z(t).$$

Then locally the input-output map has the generating series $c = \sum_{j\geq 0} x_1^j = (1-x_1)^{-1}$, which is clearly rational. The claim is that the series $c \circ c = \sum_{j\geq 0} x_1^j \circ c$, where

$$x_1^j \circ c = x_0[c \sqcup\!\sqcup (x_1^{j-1} \circ c)],\ \ j > 0,$$

is *not* rational. One can show by induction that

$$x_0^{-k}(x_1^j \circ c) = \begin{cases} c^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^{j-k} \circ c) & : \ j \geq k \\ 0 & : \ j < k, \end{cases}$$

where x_0^{-1} denotes the *left-shift operator*, $x_0^{-k} := (x_0^k)^{-1}$, and $c^{\sqcup\!\sqcup\, k}$ is the *shuffle power* of c. In which case,

$$\begin{aligned} (c \circ c, x_0^k x_1^k) &= \sum_{j=0}^{\infty} (x_1^{-k} x_0^{-k}(x_1^j \circ c), \emptyset) \\ &= \sum_{j=k}^{\infty} (x_1^{-k}(c^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^{j-k} \circ c)), \emptyset) \\ &= \sum_{j=k}^{\infty} (x_1^{-k}(c^{\sqcup\!\sqcup\, k}), \emptyset)\ (x_1^{j-k} \circ c, \emptyset) \\ &= x_1^{-k}(c^{\sqcup\!\sqcup\, k}, \emptyset) \\ &= (c^{\sqcup\!\sqcup\, k}, x_1^k). \end{aligned}$$

The identity

$$\left(\sum_{j=0}^{\infty} x_1^j\right)^{\sqcup\!\sqcup\, k} = \sum_{j=0}^{\infty} k^j x_1^j,\ \ k \geq 1$$

yields the final expression

$$(c \circ c, x_0^k x_1^k) = k^k,\ \ k \geq 1.$$

The key observation is that these coefficients are growing faster than any sequence of coefficients from a rational series can possibly grow, namely, at a rate $KM^{|\eta|}$, where $K, M > 0$ (e.g., see [12]). Hence, the series $c \circ c$ can not be rational. □

While the composition product is not a rational operation in general, it will preserve rationality under certain conditions.

Definition 5: [4], [5] A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is **limited relative to x_i** if there exists an integer $\mathcal{N}_i \geq 0$ such that

$$\sup_{\eta \in \mathrm{supp}(c)} |\eta|_{x_i} = \mathcal{N}_i < \infty.$$

If c is limited relative to x_i for every $i = 1, \ldots, m$ then c is said to be **input-limited**. In such cases, let $\mathcal{N}_c := \max_i \mathcal{N}_i$. A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is input-limited if each component series, c_j, is input-limited for $j = 1, \ldots, \ell$. In this case, $\mathcal{N}_c := \max_j \mathcal{N}_{c_j}$.

The following is Ferfera's sufficient condition for preserving rationality.

Proposition 1: [4], [5] Let $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ be two rational series. If c is input-limited then the series $c \circ d$ is rational.

$$\begin{array}{ccc} \Sigma_c \text{ is Lie nilpotent} & \Longleftarrow & \Sigma_c \text{ is input nilpotent} \\ \Updownarrow & & \Updownarrow \\ \Sigma_c \text{ has a finite Volterra series} & \Longleftarrow & c \text{ is input-limited} \end{array}$$

Fig. 1. Properties related to the input-limited property for a minimal bilinear system $\Sigma_c := (N_0, N_1, C, \gamma)$ with generating series c.

Example 2: Let $c = \sum_{j\geq 0} x_1^j = (1-x_1)^{-1}$, which is rational but obviously *not* input-limited, and $d = 1$. Trivially $c \circ d = (1-x_0)^{-1}$. Thus, having c input-limited is a sufficient but not necessary condition for the composition product to preserve rationality. On the other hand, if one sets $d = x_1$ then it can be verified that $(c \circ d, x_0^k x_1^k) = k!$, $k \geq 1$ [4]. In which case, requiring d to be input-limited instead of c is not a sufficient condition for preserving rationality. □

Example 3: Consider a minimal bilinear realization (1)-(2) with $m = 1$. The system is said to be *Lie nilpotent* if there exists an integer $l > 0$ such that the product of any l matrices from the set $\{N_1, [N_0, N_1], \ldots, ad_{N_0}^k(N_1), \ldots\}$ is zero. It can be shown that a minimal realization has a finite Volterra series representation if and only if it is Lie nilpotent [3]. On the other hand, the system is said to be *input nilpotent* if there exists an integer $k_1 > 0$ such that the product of any $k \geq k_1$ matrices from the set $\{N_0, N_1\}$ containing k_1 copies of N_1 is zero. It is known that a minimal realization is input nilpotent if and only if its generating series is input-limited [4]. If is also easily verified that *any* system which is input-limited and locally convergent has a finite Volterra series representation. The situation for the system under consideration is summarized in Fig. 1. □

III. RATIONAL TRANSDUCTIONS

To prove Proposition 1, some results are needed from the theory of rational transductions. For brevity, the focus is on scalar output case, i.e., when $\ell = 1$. Let X and Y be two arbitrary alphabets. Any $\mathbb{R}$-linear mapping $\tau : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle Y \rangle\rangle$ is called a *transduction* [6], [16], [19], [23]. It is completely specified by

$$\tau(\eta) = \sum_{\xi \in Y^*} (\tau(\eta), \xi)\xi,\ \ \forall \eta \in X^*,$$

if for each $c \in \mathbb{R}\langle\langle X \rangle\rangle$, the sum $\sum_{\eta \in X^*} (\tau(\eta), \xi)(c, \eta)$ is finite for all $\xi \in Y^*$. Otherwise, it is only partially defined on $\mathbb{R}\langle\langle X \rangle\rangle$. One can canonically associate with any transduction τ a series in $\mathbb{R}\langle\langle X \otimes Y \rangle\rangle$, namely

$$\hat{\tau} = \sum_{\eta \in X^*} \eta \otimes \tau(\eta) = \sum_{\eta \in X^*,\ \xi \in Y^*} (\tau(\eta), \xi)\ \eta \otimes \xi.$$

From $\hat{\tau}$ one can define a second transduction $\tau' : \mathbb{R}\langle\langle Y \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle$ via

$$\tau'(\xi) = \sum_{\eta \in X^*} (\tau(\eta), \xi)\eta,\ \ \forall \xi \in Y^*.$$

τ' is called the *inverse* of τ. A transduction τ is called *rational* if the series $\hat{\tau}$ is a rational series in $\mathbb{R}\langle\langle X \otimes Y \rangle\rangle$, in which case, every rational series in $\mathbb{R}\langle\langle X\rangle\rangle$ is mapped to a rational series in $\mathbb{R}\langle\langle Y\rangle\rangle$ (if it is well-defined). Clearly, if τ is a rational transduction, then τ' is also a rational transduction.

Theorem 3: [6] A transduction $\tau : \mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$, defined everywhere on $\mathbb{R}\langle\langle X\rangle\rangle$, is rational if and only if there exists an alphabet W, two monoid morphisms $\varrho_1 : W^* \rightarrow X^*$ and $\varrho_2 : W^* \rightarrow Y^*$, and a rational series $c_W \in \mathbb{R}\langle\langle W\rangle\rangle$ such that for all series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ the following identity is satisfied

$$\tau(c) = \varrho_2(\varrho_1^{-1}(c) \odot c_W),$$

where

$$\varrho_1^{-1}(\eta) := \sum_{\substack{\nu \in W^* \\ \varrho_1(\nu)=\eta}} \nu, \quad \forall \eta \in X^*$$

is $\mathbb{R}$-linearly extended to $\mathbb{R}\langle\langle X\rangle\rangle$ (likewise for ϱ_2), and $e \odot f$ is the Hadamard product on $\mathbb{R}\langle\langle W\rangle\rangle$, i.e.,

$$e \odot f = \sum_{\nu \in W^*} (e,\nu)(f,\nu)\nu.$$

Theorem 4: [1] Suppose X and Y are two arbitrary alphabets. Let $\varrho : X \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$ be a map such that the series $\varrho(x_i)$ is a proper rational series for all $x_i \in X$. Then ϱ can be extended uniquely to a monoid morphism and then extended again uniquely to a morphism of semirings $\mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$, which is the identity on $\mathbb{R}$ (i.e, $\varrho(\alpha) = \alpha,\ \forall \alpha \in \mathbb{R}$) and a rational transduction defined everywhere on $\mathbb{R}\langle\langle X\rangle\rangle$.

A transduction is $\mathbb{R}\langle X\rangle$-*recognizable* if there exists a linear representation $\mu : X^* \rightarrow (\mathbb{R}\langle X\rangle)^{N\times N}$, γ, $\lambda^T \in (\mathbb{R}\langle X\rangle)^{N\times 1}$ such that $\tau(\eta) = \lambda\mu(\eta)\gamma$ for all $\eta \in X^*$.

Theorem 5: [16] The inverse of a $\mathbb{R}\langle X\rangle$-recognizable transduction $\tau : \mathbb{R}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}\langle\langle Y\rangle\rangle$ is a rational transduction which is defined everywhere on $\mathbb{R}\langle\langle Y\rangle\rangle$.

For the next pair of lemmas, it is necessary to introduce the following framework. Let Y and Y' be two finite disjoint alphabets and define a third alphabet $Z = Y \cup Y'$. It is easy to see that Z^* can be partitioned as $Z^* = Y^* \cup Z^*Y'Y^*$, i.e., a word $\eta \in Z^*$ is either from Y^* or can be constructed by a catenation of a word from Z^*, a letter from Y' and a word from Y^*. In what follows, $\bar{Z}$ and $\bar{\bar{Z}}$ will be two distinct copies of Z. Let $\bar{\bar{\mathbf{Z}}}^*$ be the characteristic series of the language $\bar{\bar{Z}}^*$, i.e., $\bar{\bar{\mathbf{Z}}}^* = \sum_{\theta\in\bar{\bar{Z}}^*}\theta$, and let $\bar{\varrho} : Z^* \mapsto \bar{Z}^*$ be the natural monoid isomorphism defined by $\bar{\varrho}(z_i) = \bar{z}_i$ for all $z_i \in Z$. Define the transduction τ_1 as the $\mathbb{R}$-linear extension of the mapping

$$\tau_1 : Z^* \rightarrow \mathbb{R}\langle\langle Z \cup \bar{Z} \cup \bar{\bar{Z}}\rangle\rangle$$
$$: \eta \mapsto \begin{cases} \bar{\bar{\mathbf{Z}}}^*(y_{j_n}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{j_1}\bar{\bar{\mathbf{Z}}}^*) & : \eta = y_{j_n}\cdots y_{j_1} \in Y^* \\ \bar{\varrho}(\nu)\bar{\bar{\mathbf{Z}}}^*(y_{j_n}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{j_1}\bar{\bar{\mathbf{Z}}}^*) & : \eta = \nu\xi,\ \nu \in Z^*Y', \\ & \quad \xi = y_{j_n}\cdots y_{j_1} \in Y^*. \end{cases}$$

Similarly, define the transduction τ_2 from the mapping

$$\tau_2 \ : \ Z^* \rightarrow \mathbb{R}\langle\langle Z \cup \bar{\bar{Z}}\rangle\rangle$$
$$: \ \eta = z_{i_n}\cdots z_{i_1} \mapsto (\bar{\bar{z}}_{i_n}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*).$$

By definition $\tau_1(\emptyset) = \bar{\bar{\mathbf{Z}}}^*$ and $\tau_2(\emptyset) = \emptyset$. The following result is essential.

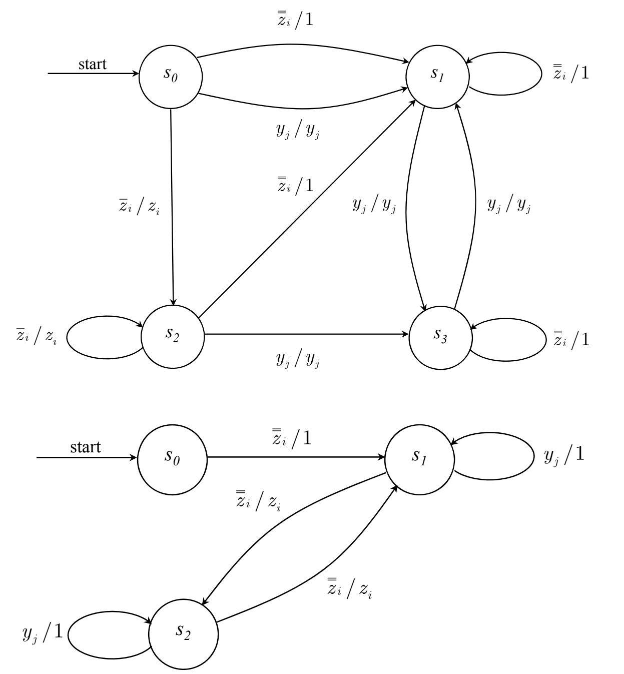

Fig. 2. Mealy machines for the linear representation of τ_1' (top) and τ_2' (bottom). A path labeled by $y_j/\cdot$, for example, means that the state transition is undertaken for *any* $y_j \in Y$.

Lemma 1: [4] The transductions τ_1 and τ_2 are rational and defined everywhere on $\mathbb{R}\langle\langle Z\rangle\rangle$.

Proof: Consider their respective inverse transductions τ_1' and τ_2'. One can construct a linear representation for each via the Mealy finite-state machines shown in Fig. 2. Specifically, for τ_1':

$$\mu_1(y_j) = \begin{bmatrix} 0 & y_j & 0 & 0 \\ 0 & 0 & 0 & y_j \\ 0 & 0 & 0 & y_j \\ 0 & y_j & 0 & 0 \end{bmatrix}, \quad \mu_1(y_k') = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mu_1(\bar{z}_i) = \begin{bmatrix} 0 & 0 & z_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mu_1(\bar{\bar{z}}_i) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T, \quad \lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

where $i = 1, 2, \ldots, \mathrm{card}(Z)$, $j = 1, 2, \ldots, \mathrm{card}(Y)$ and $k = 1, 2, \ldots, \mathrm{card}(Y')$. Similarly for τ_2':

$$\mu_2(y_j) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mu_2(y_k') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mu_2(\bar{\bar{z}}_i) = \begin{bmatrix} 0 & z_i & 0 \\ 0 & 0 & z_i \\ 0 & z_i & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$\lambda_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Thus, applying Theorem 5, the lemma is proved. ■

The next concept needed is a variation of the shuffle product as described below.

Definition 6: [4] Let $Z = Y \cup Y'$, where Y and Y' are two disjoint alphabets. The **restricted shuffle product** of two series $c, d \in \mathbb{R}\langle\langle Z \rangle\rangle$ is defined as

$$c \sqcup\!\sqcup_Y d = \sum_{\eta,\xi \in Z^*} (c,\eta)(d,\xi)\eta \sqcup\!\sqcup_Y \xi,$$

where

$$\eta \sqcup\!\sqcup_Y \xi = \begin{cases} \eta \sqcup\!\sqcup \xi & : \ \eta \in Y^*, \ \xi \in Z^* \\ \nu(\eta' \sqcup\!\sqcup \xi) & : \ \eta = \nu\eta', \ \nu \in Z^*Y', \\ & \ \ \ \eta' \in Y^*, \ \xi \in Z^*. \end{cases}$$

Lemma 2: [4] If $c, d \in \mathbb{R}\langle\langle Z \rangle\rangle$ are rational series then $c \sqcup\!\sqcup_Y d$ is also a rational series.

Proof: In light of Theorem 4, let $\varphi : \mathbb{R}\langle\langle Z \cup \bar{Z} \cup \bar{\bar{Z}} \rangle\rangle \to \mathbb{R}\langle\langle Z \rangle\rangle$ be the rational semiring epimorphism defined by setting $\varphi(z_i) = \varphi(\bar{z}_i) = \varphi(\bar{\bar{z}}_i) = z_i$ for all $z_i \in Z$, $\bar{z}_i \in \bar{Z}$ and $\bar{\bar{z}}_i \in \bar{\bar{Z}}$. It will be shown first that

$$\eta \sqcup\!\sqcup_Y \xi = \varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)), \ \forall \eta, \xi \in Z^*. \quad (3)$$

Since Z^* is partitioned by Y^* and $Z^*Y'Y^*$, one can divide the proof of identity (3) into two cases for a given $\eta \in Z^*$, namely, when $\eta \in Y^*$ and when $\eta \in Z^*Y'Y^*$.

Case 1: If $\eta \in Y^*$ then for all $\nu \in \text{supp}(\tau_1(\eta))$ it is clear that $|\nu|_{\bar{Z}} = 0$. In addition, all the words in $\text{supp}(\bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi))$ have prefixes belonging to $\bar{Z}^*$. So the only word from $\bar{Z}^*$ that can serve as a prefix in the set $\text{supp}(\tau_1(\eta)) \cap \text{supp}(\bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi))$ is the empty word. Therefore,

$$\varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) = \varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi)).$$

The next step is to show that for all $\eta \in Y^*$ and $\xi \in Z^*$

$$\varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi)) = \eta \sqcup\!\sqcup \xi. \quad (4)$$

This will be proven by induction on the sum of the lengths of η and ξ. Let $\ell := |\eta| + |\xi|$ and suppose that $\ell = 0$, i.e., $\eta = \emptyset$ and $\xi = \emptyset$. Applying the definition of the Hadamard product and φ, it follows directly that

$$\begin{aligned} \varphi(\tau_1(\emptyset) \odot \mathbf{Y}^*\tau_2(\emptyset)) &= \varphi(\bar{\bar{\mathbf{Z}}}^* \odot \mathbf{Y}^*) = \varphi(\emptyset) \\ &= \emptyset = \emptyset \sqcup\!\sqcup \emptyset. \end{aligned}$$

Now assume identity (4) is true up to some $\ell = n+p-1 \geq 0$, and consider the words $\eta' = y_{i_{n-1}} y_{i_{n-2}} \ldots y_{i_1} \in Y^*$ and $\xi' = z_{i_{p-1}} z_{i_{p-2}} \ldots z_{i_1} \in Z^*$. Note that if $Y = \{y_1, \ldots, y_k\}$ and $\bar{\bar{Z}} = \{\bar{\bar{z}}_1, \bar{\bar{z}}_2, \ldots, \bar{\bar{z}}_l\}$ then

$$\begin{aligned} \mathbf{Y}^* &= \emptyset + y_1\mathbf{Y}^* + \cdots + y_k\mathbf{Y}^* \\ \bar{\bar{\mathbf{Z}}}^* &= \emptyset + \bar{\bar{z}}_1\bar{\bar{\mathbf{Z}}}^* + \cdots + \bar{\bar{z}}_l\bar{\bar{\mathbf{Z}}}^*. \end{aligned}$$

In which case, if $\eta = y_{i_n}\eta'$ and $\xi = z_{i_p}\xi'$ then

$$\begin{aligned} &\varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi)) \\ &= \varphi(\bar{\bar{\mathbf{Z}}}^*(y_{i_n}\bar{\bar{\mathbf{Z}}}^*)(y_{i_{n-1}}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) \odot \\ &\qquad \mathbf{Y}^*(\bar{\bar{z}}_{i_p}\mathbf{Y}^*)(\bar{\bar{z}}_{i_{p-1}}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*)) \\ &= \varphi(y_{i_n}(\bar{\bar{\mathbf{Z}}}^*(y_{i_{n-1}}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) \odot \\ &\qquad \mathbf{Y}^*(\bar{\bar{z}}_{i_p}\mathbf{Y}^*)(\bar{\bar{z}}_{i_{p-1}}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*)) + \\ &\qquad \bar{\bar{z}}_{i_p}(\bar{\bar{\mathbf{Z}}}^*(y_{i_n}\bar{\bar{\mathbf{Z}}}^*)(y_{i_{n-1}}\bar{\bar{\mathbf{Z}}}^*)\cdots(y_{i_1}\bar{\bar{\mathbf{Z}}}^*) \odot \\ &\qquad \mathbf{Y}^*(\bar{\bar{z}}_{i_{p-1}}\mathbf{Y}^*)\cdots(\bar{\bar{z}}_{i_1}\mathbf{Y}^*))) \\ &= y_{i_n}\varphi(\tau_1(\eta') \odot \mathbf{Y}^*\tau_2(\xi)) + z_{i_p}\varphi(\tau_1(\eta) \odot \mathbf{Y}^*\tau_2(\xi')) \\ &= y_{i_n}(\eta' \sqcup\!\sqcup \xi) + z_{i_p}(\eta \sqcup\!\sqcup \xi') \\ &= \eta \sqcup\!\sqcup \xi. \end{aligned}$$

Hence, identity (4) is proved for words of arbitrary length. Observe that since $\eta \in Y^*$ then $\eta \sqcup\!\sqcup_Y \xi = \eta \sqcup\!\sqcup \xi$, which implies that identity (3) is also proved.

Case 2: If $\eta = \nu\eta'$, where $\nu \in Z^*Y'$ and $\eta' \in Y^*$, then from the definition of τ_1 it follows that

$$\varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) = \varphi(\bar{\nu}\tau_1(\eta') \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)).$$

Applying φ and using the definition of the Hadamard product, it is clear that

$$\varphi(\bar{\nu}\tau_1(\eta') \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) = \nu\varphi(\tau_1(\eta') \odot \mathbf{Y}^*\tau_2(\xi)).$$

The remainder of the proof for identity (3) is now identical to that of Case 1.

To extend identity (3) to rational $c, d \in \mathbb{R}\langle\langle Z \rangle\rangle$, observe from the fact that φ is a semiring epimorphism which induces the identity over $\mathbb{R}$, one can write

$$\begin{aligned} &c \sqcup\!\sqcup_Y d \\ &= \sum_{\eta,\xi \in Z^*} (c,\eta)(d,\xi)\eta \sqcup\!\sqcup_Y \xi \\ &= \sum_{\eta,\xi \in Z^*} (c,\eta)(d,\xi)\varphi(\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(\xi)) \\ &= \varphi\left(\sum_{\eta,\xi \in Z^*} (c,\eta)\tau_1(\eta) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*(d,\xi)\tau_2(\xi)\right) \\ &= \varphi\left(\left(\sum_{\eta \in Z^*} (c,\eta)\tau_1(\eta)\right) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\left(\sum_{\xi \in Z^*} (d,\xi)\tau_2(\xi)\right)\right) \\ &= \varphi(\tau_1(c) \odot \bar{\mathbf{Z}}^*\mathbf{Y}^*\tau_2(d)). \end{aligned}$$

Finally, since the Hadamard product preserves rationality [1], and the transductions φ, τ_1 and τ_2 are rational (cf. Lemma 1), the restricted shuffle product also preserves rationality. ■

IV. Proof of Proposition 1

For brevity, only the single-input, single-output case is considered. Hence, the underlying alphabet is $X = \{x_0, x_1\}$. If c is input-limited, then there exists an $\mathcal{N}_c \in \mathbb{N}$ such that

$$\text{supp}(c) \subset \bigcup_{j=0}^{\mathcal{N}_c} x_0^*(x_1x_0^*)^j.$$

Let c_j be the restriction of c to $x_0^*(x_1x_0^*)^j$, that is, $c_j = c \odot x_0^*(x_1x_0^*)^j$ for $j = 0, \ldots, \mathcal{N}_c$. (No notational distinction is made here between the language $x_0^*(x_1x_0^*)^j$ and its characteristic series.) If c is rational, it is easy to see that c_j is also rational since $x_0^*(x_1x_0^*)^j$ is trivially rational and the Hadamard product is a rational operation. The proof thus reduces to verifying the rationality of $c \circ d = \sum_{j=0}^{\mathcal{N}_c} c_j \circ d$ by showing that each $c_j \circ d$ term is rational when c and d are rational. For a specific j, it is possible to write

$$c_j = \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0} \, x_0^{k_j}x_1 \cdots x_0^{k_1}x_1x_0^{k_0},$$

where for brevity $\alpha_{k_j,\ldots,k_0} := (c, x_0^{k_j}x_1 \cdots x_0^{k_1}x_1x_0^{k_0})$. Introducing the alphabets $Z_0 = \{x_0\}$, $Z_1 = \{x_0, x_1, y_1, y_1'\}$, $Z_2 = \{x_0, x_1, y_1, y_2, y_1', y_2'\}$, $\ldots$,

$Z_j = \{x_0, x_1, y_1, \ldots, y_j, y_1', \ldots, y_j'\}$, define iteratively the series $\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_j$ as :

$$\bar{c}_0 = \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' x_0^{k_0} \qquad (5)$$
$$\bar{c}_i = \bar{c}_{i-1} \sqcup\!\sqcup_{\,z_{i-1}} d, \quad i = 1, 2, \ldots, j.$$

Also let $\psi : \mathbb{R}\langle\langle Z_j \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle$ be the rational semiring epimorphism defined by: $\psi(x_0) = x_0$, $\psi(x_1) = x_1$, and $\psi(y_i) = \psi(y_i') = x_0$, for $i = 1, 2, \ldots, j$ (cf. Theorem 4). The first objective is to show that $c_j \circ d = \psi(\bar{c}_j)$. This can be done by an inductive procedure. From the commutativity of the (normal) shuffle product, observe

$$\begin{aligned} \bar{c}_1 &= \bar{c}_0 \sqcup\!\sqcup_{\,z_0} d \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0} \left[y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' x_0^{k_0} \sqcup\!\sqcup_{\,z_0} d \right] \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' \left[x_0^{k_0} \sqcup\!\sqcup_{\,z_0} d \right] \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' \left[d \sqcup\!\sqcup x_0^{k_0} \right]. \end{aligned}$$

Now, assume that the procedure has been applied up to $\bar{c}_{j-1}$. Then

$$\begin{aligned} \bar{c}_j &= \bar{c}_{j-1} \sqcup\!\sqcup_{\,z_{j-1}} d \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0} \Big[y_j^{k_j} y_j' y_{j-1}^{k_{j-1}} y_{j-1}' \Big[d \sqcup\!\sqcup y_{j-2}^{k_{j-2}} y_{j-2}' \cdot \\ &\qquad \Big[\cdots \Big[d \sqcup\!\sqcup y_1^{k_1} y_1' \Big[d \sqcup\!\sqcup x_0^{k_0} \Big] \cdots \Big] \sqcup\!\sqcup_{\,z_{j-1}} d \Big]\Big]\Big] \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \Big[y_{j-1}^{k_{j-1}} y_{j-1}' \Big[d \sqcup\!\sqcup y_{j-2}^{k_{j-2}} y_{j-2}' \cdot \\ &\qquad \Big[\cdots \Big[d \sqcup\!\sqcup y_1^{k_1} y_1' \Big[d \sqcup\!\sqcup x_0^{k_0} \Big] \cdots \Big] \sqcup\!\sqcup_{\,z_{j-1}} d \Big]\Big]\Big] \\ &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \Big[d \sqcup\!\sqcup y_{j-1}^{k_{j-1}} y_{j-1}' \cdot \\ &\qquad \Big[d \sqcup\!\sqcup y_{j-2}^{k_{j-2}} y_{j-2}' \Big[\cdots \Big[d \sqcup\!\sqcup y_1^{k_1} y_1' \Big[d \sqcup\!\sqcup x_0^{k_0} \Big] \cdots \Big]\Big]\Big]\Big]. \end{aligned}$$

With the form of $\bar{c}_j$ established, apply the epimorphism ψ to $\bar{c}_j$:

$$\begin{aligned} \psi(\bar{c}_j) &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, x_0^{k_j+1} \cdot \\ &\qquad \Big[d \sqcup\!\sqcup x_0^{k_{j-1}+1} \Big[\cdots \Big[d \sqcup\!\sqcup x_0^{k_1+1} \Big[d \sqcup\!\sqcup x_0^{k_0} \Big] \cdots \Big]\Big]\Big]. \end{aligned}$$

But recall from the definition of the composition product, if $\xi = x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0}$ then

$$\xi \circ d = x_0^{k_j+1} \Big[d \sqcup\!\sqcup x_0^{k_{j-1}+1} \Big[\cdots \Big[d \sqcup\!\sqcup x_0^{k_1+1} \Big[d \sqcup\!\sqcup x_0^{k_0} \Big] \cdots \Big]\Big]\Big].$$

Therefore, $c_j \circ d = \psi(\bar{c}_j)$.

The final step is to show that $\bar{c}_j$ is rational. This can be done by showing that $\bar{c}_0$ is rational, since $\bar{c}_j$ is then computed using only rational operations applied to rational series. Let $\tilde{Z}_j = Z_j - \{x_1\}$. Theorem 3 will be used to construct a rational transduction $\delta_j : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle \tilde{Z}_j \rangle\rangle$ such that $\delta_j(c_j) = \bar{c}_0$. Consider the alphabet $W = A \cup A'$, where $A = \{a_0, \ldots, a_j\}$ and $A' = \{a_1', \ldots, a_j'\}$. Define two monoid morphisms ϱ_1 and ϱ_2 in the following way: $\varrho_1(a_0) = x_0$, $\varrho_1(a_i) = x_0$, $\varrho_1(a_i') = x_1$, $\varrho_2(a_0) = x_0$, $\varrho_2(a_i) = y_i$, and $\varrho_2(a_i') = y_i'$, where $i = 1, \ldots, j$. Clearly

$$\varrho_1^{-1}(x_0) = \sum_{k=0}^{j} a_k, \quad \varrho_1^{-1}(x_1) = \sum_{k=1}^{j} a_k'.$$

If $c_W := a_j^* a_j' \cdots a_1^* a_1' a_0^* \in \mathbb{R}\langle\langle W \rangle\rangle$ then

$$\varrho_2(c_W) = y_j^* y_j' \cdots y_1^* y_1' x_0^*.$$

Now observe that if $\tilde{X}^j := \{\xi = x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0} \in X^* : k_i \geq 0,\ i = 0, 1, \ldots, j\}$ then

$$\varrho_1^{-1}(\xi) = \sum_{\substack{\eta_i \in A^{k_i} \\ \ell_i \in \{1,\ldots,j\} \\ i=0,\ldots,j}} \eta_j a_{\ell_j}' \cdots \eta_1 a_{\ell_1}' \eta_0, \quad \forall \xi \in \tilde{X}^j$$

with $a_{\ell_0}' := \emptyset$ (suppressed). By $\mathbb{R}$-linearity

$$\varrho_1^{-1}(c_j) = \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, \varrho_1^{-1}(x_0^{k_j} x_1 \cdots x_0^{k_1} x_1 x_0^{k_0}),$$

and thus, using equation (5):

$$\begin{aligned} \varrho_1^{-1}(c_j) \odot c_W &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, a_j^{k_j} a_j' \cdots a_1^{k_1} a_1' a_0^{k_0} \\ \varrho_2(\varrho_1^{-1}(c_j) \odot c_W) &= \sum_{k_j,\ldots,k_0 \geq 0} \alpha_{k_j,\ldots,k_0}\, y_j^{k_j} y_j' \cdots y_1^{k_1} y_1' x_0^{k_0} \\ &= \bar{c}_0 = \delta_j(c_j). \end{aligned}$$

By design, the transduction $\delta_j : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle \tilde{Z}_j \rangle\rangle : c \mapsto \varrho_2(\varrho_1^{-1}(c) \odot c_W)$ is rational and defined everywhere on $\mathbb{R}\langle\langle X \rangle\rangle$. Therefore, $\bar{c}_0$ is rational, and the sufficient condition of Ferfera is proved.

V. A Generalization of Ferfera's Condition

In this final section, a generalization of Ferfera's sufficient condition for the rationality of the composition product is described in terms of the Hankel rank of a formal power series. This well known concept provides an alternative characterization of a rational series as briefly summarized below [7].

Definition 7: For any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, the $\mathbb{R}$-linear mapping $\mathcal{H}_c : \mathbb{R}\langle X \rangle \mapsto \mathbb{R}^\ell\langle\langle X \rangle\rangle$ on the vector space $\mathbb{R}\langle X \rangle$ uniquely specified by

$$(\mathcal{H}_c(\eta), \xi) = (c, \xi\eta), \quad \forall \xi, \eta \in X^*$$

is called the ***Hankel mapping*** of c.

$\mathcal{H}_c$ has a matrix representation, whose (ξ, η) component is given by $(\mathcal{H}_c)_{\xi,\eta} = (c, \xi\eta)$ for all $\xi, \eta \in X^*$. Its range space, $\mathcal{H}_c(\mathbb{R}\langle X \rangle)$, is an $\mathbb{R}$-vector subspace of $\mathbb{R}^\ell\langle\langle X \rangle\rangle$, which is not necessarily finite dimensional. Consider the following definition and theorem.

Definition 8: The ***Hankel rank*** of $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is $\rho_H(c) = \dim(\mathcal{H}_c(\mathbb{R}\langle X \rangle))$.

Theorem 6: A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is rational if and only if its Hankel rank is finite.

Now consider the multivariable version of the decomposition of c used in the proof of Proposition 1, namely, $c = \sum_{j\geq 0} c_j$, where

$$\mathrm{supp}(c_j) = \left\{ \eta \in X^* : \sum_{i=1}^{m} |\eta|_{x_i} = j \right\}.$$

Define the corresponding partial sum $\tilde{c}_r = \sum_{j=0}^{r} c_j$. Clearly each $\tilde{c}_r$ is input-limited. Thus, it follows from Ferfera's condition, that for any rational $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$, each series in the sequence $\{\tilde{c}_r \circ d\}_{r\geq 0}$ is rational, or equivalently, in light of Theorem 6, $\{\rho_H(\tilde{c}_r \circ d)\}_{r\geq 0}$ is a well-defined sequence of nonnegative integers. In this context, consider the following theorem.

Theorem 7: Let $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ be two rational series. If the sequence $\{\rho_H(\tilde{c}_r \circ d)\}_{r\geq 0}$ has a limit then $c \circ d$ is rational.

Proof: The claim follows directly from Theorem 6 and the following (easily verified) property concerning infinite matrices. Let $\{M_r\}_{r\geq 0}$ be a sequence of doubly infinite matrices with real coefficients. Assume that $\lim_{r\to\infty} M_r = M$ (componentwise in the usual topology). If each M_r has finite rank (meaning that $\rho_H(M_r) := \dim(M_r(\mathbb{R}\langle X \rangle))$ is finite), it is not necessarily the case that M has finite rank (such examples abound). But if the sequence $\{\rho_H(M_r)\}_{r\geq 0}$ has a limit, then it does follow that $\rho_H(M) \leq \lim_{r\to\infty} \rho_H(M_r)$, that is, M must have finite rank. ■

Example 4: Suppose $X = \{x_0, x_1, x_2\}$, and let $c = 1 + (1 - x_1)^{-1} - (1 - x_2)^{-1}$ and $d = [1\ 1]$. Clearly in this case c is not input-limited since $c_0 = 1$ and $c_j = x_1^j - x_2^j$ for all $j \geq 1$. Now observe that $c_0 \circ d = 1$ and $c_j \circ d = 0$ for all $j \geq 1$. Thus, $\tilde{c}_r \circ d = 1$ for all $r \geq 0$, which in turn implies that $\rho_H(\tilde{c}_r \circ d) = 1$ for all $r \geq 0$. From Theorem 7 it then follows that $c \circ d$ must be rational. It is trivial to verify that indeed $c \circ d = 1$. □

If c is input-limited, then obviously $\lim_{r\to\infty} \rho_H(\tilde{c}_r \circ d) = \rho_H(\tilde{c}_r \circ d)|_{r=\mathcal{N}_c} = \rho_H(c \circ d)$. Conversely if $\lim_{r\to\infty} \rho_H(\tilde{c}_r \circ d)$ exists, then there must exist an integer $r^* \geq 0$ such that $\rho_H(\tilde{c}_r \circ d) = \rho_H(c \circ d)$ for all $r \geq r^*$ (which, of course, does not imply that $\tilde{c}_r \circ d = c \circ d$). In general, however, it is possible for $\mathcal{H}_{\tilde{c}_r \circ d} \to \mathcal{H}_{c \circ d}$ as $r \to \infty$, while at the same time the integer sequence $\rho_H(\tilde{c}_r \circ d)$ diverges. In which case, Theorem 7 would clearly not apply.

Example 5: Reconsider Example 2, where $c = (1 - x_1)^{-1}$, $d = 1$, and $c \circ d = (1 - x_0)^{-1}$. As noted earlier, c is not input-limited, but rationality is still preserved, specifically $\rho_H(c \circ d) = 1$. Now observe that $c_j = x_1^j$, $c_j \circ d = x_0^j$ for $j \geq 0$, and thus, $\tilde{c}_r \circ d = \sum_{j=0}^{r} x_0^j$. In which case, $\rho_H(\tilde{c}_r \circ d) = r$ for all $r \geq 0$. This example clearly falls outside the realm of Proposition 1 and its generalization, Theorem 7, even though rationality is in fact preserved. □

It is worth noting in the previous example that if each Hankel matrix $\mathcal{H}_{\tilde{c}_r \circ d}$ is truncated to an $r \times (r + 1)$ matrix, then the resulting matrix always has rank equivalent to that of $\mathcal{H}_{c \circ d}$ for every $r > 0$. This is reminiscent of classical Hankel matrix analysis done for the partial (linear) realization problem [17]. In fact, when $c \circ d$ is not rational, $\tilde{c}_r \circ d$ can be viewed a rational approximation of $c \circ d$, i.e., a type of partial bilinear realization problem or a noncommutative Padé approximation (e.g., see [13], [14]). It therefore seems unlikely that any *finite* test for rationality can be devised by considering only the ranks of truncated Hankel matrices.

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