Automatica 50 (2014) 2381–2388

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Left inversion of analytic nonlinear SISO systems via formal power series methods[☆]

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ARTICLE INFO

Article history:
Received 26 June 2013
Received in revised form
1 February 2014
Accepted 3 May 2014
Available online 13 August 2014

Keywords:
Nonlinear systems
Chen–Fliess series
Feedback linearization
Interconnected systems
Algebraic systems theory

ABSTRACT

Given a single-input, single-output (SISO) system, F, and a function y in the range of F, the left inversion problem is to determine an input u such that $y = F[u]$. The goal of this paper is to provide an exact and explicit analytical solution to this problem in the case where F is an analytic mapping in the sense that it has a convergent Chen–Fliess functional expansion, and y is a real analytic function. In particular, it will be shown that given a certain condition on the generating series c of F, a corresponding unique analytic u can always be determined via operations on formal power series. The condition on c turns out to be equivalent to having a well-defined relative degree when F has an input-affine analytic state space realization with finite dimension. But the method is applicable even when F does *not* have such a realization. The technique is demonstrated on four examples, including a continuous stirred chemical reactor.

1. Introduction

Given a single-input, single-output (SISO) system, F, and a function y in the range of F, the left inversion problem is to determine an input u such that $y = F[u]$. Some version of this problem is implicit in virtually all output tracking and motion planning algorithms. The existence of a local left (and right) inverse *operator* in a Volterra series setting is assured when F has an invertible linear part (Boyd, Chua, & Desoer, 1984; Rugh, 1986). But this is a stronger definition and condition than what will be considered here. In the event that F has a smooth input-affine state space realization with finite dimension n, it is known that a well defined relative degree at a point $z_0 \in \mathbb{R}^n$ is a sufficient (but not necessary) condition for solving the left inversion problem at z_0 (Respondek, 1990; Tanwani & Liberzon, 2010) and on a neighborhood of z_0 (Hirschorn, 1979). In such a context, this is accomplished via the method of feedback linearization (Isidori, 1995; Krener, 1999), which is a type of dynamic system inversion (Getz, 1995). The problem is significantly simplified when the system has *full* relative degree, i.e., is differentially flat (Fliess, Lévine, Martin, & Rouchon, 1995). The goal of this paper is to provide an exact and explicit analytical solution to this problem in the case where F is an analytic mapping in the sense that it has a convergent Chen–Fliess functional expansion (Fliess, 1981, 1983), and where y is a real analytic function. In which case, the Fliess operator F is completely characterized by a noncommutative formal power series c. In particular, it will be shown that given a certain condition on c, a corresponding unique analytic u can be computed by an explicit formula involving operations on formal power series. The condition on c turns out to be equivalent to having a well-defined relative degree when c has an input-affine analytic state space realization. The method, however, is applicable even when c does *not* have such a realization, and thus, it is coordinate free. The derivation relies heavily on recent advances in the analysis of interconnected nonlinear systems, in particular, the tools presented in Gray and Duffaut Espinosa (2011, in press) which exploit the underlying combinatorial structures found in feedback systems via the theory of Hopf algebras. Recent advances in algorithms for computing the antipodes of such algebras has decreased the computation time by more than an order of magnitude (Gray, Duffaut Espinosa, & Ebrahimi-Fard, 2014). Thus,

☆ The second author was supported by the Australian Research Council and the Air Force Office of Scientific Research under agreement FA2386-09-1-4089. Prof. Helton's participation was supported by National Science Foundation grants DMS 0757212 and DMS 0700758. The material in this paper was partially presented at the 15th Latin American Control Conference (CLCA 2012), October 23–26, 2012, Lima, Perú. This paper was recommended for publication in revised form by Associate Editor Alessandro Astolfi under the direction of Editor Andrew R. Teel.

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http://dx.doi.org/10.1016/j.automatica.2014.07.017
0005-1098/

the method presented here has the potential to explicitly and efficiently compute high precision system inverses for complex systems. It can also be viewed as a stepping stone to multivariable inversion in a purely input–output setting.

The paper is organized as follows. Some preliminaries concerning Fliess operators and their interconnections are briefly reviewed in the next section. However, Section 2.3, addressing the quotient connection, is entirely new material motivated by the system inversion problem. Section 3 describes the main result of the paper, a nonlinear system inversion formula. The paper's conclusions are given in the final section. It should be noted that this paper is an expanded version of Gray, Thitsa, and Duffaut Espinosa (2012). Here a general convergence analysis theory is presented and employed in all the examples. In addition, the uniqueness argument concerning the left inverse in Theorem 12 is presented, and an application involving temperature control in an exothermic chemical reactor is given in Example 16 (Doyle & Henson, 1997; Uppal, Ray, & Poore, 1974).

2. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Define $X^+ = X^* - \{\emptyset\}$. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. If $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. Typically, c is represented as the formal sum $c = \sum_{\eta\in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product, namely, $cd := \sum_{\eta\in X^*}(cd, \eta)\eta$, where $(cd, \eta) = \sum_{\eta=\xi\nu} (c, \xi)(d, \nu)$, and the product $(c, \xi)(d, \nu)$ is computed componentwise in $\mathbb{R}^\ell$. In addition, $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by the Cyrillic letter *sha* ($\sqcup\!\sqcup$). The shuffle product on $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is derived from the componentwise $\mathbb{R}$-bilinear extension of the shuffle product of two words, which is defined inductively by

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi)$$

with $\eta \sqcup\!\sqcup \emptyset = \eta$ for all $\eta, \xi \in X^*$ and $x_i, x_j \in X$ (Fliess, 1981).

2.1. Fliess operators and their convergence

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \ge 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \le i \le m\}$, where $u = [u_1\ u_2 \cdots u_m]^T$ and $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \le R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input–output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta\in X^*} (c, \eta) E_\eta[u](t, t_0) \tag{1}$$

(Fliess, 1981, 1983). If there exists real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \le K_c M_c^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{2}$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}+1/\mathfrak{q} = 1$ (Gray & Wang, 2002). (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. In particular, when $\mathfrak{p} = 1$, the series (1) converges absolutely and uniformly if $\max\{R, T\} < 1/M_c(m + 1)$ (Duffaut Espinosa, 2009; Duffaut Espinosa, Gray, & González, 2009). It is important in applications to identify the *smallest* possible geometric growth constant, M_c, in order to avoid over restricting the domain of F_c. So let $\pi : \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}^+ \cup \{0\}$ take each series c to the infimum of all M_c satisfying (2). Therefore, $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ can be partitioned into equivalence classes, and the number $1/M_c(m+1)$ will be referred to as the *radius of convergence* for the class $\pi^{-1}(M_c)$. This is in contrast to the usual situation where a radius of convergence is assigned to individual series. That is, for a specific series in a given class not all the coefficients need to grow at the maximum rate. In which case, the radius of convergence for the class acts only as a lower bound on the radius of convergence for the series. It can be very conservative, for example, when many of the coefficients of the series are zero. When c satisfies the more stringent growth condition $|(c, \eta)| \le K_c M_c^{|\eta|}$, $\eta \in X^*$, the series (1) defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $C[t_0, \infty)$, where

$$L^m_{\mathfrak{p},e}(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_\mathfrak{p}[t_0, t_1], \forall t_1 \in (t_0, \infty)\},$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ (Gray & Wang, 2002). The set of all such *globally convergent* series is designated by $\mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$.

2.2. Elementary system interconnections

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_cF_d = F_{c \sqcup\!\sqcup d}$, respectively (Fliess, 1981). When $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the cascade connection satisfies $F_c \circ F_d = F_{c\circ d}$, where $c \circ d$ denotes the *composition product* (Ferfera, 1979, 1980). To describe this product, it is convenient to first define a family of mappings $D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup e)$, where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Let $D_\emptyset$ be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_i}D_{x_j} := D_{x_ix_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the catenation product.

Definition 1. (Ferfera, 1979, 1980) The **composition product** of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as $\eta \circ d = D_\eta(1)$. For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, $c \circ d := \sum_{\eta\in X^*} (c, \eta)\eta \circ d$.

The composition product is associative and distributes to the left over the shuffle product. While the composition product by design is $\mathbb{R}$-linear in its left argument c, it is linear in its right argument if and only if its left argument is a *linear series*, that is, $\mathrm{supp}(c) \subseteq L$, where $L := \{\eta \in X^* : \eta = x_0^{n_1}x_ix_0^{n_0},\ i \ne 0,\ n_j \ge 0\}$ is the set of *linear words*. In addition, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ under the ultrametric dist : $(c, d) \mapsto \sigma^{\mathrm{ord}(c-d)}$ (Ferfera, 1979; Gray & Li, 2005). Here σ is any real number $0 < \sigma < 1$, and the *order* of a series c, $\mathrm{ord}(c)$, is taken as the length of the smallest word in the support of c.

In the event that two Fliess operators are interconnected to form a feedback system as shown in Fig. 1, the output y must satisfy the feedback equation $y = F_c[v] = F_c[u + F_d[y]]$ for every admissible input u. It was shown by Gray and Li (2005) that there

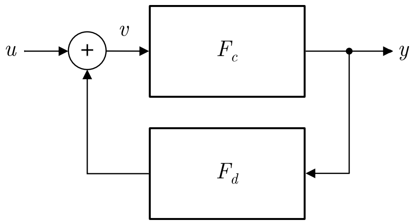

Fig. 1. Feedback connection.

always exists a unique generating series e so that $y = F_e[u]$ whenever $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. In which case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d\circ e}[u]] = F_{c\tilde{\circ}(d\circ e)}[u],$$

where $\tilde{\circ}$ denotes the *modified* composition product, that is, the product $c\tilde{\circ}d = \sum_{\eta\in X^*}(c,\eta)\eta\tilde{\circ}\, d$, where $\eta\tilde{\circ}\, d = \tilde{D}_\eta(1)$ with $\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup e)$ for each letter $x_i \in X$ and $d_0 := 0$. The mapping $d \mapsto c\tilde{\circ}\, d$ is also an ultrametric contraction (Gray & Li, 2005). Therefore, the *feedback product* of c and d, namely $c@d$, is defined as the unique fixed point of the contractive iterated map $\tilde{S} : e_i \mapsto e_{i+1} = c\tilde{\circ}(d \circ e_i)$. Specifically, $c@d = e$, where $e = c\tilde{\circ}(d\circ e)$. Given arbitrary c and d, the feedback product can be computed explicitly using the theory of Hopf algebras (Gray & Duffaut Espinosa, 2011, in press). Henceforth, the focus will be exclusively on the SISO case, that is, $X = \{x_0, x_1\}$. Define the set of operators $\mathscr{F}_\delta = \{I + F_c \;:\; c \in \mathbb{R}\langle\langle X\rangle\rangle\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. In which case, $c\tilde{\circ}\, d = c \circ (\delta + d)$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta\rangle\rangle$. The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group of operators under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c\tilde{\circ}\, d$. In which case, the corresponding group of generating series $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ has a Faà di Bruno type Hopf algebra with antipode, α, satisfying

$$c_\delta^{\circ-1} = \delta + c^{\circ-1} = \delta + \sum_{\eta\in X^*}(\alpha a_\eta)(c)\eta,$$

where $c^{\circ-1}$ denotes the composition inverse of c, $a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta)$ is the coordinate function for $\eta \in X^*$, and $a_\delta(c_\delta) := 1$.[2] The antipode can be computed systematically by either a series expansion or via a matrix representation of the group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ (Gray & Duffaut Espinosa, in press). The first few antipode terms are:

$$\alpha 1 = 1 \tag{3a}$$
$$\alpha a_\emptyset = -a_\emptyset \tag{3b}$$
$$\alpha a_{x_0} = -a_{x_0} + a_\emptyset a_{x_1} \tag{3c}$$
$$\alpha a_{x_1} = -a_{x_1} \tag{3d}$$
$$\alpha a_{x_0^2} = -a_{x_0^2} + a_\emptyset a_{x_0x_1} + a_{x_0}a_{x_1} + a_\emptyset a_{x_1x_0} - a_\emptyset a_{x_1}^2 - a_\emptyset^2 a_{x_1^2} \tag{3e}$$
$$\alpha a_{x_0x_1} = -a_{x_0x_1} + a_{x_1}^2 + a_\emptyset a_{x_1^2} \tag{3f}$$
$$\alpha a_{x_1x_0} = -a_{x_1x_0} + a_\emptyset a_{x_1^2} \tag{3g}$$
$$\alpha a_{x_1^2} = -a_{x_1^2}. \tag{3h}$$

The following theorem holds.

[2] A bialgebra, H, is vector space with a product and coproduct, Δ, satisfying $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a, b \in H$. A Hopf algebra is a bialgebra with an antipode, i.e., a linear mapping $\alpha : H \to H$ satisfying $\alpha(ab) = \alpha(b)\alpha(a)$.

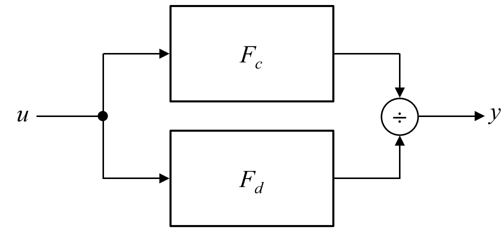

Fig. 2. Quotient connection.

Theorem 2. *For any $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ it follows that $c@d = c\tilde{\circ}(-d \circ c)^{\circ-1} = c \circ (\delta - d \circ c)^{\circ-1}$.*

Finally, a few comments about the convergence of interconnected systems. It is known that the parallel, product, cascade and feedback connections all preserve local convergence, and in fact their radius of convergence is known in each case (Gray & Duffaut Espinosa, 2011; Gray & Li, 2005; Thitsa, 2011; Thitsa & Gray, 2012). On the other hand, only the parallel and product connections preserve global convergence (Gray, Herencia-Zapana, Duffaut Espinosa, & González, 2009). But it is known that if the subsystems of a cascade or feedback connection are globally convergent, the radius of convergence of the locally convergent composite system will generally be larger than if the subsystems were only locally convergent (Thitsa, 2011; Thitsa & Gray, 2012). Finally, one can verify that $c@\delta = (-c)^{\circ-1}$. Thus, the radius of convergence for the composition inverse can be computed from known formulas applicable to unity feedback systems (Gray & Duffaut Espinosa, in press).

2.3. Quotient connection

In addition to the elementary system interconnections described above, the quotient interconnection shown in Fig. 2 will be needed to solve the system inversion problem. Its corresponding generating series is described in terms of the *shuffle group*. The following theorem appears to be known (Bacher, 2007, 2010), but no explicit proof is readily available to the authors' knowledge.

Theorem 3. *The set of non proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ is a group under the shuffle product. In particular, the shuffle inverse of any such series c is*

$$c^{\sqcup\!\sqcup -1} = ((c,\emptyset)(1 - c'))^{\sqcup\!\sqcup -1} = (c,\emptyset)^{-1}(c')^{\sqcup\!\sqcup *}$$

where $c' = 1 - c/(c,\emptyset)$ is proper and $(c')^{\sqcup\!\sqcup *} := \sum_{k\geq 0}(c')^{\sqcup\!\sqcup k}$.

Proof. Since the set of non proper series is a subalgebra of the shuffle algebra, it only needs to be shown that $c^{\sqcup\!\sqcup -1}$ is the shuffle inverse of any non proper series c. Since c' as defined above is proper, it is easy to verify that $\mathrm{ord}\left((c')^{\sqcup\!\sqcup k}\right) \geq k \geq 0$. In which case, the set of series $\{(c')^{\sqcup\!\sqcup k} : k \geq 0\}$ is locally finite, and therefore, summable (Berstel & Reutenauer, 1988). Hence, the series $(c')^{\sqcup\!\sqcup *}$ is well defined. It is clear that $c^{\sqcup\!\sqcup -1}$ is non proper, and the bilinearity of the shuffle product gives immediately that $c \sqcup\!\sqcup c^{\sqcup\!\sqcup -1} = c^{\sqcup\!\sqcup -1} \sqcup\!\sqcup c = 1$. In fact, since the shuffle algebra is an integral domain, i.e., $c \sqcup\!\sqcup d = 0$ if and only if c and/or d is zero (Wang, 1990), $c^{\sqcup\!\sqcup -1}$ is the only series with this property.

The next theorem states that local convergence is preserved under the shuffle inverse and provides the radius of convergence for the resulting series. The following lemma is needed for the proof.

Lemma 4. *For any proper $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with minimal growth constants $K_c, M_c > 0$ it follows that*

$$|(c^{\sqcup\!\sqcup *}, \eta)| \leq K\,((K_c + 1)M_c)^{|\eta|}\,|\eta|!, \quad \eta \in X^*, \tag{4}$$

for some $K > 0$. Furthermore, no geometric growth constant smaller than $(K_c + 1)M_c$ can satisfy (4).

Proof. Consider the proper series

$$\bar{c} = \sum_{\eta \in X^+} K_c M_c^{|\eta|} \, |\eta|! \eta = K_c \sum_{k=1}^{\infty} [M_c(x_0 + x_1)]^{\sqcup\!\sqcup\, k}$$
$$= K_c[(M_c(x_0 + x_1))^{\sqcup\!\sqcup\, *} - 1].$$

Since $|(c, \eta)| \leq (\bar{c}, \eta)$ for all $\eta \in X^*$ it follows that

$$\left|(c^{\sqcup\!\sqcup\, *}, \eta)\right| \leq \sum_{k=0}^{\infty} \left|(c^{\sqcup\!\sqcup\, k}, \eta)\right| \leq \sum_{k=0}^{\infty} (\bar{c}^{\sqcup\!\sqcup\, k}, \eta) = (\bar{c}^{\sqcup\!\sqcup\, *}, \eta). \quad (5)$$

The goal now is to construct a state space realization for F_d, where $d = \bar{c}^{\sqcup\!\sqcup\, *}$. Letting $z = F_{x_0+x_1}$ observe that $F_{\bar{c}} = K_c \sum_{k\geq 1} M_c^k F_{x_0+x_1}^k = K_c M_c z/(1 - M_c z)$. Thus,

$$F_d = \sum_{k=0}^{\infty} F_{\bar{c}}^k = \frac{1}{1 - F_{\bar{c}}} = \frac{1 - M_c z}{1 - (K_c + 1) M_c z}$$

has the realization

$$\dot{z} = 1 + u, \qquad z(0) = 0, \qquad h(z) = \frac{1 - M_c z}{1 - (K_c + 1) M_c z}. \quad (6)$$

Now applying the theorems of Wilf and Pringsheim (see Thitsa and Gray (2012)), it follows that the minimal geometric growth constant of d is $M = 1/t^*$, where $t^* > 0$ is the singularity nearest $t = 0$ of the zero-input response of (6), namely, $t^* = 1/(K_c+1)M_c$. Finally, note that the upper bound in (5) is achieved when $c = \bar{c}$. Thus, the theorem is proved.

Theorem 5. *For any non proper $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ with minimal growth constants $K_c, M_c > 0$ it follows that*

$$|(c^{\sqcup\!\sqcup\, -1}, \eta)| \leq K \left(\left(\frac{K_c}{|(c, \emptyset)|} + 1 \right) M_c \right)^{|\eta|} |\eta|!, \quad \eta \in X^*, \quad (7)$$

for some $K > 0$. Furthermore, no geometric growth constant smaller than $(K_c/|(c, \emptyset)| + 1)M_c$ can satisfy (7). Therefore, the radius of convergence of $c^{\sqcup\!\sqcup\, -1}$ is $1/2M_c(K_c/|(c, \emptyset)| + 1)$.

Proof. From Lemma 4 it follows directly for any $\eta \in X^*$ that

$$\left|(c^{\sqcup\!\sqcup\, -1}, \eta)\right| = \frac{\left|((c')^{\sqcup\!\sqcup\, *}, \eta)\right|}{|(c, \emptyset)|} \leq \frac{K'}{|(c, \emptyset)|} ((K_{c'} + 1) M_{c'})^{|\eta|} |\eta|!.$$

Inequality (7) follows from the fact that $K_{c'} = K_c/|(c, \emptyset)|$ and $M_{c'} = M_c$.

The following example demonstrates that global convergence is *not* preserved under the shuffle inverse. The radius of convergence when c is globally convergent is computed in the subsequent theorem.

Example 6. The polynomial $c = 1 - (x_0 + x_1)$ is clearly globally convergent. But the series $c^{\sqcup\!\sqcup\, -1} = \sum_{k\geq 0} (x_0 + x_1)^{\sqcup\!\sqcup\, k} = \sum_{\eta \in X^*} |\eta|! \eta$ is only locally convergent.

Theorem 7. *For any non proper $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ it follows that*

$$|(c^{\sqcup\!\sqcup\, -1}, \eta)| \leq K \left(\frac{M_c}{\ln\left(1 + \frac{|(c,\emptyset)|}{K_c}\right)} \right)^{|\eta|} |\eta|!, \quad \eta \in X^*, \quad (8)$$

for some $K > 0$. Furthermore, no geometric growth constant smaller than $(M_c/\ln(1 + |(c, \emptyset)|/K_c))$ can satisfy (8). Therefore, the radius of convergence of $c^{\sqcup\!\sqcup\, -1}$ is $\ln(1 + |(c, \emptyset)|/K_c)/2M_c$.

Table 1
Radii of convergence for operations needed to compute the SISO left inverse of F_c.

Operation	$c, d \in \mathbb{R}_{LC}^{\ell}\langle\langle X \rangle\rangle$	$c, d \in \mathbb{R}_{GC}^{\ell}\langle\langle X \rangle\rangle$
$c + d$	$\frac{1}{2\max\{M_c, M_d\}}$	∞ (GC)
$c \sqcup\!\sqcup\, d$	$\frac{1}{2\max\{M_c, M_d\}}$	∞ (GC)
$c^{\sqcup\!\sqcup\, -1}$	$\frac{1}{2M_c\left(\frac{K_c}{\lvert(c,\emptyset)\rvert}+1\right)}$	$\frac{1}{2M_c} \ln\left(1 + \frac{\lvert(c,\emptyset)\rvert}{K_c}\right)$
$c^{\circ -1}$	$\frac{1}{2M_c}\left[1 - K_c \ln\left(1 + \frac{1}{K_c}\right)\right]$	$\frac{1}{2M_c} \ln\left(1 + \frac{1}{K_c}\right)$

Proof. The proof is similar to that of Lemma 4 and Theorem 5 except here $d = \bar{c}^{\sqcup\!\sqcup\, *}$ with

$$\bar{c} = \sum_{\eta \in X^+} K_c M_c^{|\eta|} \eta = K_c \sum_{k=1}^{\infty} [M_c(x_0 + x_1)]^{\sqcup\!\sqcup\, k} \frac{1}{k!}.$$

Thus, the corresponding realization for F_d is

$$\dot{z} = 1 + u, \qquad z(0) = 0, \qquad h(z) = \frac{1}{1 - K_c(e^{M_c z} - 1)}. \quad (9)$$

Again the singularity t^* of the zero-input response of (9) gives the minimal geometric growth constant as expressed in (8) after K_c is replaced by $K_c/|(c, \emptyset)|$. So the theorem is proved.

When $K_c \approx |(c, \emptyset)|$ the radius of convergence of $c^{\sqcup\!\sqcup\, -1}$ for a locally convergent c is about $1/4M_c$ as compared to $\ln(\sqrt{2})/M_c = 0.347M_c$ when c is globally convergent. Hence, the difference between the two cases is minuscule when the same K_c applies. In light of the expansion about $K_c/|(c, \emptyset)| = \infty$,

$$\frac{1}{\ln\left(1 + \frac{|(c,\emptyset)|}{K_c}\right)} = \frac{1}{2} + \frac{K_c}{|(c, \emptyset)|} + O\left(\frac{|(c, \emptyset)|}{K_c}\right),$$

this difference is further diminished when $K_c \gg |(c, \emptyset)|$. The main results are summarized in Table 1 in addition to some other radii of convergence formulas useful in the forthcoming examples.

The most important theorem of this subsection is given below.

Theorem 8. *For $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$, the quotient connection has a Fliess operator representation if and only if d is non proper. In particular, $F_c/F_d = F_{c/d}$, where $c/d := c \sqcup\!\sqcup\, d^{\sqcup\!\sqcup\, -1}$. In addition, the quotient c/d preserves local convergence.*

Proof. Observe that at least formally

$$\frac{F_c}{F_d} = \frac{F_c}{F_{(d,\emptyset)(1-d')}} = F_c \left((d, \emptyset)^{-1} \sum_{k=0}^{\infty} F_{d'}^k \right)$$
$$= F_c \left((d, \emptyset)^{-1} \sum_{k=0}^{\infty} F_{(d')^{\sqcup\!\sqcup\, k}} \right) = F_{c \sqcup\!\sqcup\, d^{\sqcup\!\sqcup\, -1}}.$$

From Theorem 3, if d is not proper then F_c/F_d has a well-defined generating series. On the other hand, if F_c/F_d has a well-defined Fliess operator representation then $F_d[u](t) \neq 0$ on some interval $[t_0, t_0+T)$, $T > 0$. So, in particular, $F_d[u](t_0) = (d, \emptyset) \neq 0$. Hence, d must be non proper. Finally, the convergence claim follows directly from Theorem 5 and the fact that the shuffle product preserves local convergence.

3. Left system inversion

It was shown by Wang (1990) that F_c will map every input which is analytic at t_0 to an output which is also analytic at t_0 provided $c \in \mathbb{R}_{LC}^{\ell}\langle\langle X \rangle\rangle$. In this section, the problem of computing a left inverse of F_c is considered given a real analytic function in its range. The treatment is again restricted to the SISO case and without loss of generality assume $t_0 = 0$. Note that every $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$. The following definition will provide a sufficient condition under which the left inverse of F_c exists.

Definition 9. Given $c \in \mathbb{R}\langle\langle X\rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1}X^*$. Then c has **relative degree** r if the linear word $x_0^{r-1}x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

Equivalently, c has relative degree r if $(c, x_0^k x_1\eta) = 0$ for all $0 \leq k < r-1$ and $\eta \in X^*$ and $(c, x_0^{r-1}x_1) \neq 0$. Now define for any $x_i \in X$, the left-shift operator, $x_i^{-1}(\cdot)$, on X^* by $x_i^{-1}(x_i\eta) = \eta$ with $\eta \in X^*$ and zero otherwise. Higher order shifts are defined inductively via $(x_i\xi)^{-1}(\cdot) = \xi^{-1}x_i^{-1}(\cdot)$, where $\xi \in X^*$. The left-shift operator is assumed to act linearly on $\mathbb{R}\langle\langle X\rangle\rangle$. The next lemma is trivial to prove.

Lemma 10. *If $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r then $(x_0^{r-1}x_1)^{-1}(c)$ is non proper.*

Observe that this notion of relative degree coincides with the usual definition given in a state space setting (Isidori, 1995). Specifically, in light of the identity $\dot{y} = F_{x_0^{-1}(c)}[u] + uF_{x_1^{-1}(c)}[u]$, it follows that

$$y = F_c[u] \tag{10a}$$

$$y^{(1)} = F_{x_0^{-1}(c)}[u] \tag{10b}$$

$$\vdots$$

$$y^{(r-1)} = F_{(x_0^{r-1})^{-1}(c)}[u] \tag{10c}$$

$$y^{(r)} = F_{(x_0^{r})^{-1}(c)}[u] + uF_{(x_0^{r-1}x_1)^{-1}(c)}[u]. \tag{10d}$$

From Lemma 10, $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](0) = (c, x_0^{r-1}x_1) \neq 0$ for any admissible u, and furthermore $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](t) \neq 0$ over some interval $[0, T)$, $T > 0$. Hence, any corresponding state space system will have relative degree r in the classical sense. On the other hand, if the word $x_0^{r-1}x_1 \notin \mathrm{supp}(c)$ then $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](0) = 0$ and $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](t) \neq 0$ for some $t > 0$ and input u. This implies that any corresponding state space system does not have a well defined relative degree.

Example 11. Consider the state space system in Example 4.1.5 of Isidori (1995):

$$\dot{z} = \begin{bmatrix} z_1z_2 - z_1^3 \\ z_1 \\ -z_3 \\ z_1^2 + z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2+2z_3 \\ 1 \\ 0 \end{bmatrix} u, \qquad y = z_4.$$

The system is known by the conventional definition to have relative degree $r = 2$ at any point where $z_3(0) \neq -1$, and r is undefined otherwise. It can be verified by computing iterated Lie derivatives of the output function with respect to the vector fields defining the state equation that the corresponding generating series is

$$c = 2x_0x_1 + 2x_0x_1^2 - 2x_0x_1x_0x_1 + 2x_0x_1x_0^2x_1 - \cdots$$

when $z(0) = 0$. In this case, $\mathrm{supp}(c_F) = \mathrm{supp}(c) \subset x_0X^*$, and the word $x_0x_1 \in \mathrm{supp}(c)$. On the other hand, if $z(0) = [0\,0\,-1\,0]^T$ then

$$c = 2x_0x_1x_0 + 2x_0x_1^2 - 2x_0x_1x_0^2 - 2x_0x_1x_0x_1 + \cdots$$

Here also $\mathrm{supp}(c_F) = \mathrm{supp}(c) \subset x_0X^*$, but the word $x_0x_1 \notin \mathrm{supp}(c)$. So both cases are consistent with Definition 9.

The next theorem is the main result of the paper. Here $X_0 := \{x_0\}$, and $\mathbb{R}[[X_0]]$ denotes the set of all commutative series over X_0. When $c \in \mathbb{R}[[X_0]]$, $F_c[u](t)$ reduces to the Taylor series $\sum_{k\geq 0}(c, x_0^k)E_{x_0^k}[u](t) = \sum_{k\geq 0}(c, x_0^k)t^k/k!$.

Theorem 12. *Suppose $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r. Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}_{LC}[[X_0]]$ satisfying $(c_y, x_0^k) = (c, x_0^k)$, $k = 0, \ldots, r-1$. Then the input*

$$u(t) = \sum_{k=0}^{\infty}(c_u, x_0^k)\frac{t^k}{k!}, \tag{11}$$

where

$$c_u = \left(\left(\frac{(x_0^r)^{-1}(c - c_y)}{(x_0^{r-1}x_1)^{-1}(c)}\right)^{\circ -1}\right)_N, \tag{12}$$

is the unique analytic solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$.

Proof. First observe from (10a)–(10c) that $(c_y, x_0^k) = y^{(k)}(0) = (c, x_0^k)$ for $k = 0, 1, \ldots, r-1$ due to the assumption that c has relative degree r. In effect, this is describing a set of analytic functions in the range of F_c. The relative degree assumption also ensures that the quotient $1/F_{(x_0^{r-1}x_1)^{-1}(c)}$ is well defined. If the feedback

$$u = \frac{v - F_{(x_0^r)^{-1}(c)}[u]}{F_{(x_0^{r-1}x_1)^{-1}(c)}[u]}$$

is applied, it is clear from (10d) that $y^{(r)} = v$. Noting that $v = F_{(x_0^r)^{-1}(c_y)}[u]$, it follows directly that

$$u = \frac{-F_{(x_0^r)^{-1}(c-c_y)}[u]}{F_{(x_0^{r-1}x_1)^{-1}(c)}[u]} = -F_d[u],$$

where

$$d = \frac{(x_0^r)^{-1}(c - c_y)}{(x_0^{r-1}x_1)^{-1}(c)}. \tag{13}$$

This implies that $u + F_d[u] = 0$, and taking the composition inverse gives $u = F_{d^{\circ -1}}[0] = F_{(d^{\circ -1})_N}[0]$. This last expression produces exactly (11)–(12). It is known in the state space setting that u as given by (11)–(12) is the *unique* solution of $y = F_c[u]$ (Isidori, 1995). If a state space realization is not available, then uniqueness comes from the analyticity assumptions. Specifically, if there exists two analytic inputs u and $\tilde{u}$ such that $y = F_c[u] = F_c[\tilde{u}]$ then from (10d), the identity $F_{\eta^{-1}(c)}[u](0) = (c, \eta)$ for any u and $\eta \in X^*$, and the relative degree assumption, it follows that $u(0) = \tilde{u}(0)$. A similar argument applied to $y^{(r+1)}$ gives $u^{(1)}(0) = \tilde{u}^{(1)}(0)$. A straightforward induction proves the general case.

Recall that in the usual state space tracking problem via feedback linearization, *every* analytic output is in the range of some Fliess operator induced by a suitable set of initial conditions for the variables $\xi := [y\ y^{(1)} \cdots y^{(r-1)}]$ (Isidori, 1995). But here c is fixed, so these r degrees of freedom are not available. The following corollary is fundamental in the definition of the zero dynamics of a state space realization.

Corollary 13. *Suppose that $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r and $\mathrm{supp}(c_N) \subseteq x_0^r X_0^*$. Then the input*

$$u^*(t) = \sum_{k=0}^{\infty}(c_u^*, x_0^k)\frac{t^k}{k!}, \tag{14}$$

where $c_u^* = \left(((x_0^r)^{-1}(c)/(x_0^{r-1}x_1)^{-1}(c))^{\circ -1}\right)_N$, has the property that $F_c[u^*] = 0$.

Proof. Set $c_y = 0$ in (12).

Example 14. Consider the linear time-invariant system

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = [4\ 5\ 1]z$$

initialized at $z(0) = [1\ 1\ -9]^T$. The transfer function has two finite zeros at $s = -1$ and $s = -4$, in which case, $r = 1$. Using conventional linear system theory, it can be shown for $t \geq 0$ that

$$u^*(t) = \frac{5}{3}e^{-t} + \frac{46}{3}e^{-4t} = 17 - 63t + 247\frac{t^2}{2!} - 983\frac{t^3}{3!} + 3927\frac{t^4}{4!} - \cdots. \tag{15}$$

The corresponding generating series for this system $(A, b, c, z(0))$ is $c = c_N + c_F$, where

$$\begin{aligned} c_N &= s(c(sI - A)^{-1}z(0))|_{s^{-1}\to x_0} \\ &= -17x_0 + 29x_0^2 - 53x_0^3 + 118x_0^4 - 277x_0^5 + \cdots \\ c_F &= s(c(sI - A)^{-1}b)x_1|_{s^{-1}\to x_0} \\ &= (1 + 2x_0 - 4x_0^2 + 7x_0^3 - 15x_0^4 + 35x_0^5 - \cdots)x_1. \end{aligned}$$

Since $\mathrm{supp}(c_N) \subseteq x_0X_0^*$, the function $y = 0$ is in the range of F_c. To determine u^* via Corollary 13, observe that

$$\begin{aligned} x_0^{-1}(c) &= x_0^{-1}(c_N) + x_0^{-1}(c_F) \\ &= (-17 + 29x_0 - 53x_0^2 + 118x_0^3 - 277x_0^4 + \cdots) \\ &\quad + (2 - 4x_0 + 7x_0^2 - 15x_0^3 + 35x_0^4 - \cdots)x_1 \\ x_1^{-1}(c) &= 1, \end{aligned}$$

and thus, $x_0^{-1}(c)/x_1^{-1}(c) = x_0^{-1}(c)$ and $c_u^* = \left((x_0^{-1}(c))^{\circ -1}\right)_N$. To compute the indicated composition inverse, the Faà di Bruno Hopf antipode formulas (3) are utilized:

$$\begin{aligned} (c_u^*, \emptyset) &= -(x_0^{-1}(c), \emptyset) = 17 \\ (c_u^*, x_0) &= -(x_0^{-1}(c), x_0) + (x_0^{-1}(c), \emptyset)(x_0^{-1}(c), x_1) \\ &= -63 \\ (c_u^*, x_0^2) &= -(x_0^{-1}(c), x_0^2) + (x_0^{-1}(c), \emptyset)(x_0^{-1}(c), x_0x_1) \\ &\quad + (x_0^{-1}(c), x_0)(x_0^{-1}(c), x_1) \\ &\quad + (x_0^{-1}(c), \emptyset)(x_0^{-1}(c), x_1)^2 = 247. \end{aligned}$$

As expected, these coefficients agree with those in (15). Of course, if no closed-form expression is known for the Taylor series of u^*, as will be the case for the nonlinear system in the next example, then some truncation of this series must be used. Therefore, $F_c[\hat{u}^*] \approx 0$ on $[0, T]$ provided T is small relative to the number of terms retailed in the approximation $\hat{u}^*$ of u^*. The MatLab simulated outputs shown in Fig. 3 give some idea of the accuracy of the results for approximations of (14) up to third order. A lower bound on the radius of convergence for c_u^* is found to be 0.01255 using empirical estimates of the global growth constants $K = 17$ and $M = 2.2780$ for $x_0^{-1}(c)$ and the corresponding radius of convergence formula for the composition inverse. It should be very conservative given that most of the coefficients for a linear series are zero.

Example 15. Reconsider the system in Example 11 when $z(0) = 0$. Since $r = 2$ in this case, and $c_N = 0$, the range of F_c contains all analytic outputs with generating series $c_y = \sum_{k\geq 2}(c_y, x_0^k)x_0^k$. As an illustration, select $y(t) = t^2/2$. Then $c_y = x_0^2$, $(x_0^2)^{-1}(c - c_y) = -1$, and

$$(x_0x_1)^{-1}(c) = 2 + 2x_1 - 2x_0x_1 + 2x_0^2x_1 - 2x_0^3x_1 + \cdots.$$

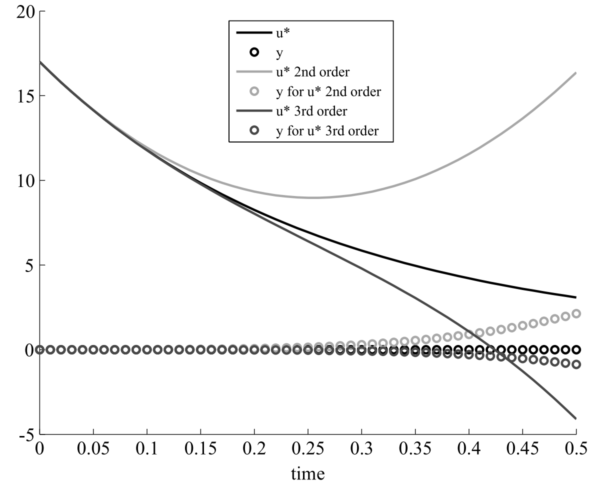

Fig. 3. Output of F_c for u^* and Taylor series approximations of u^* up to third order in Example 14.

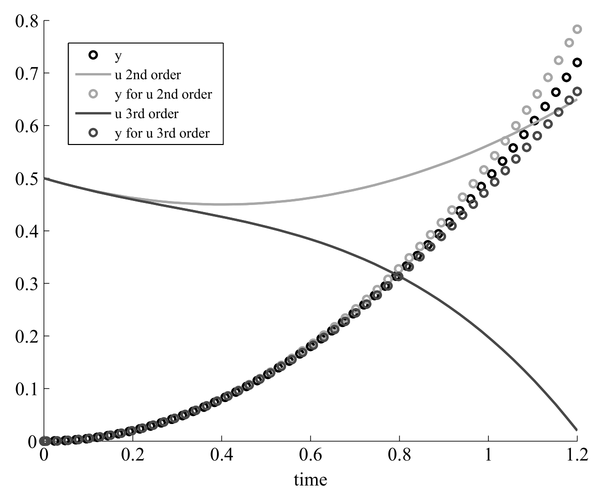

Fig. 4. Output of F_c for Taylor series approximations of u up to third order and $y(t) = t^2/2$ in Example 15.

With the assistance of a Mathematica package for manipulating noncommutative formal power series, NCFPS (which is supported by the package for noncommutative algebra NCAlgebra (2012)), it follows from (13) that

$$d = -\frac{1}{2} + \frac{1}{2}x_1 - \frac{1}{2}x_0x_1 - x_1^2 + \frac{1}{2}x_0^2x_1 + 2x_0x_1^2 + x_1x_0x_1 + 3x_1^3 - \cdots.$$

Applying (3) gives

$$u(t) = \frac{1}{2} - \frac{1}{4}t + \frac{5}{8}\frac{t^2}{2!} - \frac{35}{16}\frac{t^3}{3!} + \frac{307}{32}\frac{t^4}{4!} - \cdots.$$

If the generating series c_u is truncated to third order to produce the polynomial $\hat{c}_u$, the corresponding output error is characterized by

$$c_y - c \circ \hat{c}_u = \frac{331}{16}x_0^6 - \frac{1999}{32}x_0^7 + \frac{12855}{64}x_0^8 - \frac{83527}{128}x_0^9 + \cdots.$$

Therefore, the output approximation in this case is accurate to fifth order. Various approximate outputs computed via MatLab are compared in Fig. 4 against the desired y. A lower bound on the radius of convergence for c_u can be computed using the global growth constants $K = 2$ and $M = 1$ for $(x_0x_1)^{-1}(c)$. In which case, $K_d = 0.5$ and $M_d = 1.4427$ using Theorem 7, and the lower bound on the radius of convergence for c_u is 0.1562 using the radius of convergence formula for the composition inverse of locally convergent series given in Table 1. Again, the estimate should be very conservative given the sparseness of $(x_0x_1)^{-1}(c)$.

Example 16. Consider a first order, exothermic, irreversible reaction of a reactant in a product substance carried out in a well mixed continuous stirred chemical reactor (CSTR). The mass and energy balances give the dynamics (in dimensionless form):

$$\dot{z} = \begin{bmatrix} -z_1 + \alpha(1-z_1)e^{\frac{z_2}{1+z_2/\gamma}} \\ -(\beta+1)z_2 + \kappa\alpha(1-z_1)e^{\frac{z_2}{1+z_2/\gamma}} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u, \qquad y = z_2$$

with $z(0) = 0$. Here z_1 is the reactant concentration, z_2 is the reactor temperature, and u is the cooling reactor jacket temperature (Doyle & Henson, 1997; Uppal et al., 1974). The constants α, β, γ, and κ are all set to unity for convenience. The corresponding generating series for the input–output map is found to be

$$\begin{aligned} c &= x_0 + x_1 - 2x_0^2 - x_0x_1 - 2x_0^2x_1 - 2x_0x_1x_0 - x_0x_1^2 + 22x_0^4 \\ &\quad + 15x_0^3x_1 + 11x_0^2x_1x_0 + 4x_0^2x_1^2 + 6x_0x_1x_0^2 + 2x_0x_1x_0x_1 \\ &\quad + 2x_0x_1^2x_0 + x_0x_1^3 - 104x_0^5 + \cdots, \end{aligned}$$

which has relative degree $r = 1$. Now suppose one wants to determine an input to linearly increase the reactor temperature over time so that $y(t) = t, 0 \leq t \leq 1$. In light of Theorem 12, this desired output signal is in the range of the system since $c_y = x_0$. To compute the corresponding u via (11)–(12), first note that $x_1^{-1}(c) = 1$ and

$$\begin{aligned} d &= \frac{x_0^{-1}(c - c_y)}{x_1^{-1}(c)} = -2x_0 - x_1 - 2x_0x_1 - 2x_1x_0 - x_1^2 + 22x_0^3 \\ &\quad + 15x_0^2x_1 + 11x_0x_1x_0 + 4x_0x_1^2 + 6x_1x_0^2 + 2x_1x_0x_1 \\ &\quad + 2x_1^2x_0 + x_1^3 + \cdots. \end{aligned}$$

A direct application of (3) gives

$$u(t) = 2t + 2\frac{t^2}{2!} - 8\frac{t^3}{3!} + 14\frac{t^4}{4!} + 46\frac{t^5}{5!} - 596\frac{t^6}{6!} + \cdots.$$

The generating series of u when truncated to sixth order yields the polynomial $\hat{c}_u$ and the corresponding output error

$$c_y - c \circ \hat{c}_u = -65264x_0^8 - 878770x_0^9 - 9806016x_0^{10} - \cdots,$$

which is accurate to seventh order. The outputs generated by MatLab for u up to fourth order are compared in Fig. 5 against the desired y. Finally, empirical estimates of the global and local growth constants for $d = x_0^{-1}(c - c_y)$ are $K_d = 0.3358$, $M_d = 5.9558$ and $K_d = 1.4185$, $M_d = 1.4099$, respectively. In which case, using the corresponding expressions for the radius of convergence of the composition inverse in Table 1, lower bounds on the local and global radius of convergence for c_u are 0.0862 and 0.1158, respectively.

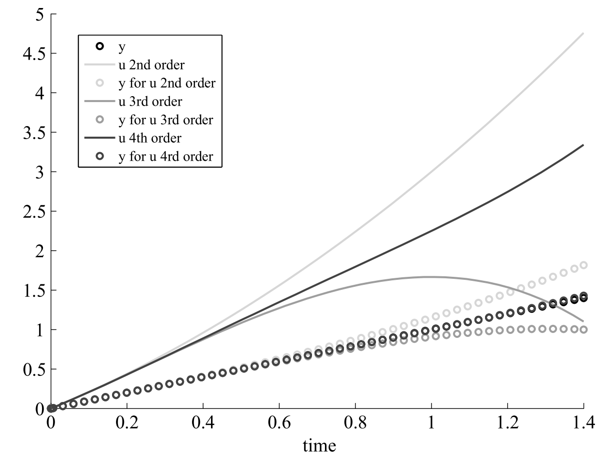

Fig. 5. Output of F_c for Taylor series approximations of u up to fourth order and $y(t) = t$ in Example 16.

4. Conclusions

The left inversion problem was solved analytically for SISO Fliess operators having well defined relative degree, as defined in terms of generating series, and real analytic outputs.

Acknowledgments

The authors thank J. William Helton and Lance Berlin for their assistance with the Mathematica package NCFPS.

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