Coherent quantum observers for n-level quantum systems

Z. Miao[†], Luis A. Duffaut Espinosa[‡], I. R. Petersen[‡], V. Ugrinovskii [‡] and M. R. James[§]

***Abstract*— The purpose of this paper is to find coherent quantum observers for open n-level quantum systems. Recently, a class of linear coherent observers has been developed for quantum harmonic oscillators. However, open n-level quantum systems, which are characterized by bilinear quantum stochastic differential equations, escape the realm of the known theory. Therefore, in this paper we show how a coherent quantum observer is designed to track the corresponding n-level quantum plant asymptotically in the sense of mean values. We also discuss suboptimal quantum observers in the sense of least mean squares estimation.**

I. INTRODUCTION

Quantum feedback control plays a vital role for the development of quantum and nano technologies. However, the majority of approaches consider a classical controller in the feedback loop although it is becoming apparent that fully quantum coherent feedback systems (not necessarily involving measurement) may have significant benefits; see [4], [12]. As a step towards better understanding fully quantum estimation and control, recently it was introduced and studied a class of coherent quantum observers, whose structure is analogous to the classical Luenberger observer, for linear quantum stochastic systems. The coherent observer, which is not a continuous measurement apparatus, does not attempt to replicate any quantum states of the corresponding quantum plant, but tracks the plant asymptotically in the sense of mean values. On the other hand, in terms of quantum correlations, entanglement can be generated in the joint plant-observer quantum systems because the output of the plant is the input to the observer, as illustrated in [14]. When the joint system gets entangled, the quantum plant and the corresponding coherent observer cannot be treated locally, which could never happen in classical control. However, in our previous work, we only considered linear quantum stochastic systems as quantum plants and gave the corresponding linear coherent observers. In contrast, here we consider n-level quantum systems as quantum plants, which can be described in the form of bilinear quantum stochastic differential equations (QSDEs); see [6]–[9], and try to find coherent quantum observes for them in this paper. In [8], we provided physical realizability conditions to ensure bilinear QSDEs to correspond to open n-level quantum systems. Those results allow us to study n-level quantum systems in the context of state space representations, which provide an effective and compact way to model and analyze systems with multiple inputs and outputs, and it turns out that using bilinear QSDEs is convenient for constructing a coherent quantum observer. In addition, we take into account the mean square error between the dynamic variables of the quantum plant and those of the corresponding observer, and thus we provide a class of suboptimal coherent observers for n-level quantum systems. Furthermore, finding an optimal filter, consistent with the laws of quantum mechanics, is a very difficult problem because of the failure of conditional expectation of quantum conditioning onto non-commutative subspaces of signals; see [5], [13]. By contrast, observer-based quantum control is a promising approach to realizing coherent feedback control since a physical observer can always be found even if the observation process is not commutative. Moreover, it is possible to manipulate a coherent observer so that we could control the corresponding quantum plant indirectly through altering the quantum correlations in the joint plant-observer system.

The paper is organized as follows. We begin in Section II by presenting the bilinear quantum stochastic differential equations under consideration, and the corresponding physical realizability conditions. In Section III, we show how to construct a coherent quantum observer for open n-level quantum systems, consistent with the laws of quantum mechanics. We also provide a suboptimal quantum observer. Section IV gives some concluding remarks and future research directions.

II. BILINEAR QUANTUM STOCHASTIC SYSTEMS

Quantum stochastic calculus allows us to express the evolution of the observable X of an open quantum system in the *Heisenberg picture* as (see [6])

$$\begin{aligned} dX &= \left(S^\dagger X S - X\right) d\Lambda + \mathcal{L}(X)\, dt \\ &\quad + S^\dagger [X, L]\, dW^\dagger + dW \left[L^\dagger, X\right] S, \end{aligned}$$

where $L(X)$ is the *Lindblad operator* defined as

$$\mathcal{L}(X) = -i [X, \mathcal{H}] + \frac{1}{2}\left(L^\dagger [X, L] + \left[L^\dagger, X\right] L\right).$$

The output field is given by

$$dY = Ldt + SdW.$$

Here W, $W^\dagger$ and Λ are the annihilation process, the creation process and the counting process, respectively, characterizing the inputs to the quantum system from a boson field. $\mathcal{H}$ is a self-adjoint operator denoting the *Hamiltonian* of the system,

[†]Research School of Engineering, Canberra, ACT 0200, Australia. zibo.miao@anu.edu.au.

[‡]School of Engineering and Information Technology, University of New South Wales at ADFA, Canberra, ACT 2600, Australia. l.duffautespinosa@gmail.com, {i.petersen, v.ougrinovski}@adfa.edu.au.

[§]ARC Centre for Quantum Computation and Communication Technology, Research School of Engineering, Australian National University, Canberra, ACT 0200, Australia. matthew.james@anu.edu.au.

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and S and L are operators which determine the *coupling* of the system to the quantum field, with S unitary. Hereafter, we assume $S = 1$ with 1 denoting the identity operator.

In this paper we consider bilinear quantum stochastic systems of the form (see [8])

$$\begin{aligned} dx &= F_0 dt + F x dt + G_1 x dw_1 + G_2 x dw_2 \\ dy &= H x dt + dw, \end{aligned} \tag{1}$$

where $F_0 \in \mathbb{R}^{n_x}$, $F \in \mathbb{R}^{n_x \times n_x}$, $G_i \in \mathbb{R}^{n_x \times n_x}$ $(i = 1, 2)$, and $H \in \mathbb{R}^{2 \times n_x}$ (here $n_x = n^2 - 1$ where n corresponds to n-level quantum systems).

The vector x, is a collection of the $n^2 - 1$ *generalized Gell-Mann matrices* generating the special unitary group $SU(n)$, which satisfies the following commutation and anti-commutation relations

$$\begin{aligned} \left[x, x^T\right] &= 2i\Theta^-(x) \\ \left\{x, x^T\right\} &= \frac{4}{n} I + 2\Theta^+(x), \end{aligned}$$

where $\Theta^-(\cdot)$ and $\Theta^+(\cdot)$ are linear mappings; see [8], [9] for more information.

The 2-dimensional input signal

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} W + W^\dagger \\ -i\left(W - W^\dagger\right) \end{bmatrix}$$

is the quantum noise corresponding to a boson quantum field with the following quantum Itô table (see [2], [6], [11])

$$dw dw^T = (I_2 + iJ)\, dt$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and I_m $(m \in \mathbb{Z}^+)$ is the m-dimensinal identity matrix.

We can also write the output as

$$dy = \begin{bmatrix} dy_1 & dy_2 \end{bmatrix}^T,$$

where

$$\begin{aligned} dy_1 &= H_1 x dt + dw_1 \\ dy_2 &= H_2 x dt + dw_2. \end{aligned}$$

It was shown in [8] that the system (1) is physically realizable (that is, it corresponds to an open n-level quantum system interacting with one boson field) if and only if the following identities hold

$$\begin{aligned} F_0 &= \frac{1}{n}(G_1 + iG_2)(H_1 + iH_2)^\dagger \\ G_1 &= \Theta^-(H_2) \\ G_2 &= -\Theta^-(H_1) \\ F + F^T + G_1 G_1^T + G_2 G_2^T &= \frac{n}{2}\Theta^+(F_0). \end{aligned}$$

Moreover, the quantum stochastic system (1) can be divided into two types based on the following definition.

Definition 1: The quantum stochastic system (1) is *degenerate* if $F_0 = 0$, otherwise (1) is *non-degenerate*.

III. COHERENT QUANTUM OBSERVERS

In many situations full knowledge of the plant is not available, and unknown quantities may be estimated on the basis of the available information. The Kalman filter, which computes the conditional expectations of the state variables of the plant, is a well-known statistical approach to state estimation based on dynamical linear-Gaussian models [1]. In quantum control, when the quantum plant is continuously monitored, the Belavkin-Carmichael quantum filter may be used to compute conditional expectations of plant variables; see [10], [13]. Mathematically, the quantum filter computes a quantum conditional expectation as a projection onto a commutative subspace of output signals. However, traditional techniques do not appear to be applicable since the conditional expectation is not well defined for non-commutative quantum random variables. That is why we consider the design of coherent quantum observers to allow for non-commutative estimation and control.

A. Construction of a coherent observer

First, we give the definition of a *coherent quantum observer* for n-level quantum systems.

Definition 2: A quantum system that can track an n-level quantum system (1) asymptotically in the sense of mean values, is called a *coherent quantum observer*.

Let us first consider a two-level quantum system interacting with one boson quantum field given by

$$\begin{aligned} dx &= F_0 dt + F x dt + G_1 x dw_1 + G_2 x dw_2 \\ dy &= H x dt + dw, \end{aligned} \tag{2}$$

where the components of $x = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}^T$ are comprised by a linear combination of *Pauli matrices*; see [7]. $F_0 \in \mathbb{R}^3$, $F, G_i \in \mathbb{R}^{3\times 3}$ $(i = 1, 2)$, and $H \in \mathbb{R}^{2\times 3}$. In this case, x satisfies the commutation relation:

$$\left[x, x^T\right] = 2i\Theta^-(x).$$

The physical realizability conditions for (2) are (see [7])

$$\begin{aligned} G_1 + G_1^T = G_2 + G_2^T &= 0 \\ G_1 G_2^T - G_2 G_1^T - \Theta^-(F_0) &= 0 \\ F^T + F + G_1 G_1^T + G_2 G_2^T &= 0. \end{aligned}$$

Equivalently, the system (2) can be rewritten as the following bilinear quantum stochastic differential equation with respect to $\bar{x}$

$$\begin{aligned} d\bar{x} &= \bar{F}\bar{x} dt + \bar{G}_1 \bar{x} dw_1 + \bar{G}_2 \bar{x} dw_2 \\ dy &= \bar{H}\bar{x} dt + dw, \end{aligned} \tag{3}$$

where $\bar{x} = \begin{bmatrix} 1 & \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}^T$, and the coeffiecients of (3) are defined from the coefficients of the system (2)

$$\begin{aligned} \bar{F} &= \begin{bmatrix} 0_{1\times 4} \\ F_0, F \end{bmatrix} \\ \bar{G}_1 &= \begin{bmatrix} 0_{1\times 4} \\ 0_{3\times 1}, G_1 \end{bmatrix} \\ \bar{G}_2 &= \begin{bmatrix} 0_{1\times 4} \\ 0_{3\times 1}, G_2 \end{bmatrix} \\ \bar{H} &= [0_{2\times 1}, H]. \end{aligned}$$

Now we consider a two-mode linear quantum system driven by the output of (2) in a cascade arrangement of the form as follows (see [3], [14])

$$\begin{aligned} d\hat{x} &= A\hat{x}dt + Kdy + B_0 dw_0 \\ d\hat{y} &= C\hat{x}dt + \begin{bmatrix} dy^T & dw_0^T \end{bmatrix}^T. \end{aligned} \tag{4}$$

Here $\hat{x} = \{q_1, p_1, q_2, p_2\}^T$ with q_i and p_i $(i = 1, 2)$ representing the position and momentum operators. w_0 denotes the additional quantum noises from n_{w_0} additive boson fields input to (4). $A, C \in \mathbb{R}^{4\times 4}$, $K \in \mathbb{R}^{4\times 2}$ and $B_0 \in \mathbb{R}^{4\times 2n_{w_0}}$. The physical realizability conditions for (4) are (see [11], [14])

$$\begin{aligned} &A\Theta_2 + \Theta_2 A^T \\ &+KJK^T + B_0 \Theta_{n_{w_0}} B_0^T = 0 \\ &C = \Theta_{n_{w_0}+1} \begin{bmatrix} K & B_0 \end{bmatrix}^T \Theta_2, \end{aligned} \tag{5}$$

where $\Theta_n = \mathrm{diag}_n(J) = I_n \otimes J$ for any positive integer n generally. Note that B_0 is added to ensure (5) holds, or rather say, (4) is guaranteed to be physical with additional quantum noise input w_0.

Considering $A = \bar{F} - K\bar{H}$ and after some calculations, the error dynamics can be given by

$$\begin{aligned} d\bar{e} \;=\; &\left(\bar{F} - K\bar{H}\right)\bar{e}dt \\ &+Kdw + B_0 dw_0 - \bar{G}_1 \bar{x} dw_1 - \bar{G}_2 \bar{x} dw_2 \end{aligned}$$

where $\bar{e} = \hat{x} - \bar{x}$, and thus the mean value (quantum expectation) of the error $\langle \bar{e} \rangle$ satisfies

$$d\langle \bar{e} \rangle = \left(\bar{F} - K\bar{H}\right)\langle \bar{e} \rangle dt.$$

The observer error is defined as $e = E\bar{e}$ with

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

since we do not need to track 1 in $\bar{x}$.

Let (2) be non-degenerate ($F_0 \neq 0$), if $\left(\bar{F}, \bar{H}\right)$ is detectable, then we can always make $\bar{F} - K\bar{H}$ Hurwitz. On the other hand, if (2) is degenerate ($F_0 = 0$), then we obtain

$$d\langle e \rangle = (F - EKH)\langle e \rangle dt. \tag{6}$$

Hence, if (F, H) is detectable, $F - EKH$ can always be made Hurwitz by appropriate choice of the matrix EK. Then the observer gain is given by

$$K = \begin{bmatrix} K_1 \\ EK \end{bmatrix},$$

where $K_1^T \in \mathbb{R}^2$ is arbitrary.

Therefore, the two-mode linear quantum observer (4) can be used to track the two-level quantum plant (2) asymptotically in the sense of mean values by appropriate choice of the observer gain matrix K.

Example 1: Consider a two-level quantum plant of the following form

$$\begin{aligned} dx &= Fxdt + G_2 x dw_2 \\ dy &= Hxdt + dw \end{aligned}$$

with

$$F_0 = 0, F = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.5 & -1 & 0.5 \\ 0.5 & 0.5 & -1 \end{bmatrix},$$

$$G_1 = 0, G_2 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Choose

$$K = \begin{bmatrix} 1 & -0.5 & -0.5 & -0.5 \\ 1 & -0.5 & -0.5 & -0.5 \end{bmatrix}^T.$$

Then

$$F - EKH = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & -1.5 \end{bmatrix},$$

which is Hurwitz. Therefore, from (6), $\langle e \rangle$ will converge to zero asymptotically.

Consider now the case for n-level quantum systems interacting with one boson field as in (1). The following definition exhibits the structure of a linear coherent quantum observer explicitly.

Definition 3: An m-mode linear coherent quantum observer ($m \in \mathbb{Z}^+$) is a system of the form

$$\begin{aligned} d\hat{x} &= A\hat{x}dt + Kdy + B_0 dw_0 \\ d\hat{y} &= C\hat{x}dt + \begin{bmatrix} dy^T & dw_0^T \end{bmatrix}^T, \end{aligned} \tag{7}$$

where y is the output of a quantum plant. The vector x is defined as $x = \begin{bmatrix} q_1 & p_1 & \dots & q_m & p_m \end{bmatrix}^T$, which is an arrangement of position and momentum operators satisfying the commutation relations

$$\left[x, x^T\right] = xx^T - \left(xx^T\right)^T = 2i\Theta_m.$$

The vector w_0 denotes the additional quantum noises from n_{w_0} different channels. In addition, A, K, B_0 and C are real matrices of appropriate dimensions.

Given the definition of a linear coherent quantum observer, the following theorems show how to determine the observer parameters in order to track an n-level quantum plant as in (1) in the sense of mean values.

Theorem 1: Assume (1) is non-degenerate and $\left(\bar{F},\bar{H}\right)$ is detectable, then there always exists a $\left[\frac{n^2+1}{2}\right]$*-mode linear coherent quantum observer* (7) for an n-level quantum plant as in (1).

Proof: Similarly to the two-level case, include 1 to form the augmented vector $\bar{x}=\left[\begin{array}{cc}1 & x^T\end{array}\right]^T$, and rewrite (1) for n-level systems as follows

$$\begin{aligned} d\bar{x} &= \bar{F}\bar{x}dt+\bar{G}_1\bar{x}dw_1+\bar{G}_2\bar{x}dw_2 \\ dy &= \bar{H}\bar{x}dt+dw. \end{aligned}$$

This rewriting is of much importance in order to make the error dynamics converge exponentially to zero in the sense of mean values. In particular,

$$\begin{aligned} \bar{F} &= \left[\begin{array}{c} 0_{1\times n^2} \\ F_0, F \end{array}\right] \\ \bar{G}_1 &= \left[\begin{array}{c} 0_{1\times n^2} \\ 0_{(n^2-1)\times 1}, G_1 \end{array}\right] \\ \bar{G}_2 &= \left[\begin{array}{c} 0_{1\times n^2} \\ 0_{(n^2-1)\times 1}, G_2 \end{array}\right] \\ \bar{H} &= \left[0_{2\times 1}, H\right]. \end{aligned}$$

If n is even, then we define $\bar{e}=\hat{x}-\bar{x}$ and let $A=\bar{F}-K\bar{H}$ with A being Hurwitz by appropriate choice of the observer gain matrix K. Hence, the observer (7) can track the corresponding n-level quantum system asymptotically in the sense of mean values.

If n is odd, then we define $\bar{e}=S_n\hat{x}-\bar{x}$ where

$$S_n=\left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{array}\right]\in\mathbb{R}^{n^2\times\left(n^2+1\right)},$$

and let $\left[\begin{array}{cccc} A_2^T & A_3^T & \cdots & A_{n^2+1}^T \end{array}\right]^T=S_nA=\left(\bar{F}-S_nK\bar{H}\right)S_n$ with A_i $\left(i=1,2,\ldots,n^2+1\right)$ denoting the i-th row of A. Thus, the mean values of the error dynamics satisfy

$$\begin{aligned} d\left\langle\bar{e}\right\rangle &= S_n d\left\langle\hat{x}\right\rangle - d\left\langle\bar{x}\right\rangle \\ &= S_nA\left\langle\hat{x}\right\rangle dt-\left(\bar{F}-S_nK\bar{H}\right)\left\langle\bar{x}\right\rangle dt \\ &= \left(\bar{F}-S_nK\bar{H}\right)\left(S_n\left\langle\hat{x}\right\rangle-\left\langle\bar{x}\right\rangle\right)dt \\ &= \left(\bar{F}-S_nK\bar{H}\right)\left\langle S_n\hat{x}-\bar{x}\right\rangle dt \\ &= \left(\bar{F}-S_nK\bar{H}\right)\left\langle\bar{e}\right\rangle dt. \end{aligned}$$

If $\left(\bar{F},\bar{H}\right)$ is detectable, then $\left(\bar{F}-\bar{K}\bar{H}\right)$ can be made Hurwitz by appropriate choice of $\bar{K}=S_nK=\left[\begin{array}{cccc} K_2^T & K_3^T & \cdots & K_{n^2+1}^T \end{array}\right]^T$, where K_i denotes the i-th row of K for $\left(i=1,2,\ldots,n^2+1\right)$.

Note that K is determined, then A is determined. It then follows that B_0 is given by

$$B_0=2i\Theta_{\left[\frac{n^2+1}{2}\right]}\left[\begin{array}{cc} -\alpha^\dagger & \alpha^T \end{array}\right]\Gamma, \tag{8}$$

where

$$\Gamma=P_{n_{w_0}}\mathrm{diag}_{n_{w_0}}\left(M\right),$$

and

$$M=\frac{1}{2}\left[\begin{array}{cc} 1 & i \\ 1 & -i \end{array}\right].$$

Here $P_{n_{w_0}}$ denotes a $2n_{w_0}\times 2n_{w_0}$ *permutation matrix*. For example, if we consider a column vector $a=\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{2m} \end{array}\right]^T$, then

$$P_m a=\left[\begin{array}{cccccccc} a_1 & a_3 & \cdots & a_{2m-1} & a_2 & a_4 & \cdots & a_{2m} \end{array}\right]^T$$

with

$$P_mP_m^T=P_m^TP_m=I_{2m}.$$

Let α be any complex $n_{w_0}\times 2\left[\frac{n^2+1}{2}\right]$ matrix such that

$$\begin{aligned} \alpha^\dagger\alpha = Q &- \frac{i}{4}\left(\Theta_{\left[\frac{n^2+1}{2}\right]}A+A^T\Theta_{\left[\frac{n^2+1}{2}\right]}\right) \\ &+ \frac{i}{4}\Theta_{\left[\frac{n^2+1}{2}\right]}KJK\Theta_{\left[\frac{n^2+1}{2}\right]}^T. \end{aligned} \tag{9}$$

Let Q be any real symmetric $2\left[\frac{n^2+1}{2}\right]\times 2\left[\frac{n^2+1}{2}\right]$ matrix such that the right hand side of (9) is nonnegative definite. We omit the details here; see [11], [14] for more information.

Finally, we can have that C is of the following form

$$C=\Theta_{n_{w_0}+1}\left[\begin{array}{cc} K & B_0 \end{array}\right]^T\Theta_{\left[\frac{n^2+1}{2}\right]}, \tag{10}$$

and therefore, one can always find a linear coherent quantum observer for an n-level quantum system if (1) is non-degenerate and $\left(\bar{F},\bar{H}\right)$ is detectable. ∎

Theorem 2: Assume (1) is degenerate and (F,H) is detectable, then there always exists a $\left[\frac{n^2}{2}\right]$*-mode linear coherent quantum observer* (7) for an n-level quantum plant as in (1).

Proof: For n odd, $n_x=n^2-1$ is even and the error e is $e=\hat{x}-x$. Since $F_0=0$, the matrix $A=F-KH$ and

$$d\left\langle e\right\rangle=\left(F-KH\right)\left\langle e\right\rangle dt.$$

Furthermore, since (F,H) is detectable, $F-KH$ can always be made Hurwitz by appropriate choice of the gain matrix

K. It then follows that the $\left[\frac{n^2}{2}\right]$-mode linear coherent quantum observer (7) can track an n-level quantum system asymptotically in the sense of mean values.

For n even, $n_x = n^2 - 1$ is odd and the error e is defined as $e = T_n\hat{x} - x$, where

$$T_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{\left(n^2-1\right)\times n^2}.$$

Let

$$A = \begin{bmatrix} A_1 \\ \left(F - T_n K H\right) T_n \end{bmatrix}$$

with $A_1 \in \mathbb{R}^{n^2}$ being an arbitrary vector which denotes the first row of A. Note that $T_n A = \left(F - T_n K H\right) T_n$, and that we can find the mean values of the error dynamics as follows

$$\begin{aligned} d\left\langle e\right\rangle &= T_n d\left\langle \hat{x}\right\rangle - d\left\langle x\right\rangle \\ &= T_n A\left\langle \hat{x}\right\rangle dt - \left(F - T_n K H\right)\left\langle x\right\rangle dt \\ &= \left(F - T_n K H\right)\left(T_n\left\langle \hat{x}\right\rangle - \left\langle x\right\rangle\right) dt \\ &= \left(F - T_n K H\right)\left\langle e\right\rangle dt. \end{aligned}$$

Since (F, H) is detectable, we can always find $T_n K$ such that $F - T_n K H$ is Hurwitz. Then

$$K = \begin{bmatrix} K_1 \\ T_n K \end{bmatrix},$$

where $K_1 \in \mathbb{R}^2$ denotes the first row of K.

Once K and A are determined, one can follow the procedures used in (8)-(10) to determine B_0 and C by considering $\left[\frac{n^2}{2}\right]$ instead of $\left[\frac{n^2+1}{2}\right]$. ∎

B. Optimization

In this section, we construct a least squares quantum observer for an n-level open quantum system. Without loss of generality, we consider the case of a non-degenerate two-level quantum plant interacting with one boson quantum field. Consider the observer error as $e = E\bar{e}$, where $\bar{e} = \hat{x} - \bar{x}$ and

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that there is no point in minimizing the error corresponding to 1. In addition, $e^T e = \bar{e}^T T \bar{e}$ and

$$T = E^T E = \begin{bmatrix} 0 & 0 \\ 0 & I_{3\times 3} \end{bmatrix}.$$

The dynamics for the mean square errors is

$$\begin{aligned} d\left\langle e^T e\right\rangle &= \left\langle \bar{e}^T\left(T\left(\bar{F} - K\bar{H}\right) + \left(\bar{F} - K\bar{H}\right)^T T\right)\bar{e}\right\rangle dt \\ &\quad + \left(f\left(K, \left\langle \bar{x}\right\rangle\right) + \mathrm{Tr}\left(K^T T K\right)\right) dt \\ &\quad + \mathrm{Tr}\left(B_0^T T B_0\right) dt, \end{aligned} \tag{11}$$

where

$$\begin{aligned} f\left(K, \bar{x}\right) &= -2\begin{bmatrix} 1 & 0 \end{bmatrix} K^T T \bar{G}_1 \bar{x} \\ &\quad -2\begin{bmatrix} 0 & 1 \end{bmatrix} K^T T \bar{G}_2 \bar{x} \\ &\quad + \bar{x}^T\left(\bar{G}_1^T T \bar{G}_1 + \bar{G}_2^T T \bar{G}_2\right)\bar{x} \\ &\quad + \bar{x}^T\left(i\bar{G}_1^T T \bar{G}_2 - i\bar{G}_2^T T \bar{G}_1\right)\bar{x} \\ &= \bar{\beta}\bar{x} + \bar{\beta}_0 \end{aligned}$$

is a linear function with respect to $\bar{x}$. Note that by applying the commutation relations of *Pauli matrices*, any quadratic term with respect to $\bar{x}$ can be simplified to a linear form. Also, $\left\langle \bar{x}\right\rangle$ is bounded because the quantum plant is a two-level system.

The optimization problem is to minimize the steady-state mean square error $\left\langle e^T e\right\rangle_\infty$ subject to the physical realizability conditions given in (5).

The discussion of a similar optimization problem is provided in [5]. However, in this paper, we set K first in order to make $\bar{F} - K\bar{H}$ Hurwitz, and then determine B_0 to guarantee the observer to be physical. Note that this procedure can result in a suboptimal observer, which is relevant since in practice the observer gain K is typically designed to satisfy some given performance objectives (e.g fast convergence requirement).

A suboptimal two-mode linear coherent quantum observer as in (4) for a two-level quantum plant as in (2) can be defined as follows.

Definition 4: Assume the observer gain K in (4) is determined in advance to satisfy some performance objectives, and there exists $\varepsilon > 0$ such that

$$T\left(\bar{F} - K\bar{H}\right) + \left(\bar{F} - K\bar{H}\right)^T T + \varepsilon T \leqslant 0. \tag{12}$$

(4) is a *suboptimal coherent quantum observer* for (2) if $\left\langle e^T e\right\rangle_\infty$ is minimized.

Theorem 3: If the condition (12) holds with the given observer gain K, then there exists a suboptimal coherent quantum observer (4) for (2) by appropriate choice of B_0. If K also satisfies

$$A\Theta_2 + \Theta_2 A^T + KJK^T = 0,$$

then we get a suboptimal observer without adding additional quantum noise to the observer (4).

Proof: If there exists a positive real number ε such that

$$T\left(\bar{F}-K\bar{H}\right)+\left(\bar{F}-K\bar{H}\right)^{T}T+\varepsilon T\leqslant 0,$$

then

$$\begin{aligned} d\left\langle e^{T}e\right\rangle \;\leqslant\; & -\varepsilon\left\langle e^{T}e\right\rangle dt \\ & +\left(\beta\left\langle x\right\rangle+\beta_{0}+\operatorname{Tr}\left(K^{T}TK\right)\right)dt \\ & +\operatorname{Tr}\left(B_{0}^{T}TB_{0}\right)dt, \end{aligned}$$

which indicates $\left\langle e^{T}e\right\rangle$ will converge to a constant $\left\langle e^{T}e\right\rangle_{\infty}$. Here $f\left(K,\left\langle\bar{x}\right\rangle\right)$ in (11) can be written as $f\left(K,\left\langle\bar{x}\right\rangle\right)=\bar{\beta}\left\langle\bar{x}\right\rangle+\bar{\beta}_{0}=\beta\left\langle x\right\rangle+\beta_{0}$ since $\bar{x}=\left[\begin{array}{cc}1 & x^{T}\end{array}\right]^{T}$. Observer that the optimization problem is only conditioned on

$$B_{0}\Theta_{n_{w_0}}B_{0}^{T}=-A\Theta_{2}-\Theta_{2}A^{T}-KJK^{T}$$

with respect to B_0, that is, minimizting $\left\langle e^{T}e\right\rangle_{\infty}$ is equivalent to minimizing $\operatorname{Tr}\left(B_{0}^{T}TB_{0}\right))=\operatorname{Tr}\left(TB_{0}B_{0}^{T}\right)\geqslant 0$. It is thus that this optimization problem can be solved ditectly by using quations (8) and (9).

Furthermore, it is obvious that if the observer gain matrix K satisfies the physical realizability condition for the coherent observer

$$A\Theta_{2}+\Theta_{2}A^{T}+KJK^{T}=0,$$

then it is not necessary to inject additional noise w_0 to the observer (4), that is, we can choose $B_0=0$ to make $\operatorname{Tr}\left(B_{0}^{T}TB_{0}\right)=0$, and thus the steady-state mean square error $\left\langle e^{T}e\right\rangle_{\infty}$ is minimized. ■

IV. CONCLUSIONS

We construct a class of coherent quantum observers which can track n-level quantum systems asymptotically in the sense of mean values. Furthermore, we show how to determine the observer parameters explicitly in Theorem 1 and Theorem 2. Also, Theorem 3 tells us that in some cases we obtain a suboptimal coherent observer without injecting additional quantum noise to the observer. Although the quantum correlations between an n-level quantum plant and the corresponding coherent observer are not studied in this paper, it is not surprising that the joint plant-observer system can get entangled; see [14]. Future work includes exploring how to use a quantum observer to implement coherent feedback control, and manipulating quantum plant indirectly through entanglement control.

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