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Discrete-Time Approximations of Fliess Operators

W. Steven Gray[†] Luis A. Duffaut Espinosa[‡] Kurusch Ebrahimi-Fard[§]

***Abstract*— A common way to represent a nonlinear input-output system in control theory is via a Chen-Fliess functional expansion or Fliess operator. The general goal of this paper is to describe how to approximate Fliess operators with iterated sums and to provide accurate error bounds for two different scenarios, one where the series coefficients are growing at a local convergence rate, and the other where they are growing at a global convergence rate. In each case, it is shown that the error bounds are asymptotically achievable in certain worst case scenarios.**

***Index Terms*— Chen-Fliess series, numerical approximation, discrete-time systems, nonlinear systems**

I. INTRODUCTION

A common way to represent a nonlinear input-output system in control theory is via a Chen-Fliess functional expansion or Fliess operator [3], [10]. This series of weighted iterated integrals of the input functions exhibits considerable algebraic structure that can be used, for example, to describe system interconnections [6], [7]. On the other hand, in the context of numerical simulation, it is less clear how such a representation can be utilized efficiently. One hint was provided in [9], where it was shown that iterated integrals can be well approximated by iterated sums. But there is a considerable jump in going from approximating a single iterated integral to approximating an infinite sum of such integrals. In particular, the error bound for each iterated integral has to be precise enough to yield an accurate error bound for the whole operator. Further complicating the picture is the fact that in practice only finite sums can be computed. So an independent truncation error also has to be accounted for.

The general goal of this paper is to describe how to approximate Fliess operators with iterated sums and to provide accurate error bounds for different scenarios. The starting point is to develop a refinement of the error bound in [9, Lemma 2] for a single iterated integral. This is done largely using Chen's Lemma [2]. After this, two specific cases are considered, one in which the series coefficients are growing at a local convergence rate, and the other where they are growing at a global convergence rate [8]. Each case yields different error bound, and several simulation examples are given to demonstrate the results. In particular, it is shown that the error bounds are asymptotically achievable in certain worst case scenarios.

[†]Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, Virginia 23529, USA.
[‡]Department of Electrical and Computer Engineering, George Mason University, Fairfax, Virginia 22030, USA.
[§]Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, no. 13-15, 28049 Madrid, Spain.

The paper is organized as follows. First some preliminaries on Fliess operators and Chen's Lemma are given to set the notation and terminology. Next, the notion of a discrete-time Fliess operator is developed in Section III as the class of approximators. Then the main approximation theorems are given in Section IV. The conclusions of the paper are given in the final section.

II. PRELIMINARIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, denoted here by $\sqcup\!\sqcup$. The latter is the $\mathbb{R}$-bilinear extension of the shuffle product of two words, which is defined inductively by

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi)$$

with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ for all $\eta, \xi \in X^*$ and $x_i, x_j \in X$.

A. Fliess Operators

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0). \tag{1}$$

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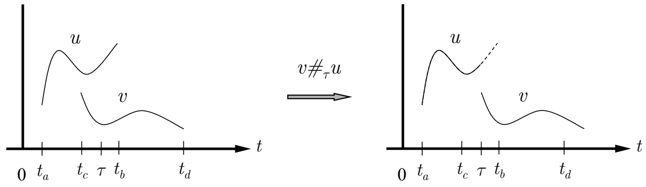

Fig. 1. The catenation of two inputs u and v at $t = \tau$.

If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \forall \eta \in X^*, \tag{2}$$

then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0, t_0 + T]$ for some $S > 0$ provided $\bar{R} := \max\{R, T\} < 1/M_c(m+1)$, and the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [8]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) In this case, the operator F_c is said to be *locally convergent* (LC), and the set of all series satisfying (2) is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$. When c satisfies the more stringent growth condition

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \quad \forall \eta \in X^*, \tag{3}$$

the series (1) defines an operator from the extended space $L_{\mathfrak{p},e}^m(t_0)$ into $C[t_0, \infty)$, where

$$L_{\mathfrak{p},e}^m(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L_{\mathfrak{p}}^m[t_0, t_1], \forall t_1 \in (t_0, \infty)\},$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ [8]. In this case, the operator is said to be *globally convergent* (GC), and the set of all series satisfying (3) is designated by $\mathbb{R}_{GC}^\ell\langle\langle X \rangle\rangle$.

B. Chen's Lemma

Consider a formal power series of the form $P[u](t) = \sum_{\eta \in X^*} \eta E_\eta[u](t, t_0)$, which is often referred to as a *Chen series*. Given two input functions $(u, v) \in L_1^m[t_a, t_b] \times L_1^m[t_c, t_d]$, their *durations* are taken to be $t_b - t_a \geq 0$ and $t_d - t_c \geq 0$, respectively, and the functions are not defined outside their corresponding intervals. The *catenation* of u and v at $\tau \in [t_a, t_b]$ is understood to be

$$(v\#_\tau u)(t) = \begin{cases} u(t) & : t_a \leq t \leq \tau \\ v((t - \tau) + t_c) & : \tau < t \leq \tau + (t_d - t_c) \end{cases}$$

(see Figure 1). It is easily verified that $L_{1,e}^m(0)$ is a monoid under the catenation operator. The identity element in this case is denoted by $\mathbf{0}$ and is equivalent to the set of functions having exactly zero duration. The following lemma is due to Chen [2].

Lemma 1: (Chen's Lemma) If $(u, v) \in L_1^m[0, T_1] \times L_1^m[0, T_2]$ and $(t_1, t_2) \in [0, T_1] \times [0, T_2]$ then

$$P[v](t_2)P[u](t_1) = P[v\#_{t_1}u](t_2 + t_1).$$

So in essence, $P : L_{1,e}^m(0) \to \mathbb{R}\langle\langle X \rangle\rangle$ acts as a monoid morphism, where $\mathbb{R}\langle\langle X \rangle\rangle$ is viewed as a monoid under the catenation product.

III. Discrete-Time Fliess Operators

This section describes the main class of discrete-time approximators used throughout the paper. The set of admissible inputs will be drawn from the real sequence space $l_\infty^{m+1}[N_0] := \{\hat{u} = (\hat{u}(N_0), \hat{u}(N_0 + 1), \ldots) : |\hat{u}(N)| < \hat{R}_{\hat{u}}, \forall N \geq N_0, 0 \leq \hat{R}_{\hat{u}} < \infty\}$, where $\hat{u} = [\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_m]^T$ and $|\hat{u}(N)| := \max_{i=0,1,\ldots,m} |\hat{u}_i(N)|$. In which case, $\|\hat{u}\|_\infty := \sup_{N \geq N_0} |\hat{u}(N)|$ is always finite. Define a ball of radius $\hat{R}$ in $l_\infty^{m+1}[N_0]$ as $B_\infty^{m+1}[N_0](\hat{R}) = \{\hat{u} \in l_\infty^{m+1}[N_0] : \|\hat{u}\|_\infty \leq \hat{R}\}$. The subset of finite sequences over $[N_0, N_f]$ (that is, $N \in \{N_0, \ldots, N_f\}$) is denoted by $B_\infty^{m+1}[N_0, N_f](\hat{R})$.

Definition 1: For any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, the corresponding **discrete-time Fliess operator** is

$$\hat{y}(N) = \hat{F}_c[\hat{u}](N) = \sum_{\eta \in X^*} (c, \eta) S_\eta[\hat{u}](N), \tag{4}$$

where $\hat{u} \in l_\infty^{m+1}[1]$, $N \geq 1$, and the iterated sum for any $x_i \in X$ and $\eta \in X^*$ is defined inductively by

$$S_{x_i\eta}[\hat{u}](N) = \sum_{k=1}^{N} \hat{u}_i(k) S_\eta[\hat{u}](k)$$

with $S_\emptyset[\hat{u}](N) := 1$.

The following lemma is essential for providing sufficient conditions for the convergence of such operators.

Lemma 2: If $\hat{u} \in B_\infty^{m+1}[1](\hat{R})$ then for any $\eta \in X^*$ and $N \geq 1$

$$|S_\eta[\hat{u}](N)| \leq \hat{R}^{|\eta|} \binom{N - 1 + |\eta|}{|\eta|} \leq 2^{N-1}(2\hat{R})^{|\eta|}.$$

Proof: If $\eta = x_{i_j} \cdots x_{i_1}$ then for any $N \geq 1$

$$\begin{aligned} &|S_\eta[\hat{u}](N)| \\ &= \left| \sum_{k_j=1}^{N} \hat{u}_{i_j}(k_j) \sum_{k_{j-1}=1}^{k_j} \hat{u}_{i_{j-1}}(k_{j-1}) \cdots \sum_{k_1=1}^{k_2} \hat{u}_{i_1}(k_1) \right| \\ &\leq \sum_{k_j=1}^{N} \left| \hat{u}_{i_j}(k_j) \right| \sum_{k_{j-1}=1}^{k_j} \left| \hat{u}_{i_{j-1}}(k_{j-1}) \right| \cdots \sum_{k_1=1}^{k_2} \left| \hat{u}_{i_1}(k_1) \right| \\ &\leq \hat{R}^{|\eta|} \sum_{k_j=1}^{N} \sum_{k_{j-1}=1}^{k_j} \cdots \sum_{k_1=1}^{k_2} 1 \\ &= \hat{R}^{|\eta|} \binom{N - 1 + |\eta|}{|\eta|}, \end{aligned}$$

using the fact that the final nested sum above has $\binom{N-1+|\eta|}{|\eta|}$ terms [1]. The remaining inequality is standard. ■

Since the upper bound on $|S_\eta[\hat{u}](N)|$ in this lemma is achievable, it is not difficult to see that when the generating series c satisfies the growth bound (2), the series (4) defining $\hat{F}_c$ can diverge. For example, if $(c, \eta) = K_c M_c^{|\eta|} |\eta|!$ for all $\eta \in X^*$, and $\hat{u}_i(N) = \hat{R}$, $N \geq 1$, $i = 0, 1, \ldots, m$ then

$$\begin{aligned} F[\hat{u}](N) &= \sum_{\eta \in X^*} K_c M_c^{|\eta|} |\eta|! \, \hat{R}^{|\eta|} \binom{N - 1 + |\eta|}{|\eta|} \\ &= K_c \sum_{j=0}^{\infty} (M_c(m + 1)\hat{R})^j j! \binom{N - 1 + j}{j}. \end{aligned}$$

Since $\lim_{j \to \infty} \binom{N-1+j}{j} = 1$, this series diverges even when $\hat{R} < 1/M_c(m+1)$. The next theorem shows that this problem is averted when c satisfies the stronger growth condition (3).

Theorem 1: Suppose $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ has coefficients which satisfy (3). Then there exists a real number $\hat{R} > 0$ such that for each $\hat{u} \in B_\infty^{m+1}[1](\hat{R})$, the series (4) converges absolutely for any $N \geq 1$.

Proof: Fix $N \geq 1$. From the assumed coefficient bound and Lemma 2, it follows that

$$\left|\hat{F}_c(\hat{u})(N)\right| \leq \sum_{j=0}^{\infty}\sum_{\eta\in X^j} |(c,\eta)|\,|S_\eta[\hat{u}](N)| \leq \sum_{j=0}^{\infty} K_c(M_c(m+1))^j\, 2^{N-1}(2\hat{R})^j = \frac{K_c 2^{N-1}}{1-2M_c(m+1)\hat{R}},$$

provided $\hat{R} < 1/2M_c(m+1)$. ■

The final convergence theorem shows that the restriction on the norm of $\hat{u}$ can be removed if an even more stringent growth condition is imposed on c.

Theorem 2: Suppose $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ has coefficients which satisfy

$$|(c,\eta)| \leq K_c M_c^{|\eta|}\frac{1}{|\eta|!}, \quad \forall \eta \in X^*$$

for some real numbers $K_c, M_c > 0$. Then for every $\hat{u} \in l_\infty^{m+1}[1]$, the series (4) converges absolutely for any $N \geq 1$.

Proof: Following the same argument as in the proof of the previous theorem, it is clear for any $\hat{u} \in l_\infty^{m+1}[1]$ and $N \geq 1$ that

$$\left|\hat{F}_c(\hat{u})(N)\right| \leq \sum_{j=0}^{\infty} K_c(M_c(m+1))^j\frac{1}{j!}\,2^{N-1}(2\|\hat{u}\|_\infty)^j = K_c 2^{N-1}\mathrm{e}^{2M_c(m+1)\|\hat{u}\|_\infty}.$$

■

Assuming the analogous definitions for local and global convergence of the operator $\hat{F}_c$, note the incongruence between the convergence conditions for continuous-time and discrete-time Fliess operators as summarized in Table I. In each case, for a fixed c, the sense in which $\hat{F}_c$ converges is *weaker* than that for F_c. The source of this dichotomy is the observation in Lemma 2 that iterated sums of $\hat{u}$ do not grow as a function of word length like $\hat{R}^{|\eta|}/|\eta|!$, which is the case for iterated integrals. As shown in the next section, however, this difference in convergence behavior does not provide any serious impediment to using discrete-time Fliess operators as approximators for their continuous-time counterparts.

TABLE I

SUMMARY OF CONVERGENCE CONDITIONS FOR F_c AND $\hat{F}_c$.

growth rate	F_c	$\hat{F}_c$
$\lvert(c,\eta)\rvert \leq K_c M_c^{\lvert\eta\rvert}\lvert\eta\rvert!$	LC	divergent
$\lvert(c,\eta)\rvert \leq K_c M_c^{\lvert\eta\rvert}$	GC	LC
$\lvert(c,\eta)\rvert \leq K_c M_c^{\lvert\eta\rvert}\frac{1}{\lvert\eta\rvert!}$	at least GC	GC

IV. Approximating Fliess Operators

A. Iterated Integrals

Let $u \in L_1^m[0,T]$ for some finite $T > 0$. Following equation (38) in [9], select some integer $L \geq 1$ and with $\Delta := T/L$ define the sequence

$$\hat{u}_i(N) = \int_{(N-1)\Delta}^{N\Delta} u_i(t)\,dt, \quad i = 0, 1\ldots, m \tag{5}$$

where $N \in [1, L]$. Observe in particular that $\hat{u}_0(N) = \Delta$. The following lemma gives a description of S_η in this setting.

Lemma 3: For any $N \in [1, L]$ and $\eta \in X^*$

$$S_\eta[\hat{u}](N) = \Delta^{|\eta|}\sum_{\xi_N\cdots\xi_1=\eta} u_{\xi_N}(N)\cdots u_{\xi_1}(1),$$

where $u_i(k) := \hat{u}_i(k)/\Delta$, $u_{x_{i_1}\cdots x_{i_r}}(k) := u_{i_1}(k)\cdots u_{i_r}(k)$, $u_\emptyset(k) := 1$, and the summation is over all partitions of η with N subwords (including partitions with empty subwords).

Proof: The proof is by induction on the length of η. For the empty word the equality holds trivially. When $\eta = x_i$ observe that

$$S_{x_i}[\hat{u}](N) = \sum_{k=1}^{N}\hat{u}_i(k) = \Delta\sum_{k=1}^{N}u_i(k) = \Delta\sum_{\xi_N\cdots\xi_1=x_i} u_{\xi_N}(N)\cdots u_{\xi_1}(1).$$

Now assume the claim holds for all words up to length $j \geq 0$. If $\eta \in X^j$ then

$$S_{x_i\eta}[\hat{u}](N) = \sum_{k=1}^{N}\hat{u}_i(k)S_\eta[\hat{u}](k) = \sum_{k=1}^{N}\Delta u_i(k)\Delta^j\sum_{\xi_k\cdots\xi_1=\eta}u_{\xi_k}(k)\cdots u_{\xi_1}(1) = \Delta^{j+1}\sum_{\xi_N\cdots\xi_1=x_i\eta}u_{\xi_N}(N)\cdots u_{\xi_1}(1),$$

which proves the lemma. ■

The starting point for the approximation theory is the observation that $S_\emptyset[\hat{u}](L) = E_\emptyset[u](T,0) = 1$, $S_{x_i}[\hat{u}](L) = E_{x_i}[u](T,0)$ for all $x_i \in X$, and the assertion of Grüne and Kloeden that for any $\eta \in X^*$ with $|\eta| \geq 2$

$$S_\eta[\hat{u}](L) = E_\eta[u](T,0) + O\left(\frac{T^{|\eta|}}{L}\right)$$

[9, Lemma 2]. The following theorem gives a higher order error bound along these lines.

Theorem 3: Let $u \in L_1^m[0,T]$ for some finite $T > 0$. Select integer $L \geq 1$, set $\Delta := T/L$, and define the sequence $\hat{u}$ as in (5). For any $\eta \in X^*$ it follows that

$$|S_\eta[\hat{u}](L) - E_\eta[u](T,0)| \leq \frac{T^{|\eta|}\|\hat{u}/\Delta\|_\infty^{|\eta|}}{|\eta|!}\left(\frac{|\eta|\,(|\eta|-1)}{2L} + O\left(\frac{1}{L^2}\right)\right)$$

as $L \to \infty$.

Proof: Since the input sequence $\hat{u}$ is computed exactly from the integration of u, there is no loss of generality in the computation of $S_\eta[\hat{u}](L)$ if one assumes a priori that u is a piecewise constant input taking values $u_i(t) := \hat{u}_i(N)/\Delta$ when $t \in [(N-1)\Delta, N\Delta]$ for $i = 0, 1, \ldots, m$. In addition, it was shown in [8, Lemma 2.1] that for any $u \in L_1^m[0,T]$

$$|E_\eta[u](N\Delta,(N-1)\Delta)| \le \frac{U_0^{|\eta|_{x_0}} \cdots U_m^{|\eta|_{x_m}}}{|\eta|_{x_0}! \cdots |\eta|_{x_m}!},$$

where $U_i := \int_{(N-1)\Delta}^{N\Delta} |u_i(\tau)|\, d\tau$, and $|\eta|_{x_i}$ denotes the number of time x_i appears in η. This upper bound is achieved when each u_i is constant over $[(N-1)\Delta, N\Delta]$. Thus, the worst case error between $E_\eta[u](T,0)$ and $S_\eta[\hat{u}](L)$ occurs for such piecewise constant inputs. Applying Chen's Lemma in this case gives

$$\begin{aligned} E_\eta[u](T,0) &= (P[u](L\Delta), \eta) \\ &= (P[u(L)](\Delta) \cdots P[u(1)](\Delta), \eta) \\ &= \sum_{\xi_L \cdots \xi_1 = \eta} E_{\xi_L}[u(L)](L\Delta,(L-1)\Delta) \cdots \\ &\qquad E_{\xi_1}[u(1)](\Delta, 0). \end{aligned}$$

But for any $\xi = x_{i_1} \cdots x_{i_r}$

$$E_\xi[u(N)](N\Delta,(N-1)\Delta) = u_{i_1}(N) \cdots u_{i_r}(N) \frac{\Delta^r}{r!},$$

and therefore,

$$E_\eta[u](T,0) = \Delta^{|\eta|} \sum_{\xi_L \cdots \xi_1 = \eta} \frac{1}{|\xi_L|! \cdots |\xi_1|!} u_{\xi_L}(L) \cdots u_{\xi_1}(1).$$

Comparing this expression to that given for $S_\eta[\hat{u}](L)$ in Lemma 3, and letting $j = |\eta|$, observe

$$\begin{aligned} &|S_\eta[\hat{u}](L) - E_\eta[u](T,0)| \\ &\le \Delta^j \sum_{\xi_L \cdots \xi_1 = \eta} \left[1 - \frac{1}{|\xi_L|! \cdots |\xi_1|!}\right] |u_{\xi_L}(L) \cdots u_{\xi_1}(1)| \\ &\le \|\hat{u}\|_\infty^j \left(\left[\sum_{\xi_L \cdots \xi_1 = \eta} 1 \right] - \left[\sum_{\xi_L \cdots \xi_1 = \eta} \frac{1}{|\xi_L|! \cdots |\xi_1|!} \right] \right) \\ &= \|\hat{u}\|_\infty^j \left(\binom{L+j-1}{j} - \frac{L^j}{j!} \right) \\ &= \frac{\|\hat{u}\|_\infty^j}{j!} \left((L)_j - L^j \right), \end{aligned}$$

where $(L)_j := (L+j-1)(L+j-2)\cdots L$ denotes the Pochhammer symbol. Using the asymptotic expansion

$$(L)_j = L^j \left(1 + \frac{j(j-1)}{2L} + O\!\left(\frac{1}{L^2}\right) \right)$$

as $L \to \infty$ [11], it follows that

$$\begin{aligned} &|S_\eta[\hat{u}](L) - E_\eta[u](T,0)| \\ &\qquad \le \frac{T^j \|\hat{u}/\Delta\|_\infty^j}{j!} \left(\frac{j(j-1)}{2L} + O\!\left(\frac{1}{L^2}\right) \right) \end{aligned}$$

as $L \to \infty$, which proves the theorem. ∎

B. Locally Convergent F_c

When c is locally convergent, it was shown in the previous section that $\hat{F}_c$ can diverge. Therefore, a truncated version of $\hat{F}_c$, namely,

$$\hat{F}_c^J[\hat{u}](N) := \sum_{j=0}^{J} \sum_{\eta \in X^j} (c, \eta) S_\eta[\hat{u}](N),$$

is considered. The following theorem states that the error in approximating $F_c[u](T)$ by $\hat{F}_c^J[\hat{u}](L)$ can be bounded by the sum of two errors, specifically, $\hat{e}(J)$, which bounds the error resulting from approximating iterated integrals by iterated sums, and $e(J)$, which bounds the tail of the series defining $F_c[u](T)$, i.e., the truncation error.

Theorem 4: Let $c \in \mathbb{R}^\ell_{LC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$. If $u \in B_1^m(R)[0,T]$ with $\bar{R} := \max\{R, T\} < 1/M_c(m+1)$ then for any fixed $J \ge 0$

$$\begin{aligned} \left| F_c[u](T) - \hat{F}_c^J[\hat{u}](L) \right| &\le \hat{e}(J) + e(J) + \\ &\qquad K_c \frac{1 - \hat{s}^{J+1}}{1 - \hat{s}} O\!\left(\frac{1}{L^2}\right) \end{aligned}$$

as $L \to \infty$, where

$$\begin{aligned} \hat{e}(J) &= \frac{K_c}{L} \left[\frac{\hat{s}^2}{(1-\hat{s})^3} - \frac{J(J+1)\hat{s}^{(J+1)}}{2(1-\hat{s})} - \frac{J\hat{s}^{(J+2)}}{(1-\hat{s})^2} - \right. \\ &\qquad \left. \frac{\hat{s}^{J+2}}{(1-\hat{s})^3} \right], \\ e(J) &= K_c \frac{s^{J+1}}{1-s}, \end{aligned}$$

$\hat{s} = M_c(m+1)L\|\hat{u}\|_\infty$, $s = M_c(m+1)\bar{R}$, and $\hat{u}$ is as defined in (5).

Proof: Applying Theorem 3 and the assumption that $s < 1$ gives the following:

$$\begin{aligned} &\left| F_c[u](T) - \hat{F}_c^J[\hat{u}](L) \right| \\ &= \left| \sum_{j=0}^{\infty} \sum_{\eta \in X^j} (c,\eta) E_\eta[u](T,0) - \sum_{j=0}^{J} \sum_{\eta \in X^j} (c,\eta) S_\eta[\hat{u}](L) \right| \\ &\le \sum_{j=0}^{J} \sum_{\eta \in X^j} |(c,\eta)|\, |E_\eta[u](T,0) - S_\eta[\hat{u}](L)| + \\ &\qquad \sum_{j=J+1}^{\infty} \sum_{\eta \in X^j} |(c,\eta)|\, |E_\eta[u](T,0)| \\ &\le \sum_{j=2}^{J} K_c M_c^j (m+1)^j j!\, \frac{T^j \|\hat{u}/\Delta\|_\infty^j}{j!} \left(\frac{j(j-1)}{2L} + \right. \\ &\qquad \left. O\!\left(\frac{1}{L^2}\right) \right) + \sum_{j=J+1}^{\infty} K_c M_c^j (m+1)^j j!\, \frac{\bar{R}^j}{j!} \\ &\le K_c \sum_{j=0}^{J} (M_c(m+1)L\|\hat{u}\|_\infty)^j \left(\frac{j(j-1)}{2L} + O\!\left(\frac{1}{L^2}\right) \right) + \\ &\qquad K_c \sum_{j=J+1}^{\infty} (M_c(m+1)\bar{R})^j \end{aligned}$$

TABLE II
SUMMARY OF SIMULATION RESULTS FOR EXAMPLE 1.

case	u	T	L	Δ	J	$\|\hat{u}\|_\infty$	s	$\hat{s}$	$y(T)$	$\hat{y}^J(L)$	$\hat{y}^J(L)-y(T)$	$\hat{e}(J)$	$e(J)$
1	1	0.5	50	0.0100	10	0.0100	0.5000	0.5000	2.0000	2.0412	0.0412	0.0387	9.7656×10^{-4}
2	1	0.5	50	0.0100	20	0.0100	0.5000	0.5000	2.0000	2.0448	0.0448	0.0400	9.5367×10^{-7}
3	1	0.5	100	0.0050	10	0.0050	0.5000	0.5000	2.0000	2.0192	0.0192	0.0193	9.7656×10^{-4}
4	$\sin(20t)$	0.5	50	0.0100	10	0.0099	0.5000	0.4975	1.1009	1.1041	0.0032	0.0378	9.7656×10^{-4}
5	$\sin(20t)$	0.5	50	0.0100	20	0.0099	0.5000	0.4975	1.1009	1.1041	0.0032	0.0390	9.5367×10^{-7}
6	$\sin(20t)$	0.5	100	0.0050	10	0.0050	0.5000	0.4994	1.1011	1.1028	0.0017	0.0192	9.7656×10^{-4}

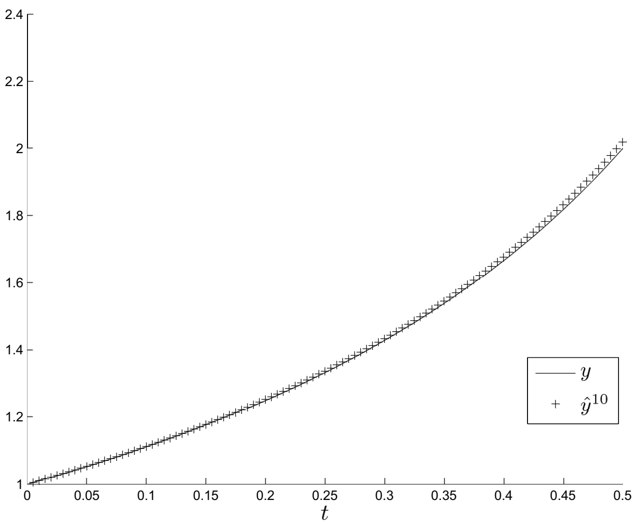

Fig. 2. Simulation comparing $y = F_c[1]$ to its approximation $\hat{y}^{10} = \hat{F}_c^{10}[\Delta]$ in Example 1, case 3.

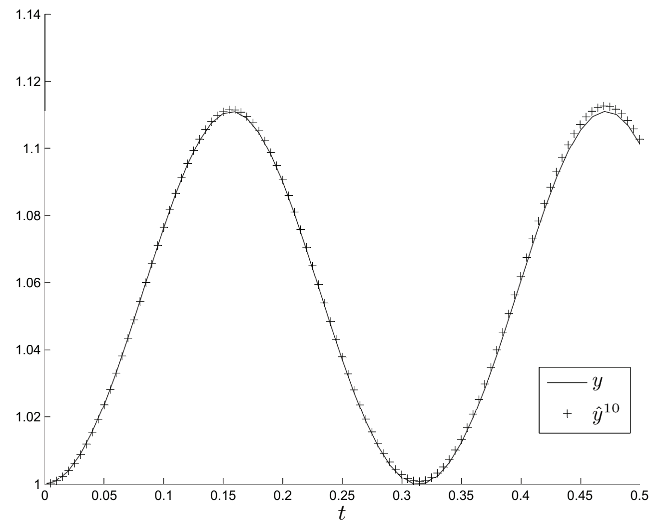

Fig. 3. Simulation comparing $y(t) = F_c[\sin(20t)]$ to its approximation $\hat{y}^{10} = \hat{F}_c^{10}[\hat{u}]$ in Example 1, case 6.

$$
\begin{aligned}
&= \frac{K_c}{L}\left[\frac{\hat{s}^2}{(1-\hat{s})^3} - \frac{J(J+1)\hat{s}^{(J+1)}}{2(1-\hat{s})} - \frac{J\hat{s}^{(J+2)}}{(1-\hat{s})^2} - \right. \\
&\qquad \left. \frac{\hat{s}^{J+2}}{(1-\hat{s})^3}\right] + K_c\frac{1-\hat{s}^{J+1}}{1-\hat{s}}\, O\!\left(\frac{1}{L^2}\right) + K_c\frac{s^{J+1}}{1-s} \\
&= \hat{e}(J) + e(J) + K_c\frac{1-\hat{s}^{J+1}}{1-\hat{s}}\, O\!\left(\frac{1}{L^2}\right),
\end{aligned}
$$

where standard formulas have been used to give closed-forms for the final series. It should also be noted that the upper bound given subsequently on the second term in the first inequality above is not trivial (it involves the shuffle product). The reader is referred to Lemma 2.3 of [5, p. 159] for details. ■

Example 1: Consider the locally convergent series $c = \sum_{k\geq 0} k!\, x_1^k$ so that $K_c = M_c = 1$. It is easy to verify that $y = F_c[u]$ has the state space realization

$$\dot{z} = u, \quad z(0) = 0, \quad y = 1/(1-z)$$

when $\bar{R} = \max\{\|u\|_1, T\} < 1$. (Effectively, $m = 0$ since c involves only one letter. This gives a tighter upper bound on $\bar{R}$. The subscripts on $\hat{u}$ are also dropped.) For example, direct substitution for z into the output equation gives

$$
\begin{aligned}
y(t) &= \sum_{j=0}^{\infty} E_{x_1}^{j}[u](t,0) = \sum_{j=0}^{\infty} E_{x_1^{\sqcup\!\sqcup\, j}}[u](t,0) \\
&= \sum_{j=0}^{\infty} j!\, E_{x_1^j}[u](t,0) = F_c[u](t).
\end{aligned}
$$

If the constant input $u = 1$ is applied over the interval $[0,T]$ with $T < 1$ then $y(T) = 1/(1-T)$ and $\bar{R} = T$. On the other hand, the discrete-time approximation $\hat{y}^J(N) = \hat{F}_c^J[\hat{u}](N)$ with $\hat{u} = \Delta$ and $N = L$ is

$$
\begin{aligned}
\hat{F}_c^J[\Delta](L) &= \sum_{j=0}^{J} j!\, S_{x_1^j}[\Delta](L) \\
&= \sum_{j=0}^{J} j!\Delta^j \sum_{k_1+k_2+\cdots k_L=j} 1 \\
&= \sum_{j=0}^{J} j!\Delta^j \binom{L+j-1}{j} = \sum_{j=0}^{J} \Delta^j (L)_j
\end{aligned}
$$

$$= \sum_{j=0}^{J} T^j + \sum_{j=0}^{J} T^j \left(\frac{j(j-1)}{2L} + O\left(\frac{1}{L^2}\right)\right)$$
$$= F_c[1](T) - e(J) + \hat{e}(J) + \frac{1-T^{J+1}}{1-T} O\left(\frac{1}{L^2}\right),$$

which is consistent with Theorem 4 ($s = \hat{s} = T$). It represents a worst case scenario in the sense that as J increases, the truncation error $e(J)$ vanishes, and the upper bound in Theorem 4 becomes achievable. The outputs y and $\hat{y}^J$ were computed numerically over the interval $[0, 0.5]$ for various choices of u, L, and J. This data is summarized in Table II, and the corresponding plots for cases 3 and 6 are shown in Figures 2 and 3, respectively. As expected, the constant input cases 1-3 yield errors that are approximately upper bounded by $\hat{e}(J)+e(J)$, while for the sinusoidal input this bound is conservative. In particular, the error in case 3 is very close to $\hat{e}(J)$ since L is large and $e(J)$ is small. □

C. Globally Convergent F_c

When c is globally convergent, the divergence problem for $\hat{F}_c$ is avoided provided $\hat{u}$ is sufficiently small. But in most cases it is usually not possible to compute the infinite sum defining $\hat{F}_c$, so once again the truncated approximator $\hat{F}_c^J$ will be utilized. The main theorem of this section is given below. It provides an upper bound on the approximation error in terms of the (upper) incomplete gamma function, $\Gamma(a,b) := \int_b^\infty t^{a-1}\mathrm{e}^{-t}\,dt/\Gamma(a)$.

Theorem 5: Let $c \in \mathbb{R}^{\ell}_{GC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$. If $u \in B_1^m(R)[0,T]$ then for any fixed $J \geq 0$

$$\left| F_c[u](T) - \hat{F}_c^J[\hat{u}](L) \right| \leq \hat{e}(J) + e(J) + K_c\,\mathrm{e}^{\hat{s}}\Gamma(J+1,\hat{s})\,O\left(\frac{1}{L^2}\right)$$

as $L \to \infty$, where

$$\hat{e}(J) = K_c\,\mathrm{e}^{\hat{s}}\Gamma(J+1,\hat{s})\frac{\hat{s}^2}{2L},$$
$$e(J) = K_c\,\mathrm{e}^{s}(1-\Gamma(J+1,s)),$$

$\hat{s} = M_c(m+1)L\|\hat{u}\|_\infty$, $s = M_c(m+1)\bar{R}$, $\bar{R} = \max\{R,T\}$, and $\hat{u}$ is as defined in (5).

Proof: Applying Theorem 3 gives the following:

$$\left| F_c[u](T) - \hat{F}_c^J[\hat{u}](L) \right|$$
$$\leq \sum_{j=0}^{J} \sum_{\eta\in X^j} |(c,\eta)|\,|E_\eta[u](T,0) - S_\eta[\hat{u}](L)| + \sum_{j=J+1}^{\infty} \sum_{\eta\in X^j} |(c,\eta)|\,|E_\eta[u](T,0)|$$
$$\leq \sum_{j=2}^{J} K_c M_c^j (m+1)^j \frac{T^j\|\hat{u}/\Delta\|_\infty^j}{j!}\left(\frac{j(j-1)}{2L} + O\left(\frac{1}{L^2}\right)\right) + \sum_{j=J+1}^{\infty} K_c M_c^j (m+1)^j \frac{\bar{R}^j}{j!}$$
$$\leq \frac{K_c}{2L}\sum_{j=0}^{J}(M_c(m+1)L\|\hat{u}\|_\infty)^{j+2}\frac{1}{j!} + K_c\sum_{j=0}^{J}(M_c(m+1)L\|\hat{u}\|_\infty)^{j}\frac{1}{j!}\,O\left(\frac{1}{L^2}\right) + K_c\sum_{j=J+1}^{\infty}(M_c(m+1)\bar{R})^j\frac{1}{j!}$$
$$= K_c\,\mathrm{e}^{\hat{s}}\Gamma(J+1,\hat{s})\frac{\hat{s}^2}{2L} + K_c\,\mathrm{e}^{\hat{s}}\Gamma(J+1,\hat{s})\,O\left(\frac{1}{L^2}\right) + K_c\,\mathrm{e}^{s}(1-\Gamma(J+1,s))$$
$$= \hat{e}(J) + e(J) + K_c\,\mathrm{e}^{\hat{s}}\Gamma(J+1,\hat{s})\,O\left(\frac{1}{L^2}\right),$$

where the identity $\sum_{j=0}^{J} s^j/j! = \mathrm{e}^s\Gamma(J+1,s)$ has been used [4, Chapter 8.35]. ■

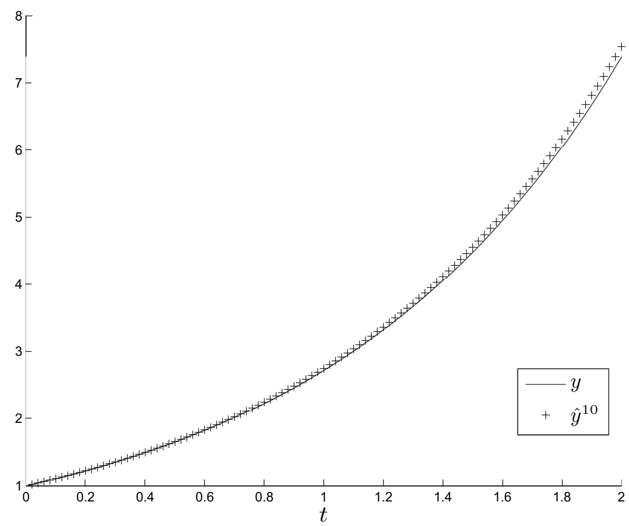

Fig. 4. Simulation comparing $y = F_c[1]$ to its approximation $\hat{y}^{10} = \hat{F}_c^{10}[\Delta]$ in Example 2, case 3.

Example 2: Consider the globally convergent series $c = \sum_{k\geq 0} x_1^k$ so that $K_c = M_c = 1$. In this case, F_c has the state space realization

$$\dot{z} = u, \quad z(0) = 0, \quad y = \mathrm{e}^z$$

since

$$y(t) = \sum_{j=0}^{\infty}(E_{x_1}[u](t,0))^j\frac{1}{j!} = \sum_{j=0}^{\infty} E_{x_1^{\sqcup\!\sqcup\, j}\frac{1}{j!}}[u](t,0)$$
$$= \sum_{j=0}^{\infty} E_{x_1^j}[u](t,0) = F_c[u](t)$$

for all $t \geq 0$. If the constant input $u = 1$ is applied over the interval $[0,T]$ then $y(T) = \mathrm{e}^T$. The discrete-time

TABLE III

SUMMARY OF SIMULATION RESULTS FOR EXAMPLE 2.

case	u	T	L	Δ	J	$\|\hat{u}\|_\infty$	s	$\hat{s}$	$y(T)$	$\hat{y}^J(L)$	$\hat{y}^J(L)-y(T)$	$\hat{e}(J)$	$e(J)$
1	1	2	50	0.0400	10	0.0400	2.0000	2.0000	7.3891	7.6989	0.3098	0.2956	6.1390×10^{-5}
2	1	2	50	0.0400	20	0.0400	2.0000	2.0000	7.3891	7.6991	0.3100	0.2956	4.5119×10^{-14}
3	1	2	100	0.0200	10	0.0200	2.0000	2.0000	7.3891	7.5403	0.1512	0.1478	6.1390×10^{-5}
4	$\sin(10t)$	2	50	0.0400	10	0.0392	2.0000	1.9601	1.0601	1.0803	0.0202	0.2728	6.1390×10^{-5}
5	$\sin(10t)$	2	50	0.0400	20	0.0392	2.0000	1.9601	1.0601	1.0803	0.0202	0.2728	4.5119×10^{-14}
6	$\sin(10t)$	2	100	0.0200	10	0.0199	2.0000	1.9899	1.0607	1.0711	0.0104	0.1448	6.1390×10^{-5}

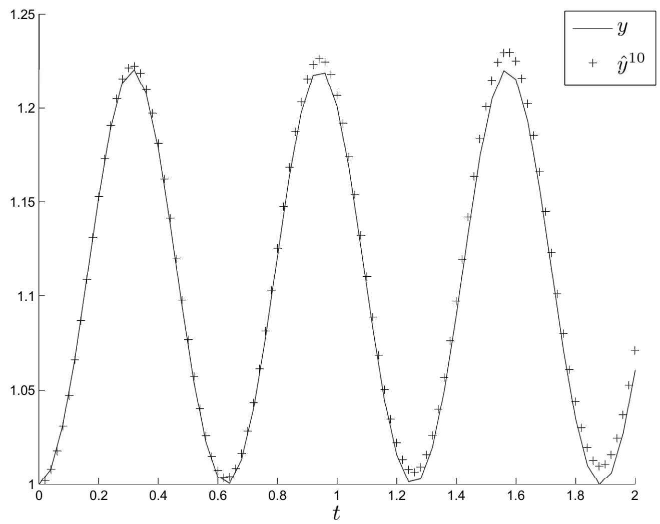

Fig. 5. Simulation comparing $y(t) = F_c[\sin(10t)]$ to its approximation $\hat{y}^{10} = \hat{F}_c^{10}[\hat{u}]$ in Example 2, case 6.

approximation at $T = L\Delta$ is

$$\begin{aligned}\hat{y}^J(L) &= \hat{F}_c^J[\Delta](L) = \sum_{j=0}^{J} S_{x_1^j}[\Delta](L) = \sum_{j=0}^{J} \frac{\Delta^j}{j!}(L)_j \\ &= \sum_{j=0}^{J} \frac{T^j}{j!} + \sum_{j=0}^{J} \frac{T^j}{j!}\left(\frac{j(j-1)}{2L} + O\left(\frac{1}{L^2}\right)\right) \\ &= F_c[1](T) - e(J) + \hat{e}(J) + \mathrm{e}^T\Gamma(J+1,T)O\left(\frac{1}{L^2}\right),\end{aligned}$$

which is consistent with Theorem 5 ($s = \hat{s} = T$). This is also a worst case scenario in the same sense described in Example 1. The outputs y and $\hat{y}$ were computed numerically over the interval $[0, 2]$ for various choices of u, L, and J. This data is summarized in Table III, and the corresponding plots for cases 3 and 6 are shown in Figures 4 and 5, respectively. As in the previous example, the constant input cases 1-3 yield errors that are approximately upper bounded by $\hat{e}(J)+e(J)$, while the error bound for the sinusoidal input is conservative. Again, case 3 has an error closest to $\hat{e}(J)$ since L is large and $e(J)$ is small. □

V. CONCLUSIONS

This paper described how to approximate Fliess operators in terms of iterated sums and gave explicit error bounds for the locally and globally convergent cases which are asymptotically achievable in certain worst scenarios.

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