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On the Preservation of Commutation and Anticommutation Relations of n-Level Quantum Systems*

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***Abstract*— The goal of this paper is to provide conditions under which a quantum stochastic differential equation (QSDE) preserves the commutation and anticommutation relations of the $SU(n)$ algebra, and thus describes the evolution of an open n-level quantum system. One of the challenges in the approach lies in the handling of the so-called anomaly coefficients of $SU(n)$. Then, it is shown that the physical realizability conditions recently developed by the authors for open n-level quantum systems also imply preservation of commutation and anticommutation relations.**

I. Introduction

The property of *physical realizability* of open quantum systems has attracted considerable interest in recent years with its main motivation being its role in *quantum coherent control* [1], [2], [11], [14]. In simple words, this property systematically characterizes the quantum nature of a control system from a state space point of view, which is relevant in many engineering areas, and is closely related to algebraic properties of the QSDE coefficients. The case of open quantum linear systems is well understood [8]. More recently, the physical realizability property was introduced for n-level quantum systems [3]. This was the product of a nontrivial extension of the two-level case presented in [4]. From the state space point of view, physical realizability has to be complemented by the preservation of commutation relations since the latter is an exhibitor of the quantum behavior of the system. In this regard, the authors provided conditions for the preservation of commutation relations in the case of QSDE's evolving in $SU(2)$. However, in order to extend the formalism to the general n-level case it is necessary to deal with the so-called *anomaly coefficients of* $SU(n)$, which forms the completely symmetric tensor d_{ijk} that appears out of the anticommutation relation of the generators of $SU(n)$. These generators form a complete orthonormal basis spanning the set of n-dimensional complex matrices and are known as the *generalized Gell-Man matrices* [10], [12], [13]. The commutation relations of these generators also generate the completely antisymmetric tensor f_{ijk}. In the case of QSDEs describing two-level quantum systems evolving in $SU(2)$, it suffices to consider preserving the commutation relations of $SU(2)$ since the anomaly tensor d is zero for all indices. On the other hand, the situation is nontrivial for $n \geq 3$ since one now requires extra machinery on the tensor d that in some cases depend on the value of n. In other words, the QSDE for an n-level quantum system must preserve both commutation and anticommutation relations for all times.

The approach followed in this manuscript is that of open quantum systems defined in terms of a triple (S, L, H), where S is an operator, known as the *scattering matrix*, describing the interaction of the environment fields among themselves, L is a vector of coupling operators expressing the interaction of the environment fields with the system variables and H is a Hamiltonian operator [6], [9]. This description implicitly takes in consideration the quantum nature of the evolution equations. For example, the (S, L, H) description will preserve, in time, the canonical commutation relations of the system under consideration. This preservation of commutation relations is necessary but not sufficient to establish the "quantumness" of a system. It is thus that the physical realizability condition relates a QSDE to an (S, L, H) description. In general, the set of QSDEs satisfying commutation relations includes the class of systems having an underlying (S, L, H) parametrization. It is thus the main goal of this paper to provide conditions for the preservation of commutation and anticommutation relations for a state space model evolving in $SU(n)$ independently of the physical realizability conditions. Then, as a second result it is shown that physical realizability *does* imply the preservation of commutation and anticommutation relations as expected from the physics of quantum system.

The paper is organized as follows. Section II summarizes the necessary tools obtained from the algebra of $SU(n)$. In Section III the basic preliminaries of open n-level quantum systems as well as the property of physical realizability for such systems are introduced. This is followed by Section IV, in which conditions for the preservation of commutation and anticommutation relations are provided. In addition, it is also shown that those conditions are implied by the physical realizability conditions. Finally, Section V gives the conclusions. The proofs of all results are included in the full version of the paper; see [5].

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II. THE ALGEBRA OF $SU(n)$

In what follows the necessary tools regarding the algebra of the special unitary group $SU(n)$ will be provided; see [10], [12], [13] for more details. Basically, this group is formed by all $n \times n$ complex matrices that are Hermitian and have zero trace. Consider the set of elementary vectors spanning $\mathbb{C}^n$, namely, $\{e_1, \cdots, e_n\}$. Define $P_{kl} \in \mathbb{C}^{n\times n}$ as $P_{k,l} = e_k e_l^T$, where $k, l = 1, \ldots, n$. A standard way of constructing a complete orthonormal basis for $SU(n)$ is

$$\begin{aligned} u_{jk} &= P_{j,k} + P_{k,j}, \\ v_{jk} &= \boldsymbol{i}\left(P_{j,k} - P_{k,j}\right), \\ w_l &= -\sqrt{\frac{2}{l(l+1)}}\left(\sum_{s=1}^{k} P_{s,s} - kP_{l+1,l+1}\right) \end{aligned}$$

for $1 \le j < k \le n$, $1 \le l \le n-1$. Note that the identity matrix I must be included in the basis in order to form a complete set. The identity, $(n^2-n)/2$ symmetric matrices u_{jk}, the $(n^2-n)/2$ antisymmetric matrices v_{jk} and the $n-1$ mutually commutative matrices w_l together form the *generators of* $SU(n)$. These generators are known as the *generalized Gell-Mann matrices*. Without any particular order, the generators are relabeled $\{I, \lambda_1, \ldots, \lambda_s\}$, where $s = n^2 - 1$. Here the orthonormality condition these matrices satisfy is $\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Their commutation and anticommutation relations are

$$[\lambda_i, \lambda_j] = 2\boldsymbol{i}\sum_{k=1}^{s} f_{ijk}\lambda_k,$$

$$\{\lambda_i, \lambda_j\} = \frac{4}{n}\delta_{ij} + 2\sum_{k=1}^{s} d_{ijk}\lambda_k.$$

Thus, the product $\lambda_i\lambda_j$ can be easily computed as

$$\begin{aligned} \lambda_i\lambda_j &= \frac{1}{2}\left([\lambda_i, \lambda_j] + \{\lambda_i, \lambda_j\}\right) \\ &= \frac{2}{n}\delta_{ij} + \sum_{k=1}^{s}\left(\boldsymbol{i}f_{ijk} + d_{ijk}\right)\lambda_k. \end{aligned} \tag{1}$$

where the real completely antisymmetric tensor f_{ijk} and the real completely symmetric tensor d_{ijk} are called the *structure constants* of $SU(n)$. The tensors f_{ijk} and d_{ijk} satisfy

$$f_{ilm}f_{mjk} + f_{jlm}f_{imk} + f_{klm}f_{ijm} = 0, \tag{2a}$$

$$f_{ilm}d_{mjk} + f_{jlm}d_{imk} + f_{klm}d_{ijm} = 0, \tag{2b}$$

$$\sum_{k=1}^{s} f_{ilk}f_{mjk} = \frac{2}{n}\left(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\right) + \sum_{k=1}^{s}\left(d_{imk}d_{ljk} - d_{ijk}d_{lmk}\right), \tag{2c}$$

$$\sum_{m,k=1}^{s} f_{imk}f_{jmk} = n\delta_{ij}. \tag{2d}$$

Define $F_i, D_i \in \mathbb{R}^{s\times s}$, $i \in \{1, \ldots, s\}$, such that their (j,k) component is $(F_i)_{jk} = f_{ijk}$ and $(D_i)_{jk} = d_{ijk}$, respectively. In particular, the set $\{-\boldsymbol{i}F_1, \ldots, -\boldsymbol{i}F_s\}$ is the adjoint representation of $SU(n)$. In [10], [12], identities (2a)-(2c) were employed to obtain the following useful relationships

$$[F_i, F_j] = -\sum_{k}^{s} f_{ijk}F_k \tag{3a}$$

$$[F_i, D_j] = -\sum_{k}^{s} f_{ijk}D_k \tag{3b}$$

$$F_iD_j + F_jD_i = \sum_{k}^{s} d_{ijk}F_k \tag{3c}$$

$$D_iF_j + D_jF_i = \sum_{k}^{s} d_{ijk}F_k \tag{3d}$$

$$(D_iD_j - F_jF_i)_{ml} = \sum_{k}^{s} d_{ijk}(D_k)_{ml} + \frac{2}{n}\left(\delta_{ij}\delta_{ml} - \delta_{im}\delta_{jl}\right). \tag{3e}$$

Definition 1: Let $\beta \in \mathbb{C}^s$. The linear mappings $\Theta^-, \Theta^+ : \mathbb{C}^s \to \mathbb{C}^{s\times s}$ are defined as

$$\Theta^-(\beta) = \left(F_1^T\beta, \cdots, F_s^T\beta\right) = \begin{pmatrix} \beta^T F_1^T \\ \vdots \\ \beta^T F_s^T \end{pmatrix},$$

$$\Theta^+(\beta) = \left(D_1^T\beta, \cdots, D_s^T\beta\right) = \begin{pmatrix} \beta^T D_1^T \\ \vdots \\ \beta^T D_s^T \end{pmatrix}.$$

Observe that the nature of the f and d-tensors make $\Theta^-(\beta)$ and $\Theta^+(\beta)$ be antisymmetric and symmetric, respectively. When β is an s-dimensional row vector then it will be understood hereafter that $\Theta^-(\beta) = \Theta^-(\beta^T)$ and $\Theta^+(\beta) = \Theta^+(\beta^T)$. Consider now the *stacking operator* $\mathrm{vec} : \mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ whose action on a matrix creates a column vector by stacking its columns below one another. With the help of vec, the matrices $\Theta^-(\beta)$ and $\Theta^+(\beta)$ can be reorganized so that

$$\mathrm{vec}(\Theta^-(\beta)) = \begin{pmatrix} \Theta_1^-(\beta) \\ \vdots \\ \Theta_s^-(\beta) \end{pmatrix} = F\beta,$$

and

$$\mathrm{vec}(\Theta^+(\beta)) = \begin{pmatrix} \Theta_1^+(\beta) \\ \vdots \\ \Theta_s^+(\beta) \end{pmatrix} = D\beta,$$

where $\beta \in \mathbb{C}^s$, $\Theta_i^-(\beta) = F_i^T\beta$, $F = (F_1, \cdots, F_s)^T$, $\Theta_i^+(\beta) = D_i\beta$ and $D = (D_1, \cdots, D_s)^T$. From (2d), F satisfies

$$F^TF = nI. \tag{4}$$

The properties of Θ^- and Θ^+ are summarized in the next lemma.

Lemma 1: Let $\beta, \gamma \in \mathbb{C}^s$. The mappings Θ^- and Θ^+ satisfy

$$\Theta^-(\beta)\gamma = -\Theta^-(\gamma)\beta, \tag{5a}$$

$$\Theta^+(\beta)\gamma = \Theta^+(\gamma)\beta, \tag{5b}$$

$$\Theta^-(\beta)\beta = 0, \tag{5c}$$

$$\Theta^-\left(\Theta^-(\beta)\gamma\right) = [\Theta^-(\beta), \Theta^-(\gamma)], \tag{5d}$$
$$\Theta^-\left(\Theta^+(\beta)\gamma\right) = \Theta^-(\beta)\Theta^+(\gamma) + \Theta^-(\gamma)\Theta^+(\beta), \tag{5e}$$
$$\Theta^+\left(\Theta^-(\beta)\gamma\right) = [\Theta^+(\beta), \Theta^-(\gamma)] = [\Theta^-(\beta), \Theta^+(\gamma)], \tag{5f}$$
$$\Theta^+\left(\Theta^+(\beta)\gamma\right) = \Theta^+(\beta)\Theta^+(\gamma) - \Theta^-(\gamma)\Theta^-(\beta) - \frac{2}{n}\left(\beta^T\gamma I - \beta\gamma^T\right). \tag{5g}$$

Some additional identities regarding matrices F and D with respect to the Kronecker product are given next. They will become useful when proving the main results of the paper in Section IV. Define the tensor permutation matrix $\mathbb{1}_\otimes \in \mathbb{R}^{s^2\times s^2}$. That is, a symmetric block matrix $\mathbb{1}_\otimes = \{\mathbb{1}_{ji}\}_{i,j=1}^s$ such that it satisfies $\mathbb{1}_\otimes(A\otimes B)\mathbb{1}_\otimes = (B\otimes A)$ for any $A, B \in \mathbb{C}^{s\times s}$.

Lemma 2: Let $F = (F_1, F_2, \cdots, F_s)^T$ and $F = (D_1, D_2, \cdots, D_s)^T$, and $A, B \in \mathbb{R}^{s\times s}$. Then

i. $F = -\mathbb{1}_\otimes F$,

ii. $D = \mathbb{1}_\otimes D$,

iii. $F^T(A\otimes B)F = F^T(B\otimes A)F$,

iv. $D^T(A\otimes B)D = D^T(B\otimes A)D$.

III. Physical Realizability of Open n-Level Quantum Systems

Consider the separable Hilbert space $\mathfrak{H}$, $\mathfrak{T}(\mathfrak{H})$ the set of operators in $\mathfrak{H}$, and $\mathfrak{T}(\mathfrak{H})^{n\times m}$ the set of $n\times m$ dimensional arrays of operators in $\mathfrak{T}(\mathfrak{H})$. The commutator of x and y in $\mathfrak{T}(\mathfrak{H})$ is $[x,y] = xy - xy$. Let $x \in \mathfrak{T}(\mathfrak{H})^{n_1}$ and $y \in \mathfrak{T}(\mathfrak{H})^{n_2}$, then $[x, y^T] \triangleq xy^T - (yx^T)^T \in \mathfrak{T}(\mathfrak{H})^{n_1\times n_2}$. The adjoint of x is denoted by $x^\dagger = (x^\#)^T$ with $x^\# \triangleq (x_1^* \cdots x_n^*)^T$ and * denotes the operator adjoint. For the case of complex vectors and matrices, * denotes the complex conjugate while † denotes the conjugate transpose.

The *Heisenberg picture* evolution of an scalar operator X interacting with a boson field is

$$dX = (S^*XS - X)\,d\Lambda_w + \mathcal{L}(X)\,dt + S^*[X,L]\,dW^* + [L^*, X]S\,dW,$$

where $\mathcal{L}(X)$ is the Lindblad operator defined as

$$\mathcal{L}(X) = -\boldsymbol{i}[X,H] + \frac{1}{2}\left(L^*[X,L] + [L^*,X]L\right).$$

The output field is given by

$$Y(t) = U(t)^*W(t)U(t),$$

which amounts to $dY = Ldt + SdW$. In summary, the dynamics of an open quantum system is uniquely determined by the parametrization (S, L, H). Hereafter, the operator S is assumed to be the identity operator ($S = \hat{I}$). If on the other hand one consider n_w interacting boson fields then the evolution equation is written as

$$dX = \mathcal{L}(X)\,dt + dW^\dagger\,[X,L] + [L^\dagger, X]\,dW,$$

where $[X, dW] = [X, dW^\dagger]^T = 0$, $L = (L_1, \ldots, L_s)^T$,

$$dW = \begin{pmatrix} dW_1 \\ \vdots \\ dW_{n_w}\end{pmatrix} \quad\text{and}\quad dW^\dagger = \left(dW_1^*, \cdots, dW_{n_w}^*\right).$$

Consider the vector of operators $x = (x_1, \ldots, x_s)^T \in \mathfrak{T}(\mathfrak{H})^s$. By stacking (column-wise) the scalar evolutions for each x_i, it follows that

$$\begin{aligned}\begin{pmatrix} dx_1 \\ \vdots \\ dx_s\end{pmatrix} &= \begin{pmatrix}\mathcal{L}(x_1)\\ \vdots \\ \mathcal{L}(x_s)\end{pmatrix} dt + \begin{pmatrix}[x_1, L^T] \\ \vdots \\ [x_s, L^T]\end{pmatrix} dW^\# + \begin{pmatrix}[L^\dagger, x_1] \\ \vdots \\ [L^\dagger, x_s]\end{pmatrix} dW \\ &= \begin{pmatrix}\mathcal{L}(x_1)\\ \vdots \\ \mathcal{L}(x_s)\end{pmatrix} dt + \begin{pmatrix}[x_1, L^T] \\ \vdots \\ [x_s, L^T]\end{pmatrix} dW^\# - \begin{pmatrix}[x_1, L^\dagger] \\ \vdots \\ [x_s, L^\dagger]\end{pmatrix} dW \\ &= \mathcal{L}(x)\,dt + [x, L^T]\,dW^\# - [x, L^\dagger]\,dW,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(x) =& -\boldsymbol{i}[x,H]\,dt + \frac{1}{2}\left(\begin{pmatrix} L^\dagger\,[x_1, L] \\ \vdots \\ L^\dagger\,[x_s, L]\end{pmatrix} + \begin{pmatrix}[L^\dagger, x_1]\,L \\ \vdots \\ [L^\dagger, x_s]\,L\end{pmatrix}\right) \\ =& -\boldsymbol{i}[x,H]\,dt + \frac{1}{2}\Big(-\left(L^\dagger\left([L,x_1], \cdots, [L, x_s]\right)\right)^T \\ & -[x, L^\dagger]\,L\Big) \\ =& -\boldsymbol{i}[x,H]\,dt + \frac{1}{2}\left(-\left(L^\dagger\,[L, x^T]\right)^T - [x, L^\dagger]\,L\right) \\ =& -\boldsymbol{i}[x,H]\,dt + \frac{1}{2}\left(\left(L^\dagger\,[x, L^T]^T\right)^T + [L^\#, x^T]^T\,L\right).\end{aligned}$$

This amounts to

$$dx = \mathcal{L}(x)\,dt + [x, L^T]\,dW^\# - [x, L^\dagger]\,dW,$$

where

$$\mathcal{L}(x) \triangleq -\boldsymbol{i}[x,H] + \frac{1}{2}\left(\left(L^\dagger\ \left[x, L^T\right]^T\right)^T + \left[L^\#, x^T\right]^T\ L\right). \tag{6}$$

The quadrature form of the evolution is attained by transforming the field as

$$\begin{pmatrix}\bar{W}_1 \\ \bar{W}_2\end{pmatrix} = \begin{pmatrix} I_{n_w} & I_{n_w} \\ -\boldsymbol{i}I_{n_w} & \boldsymbol{i}I_{n_w}\end{pmatrix}\begin{pmatrix} W \\ W^\#\end{pmatrix}, \tag{7}$$

where the operators $\bar{W}_1$ and $\bar{W}_2$ are now self-adjoint, and I_{n_w} denotes the identity matrix of dimension n_w. Thus, the evolution of $x \in \mathfrak{T}(\mathfrak{H})^s$ interacting with n_w quadrature Boson quantum fields $\bar{W}_1$ and $\bar{W}_2$ in the parametrization (S, L, H) is given by

$$dx = \mathcal{L}(x)\,dt + \frac{1}{2}\left([x, L^T] - [x, L^\dagger]\right)d\bar{W}_1 - \frac{\boldsymbol{i}}{2}\left([x, L^T] + [x, L^\dagger]\right)d\bar{W}_2. \tag{8}$$

with output field, also in quadrature form, given by

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} L + L^{\#} \\ \boldsymbol{i}(L^{\#} - L) \end{pmatrix} dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix},$$

The Itô table ([7]) for $\bar{W}_1$ and $\bar{W}_2$ is given by

$$\begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix} \begin{pmatrix} d\bar{W}_1 & d\bar{W}_2 \end{pmatrix} = \begin{pmatrix} I_{n_w} & \boldsymbol{i}I_{n_w} \\ -\boldsymbol{i}I_{n_w} & I_{n_w} \end{pmatrix} dt. \quad (9)$$

The interest in this paper is in systems evolving with respect to the special unitary group $SU(n)$. Consider the Hilbert space for these systems to be $\mathfrak{H} = \mathbb{C}^n$. It is standard to associate a vector $\beta = (\beta_1, \cdots, \beta_s) \in \mathbb{C}^s$ with the vector of operators $\hat{\beta} \in \mathfrak{T}(\mathfrak{H})^s$ by simply considering the identity operator in each component of the vector, i.e., $\hat{\beta} = (\beta_1 \hat{I}, \cdots, \beta_s \hat{I})$. In a similar manner, any complex matrix is associated to a matrix of operators by considering the identity operator $\hat{I}$ in each component. The identity operator $\hat{I}$ is usually suppressed. Therefore, abusing of the notation slightly, the mappings Θ^- and Θ^+ are allowed to act on a vector of operators. In this manner, it is valid to multiply complex matrices and operator matrices. This convention allows to express the commutation and anticommutation relations of the system variables x as

$$[x, x^T] = 2\boldsymbol{i}\Theta^-(x), \quad (10a)$$

$$\{x, x^T\} = \frac{4}{n} I + 2\Theta^+(x). \quad (10b)$$

The vector of system variables for (8) evolving in $SU(n)$ is then

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} \triangleq \begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_s \end{pmatrix},$$

where $\hat{\lambda}_1, \ldots, \hat{\lambda}_s$ are self-adjoint and spanned by the generalized Gell-Mann matrices. They are usually called *spin operators*. The initial value of the system variables can be set to $x(0) = (\lambda_1, \ldots, \lambda_s)$ with $\lambda_1, \ldots, \lambda_s$ being the generators of $SU(n)$ introduced in Section II. Due to the product relation (1) any polynomial of spin operators can be written as a linear combination of generalized Gell-Mann matrices. Therefore, assuming linearity captures a large class of Hamiltonian and coupling operators. This allows to assume, without losing generality, that the Hamiltonian is

$$\mathcal{H} = \alpha x$$

with $\alpha \in \mathbb{R}^s$, and the multiplicative coupling operator is of the form

$$L = \Lambda x$$

with $\Lambda \in \mathbb{C}^{n_w \times s}$. In general, the evolution of x in quadrature form falls into a class of bilinear QSDEs expressed as

$$dx = A_0\, dt + Ax\, dt + (B_{11}x, \cdots, B_{1n_w}x, B_{21}x, \cdots, B_{2n_w}x) \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \quad (11)$$

$$\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x\, dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}, \quad (12)$$

where $A_0 \in \mathbb{R}^s$, $A, B_{1k} \triangleq \bar{B}_{1k} + \bar{B}_{2k}, B_{2k} \triangleq \boldsymbol{i}(\bar{B}_{2k} - \bar{B}_{1k}) \in \mathbb{R}^{s \times s}$ and $C_1, C_2 \in \mathbb{R}^{n_w \times s}$, $k = 1, \ldots, n_w$. It is assumed that all matrices in (11) and (12) are real due to the fact that the class of quantum systems considered in this paper are in quadrature form. The next results in physical realizability were given in [3] and are employed later in Section IV.

Definition 2: [3] A system described by equations (11) and (12) is said to be physically realizable if there exist $\mathcal{H}$ and L such that (11) can be written as in (8).

The explicit form of matrices $A_0, A, B_{1k}, B_{2k}, C_1$ and C_2 in terms of the Hamiltonian and coupling operator is given next.

Theorem 1: [3] Let $\mathcal{H} = \alpha x$, with $\alpha^T \in \mathbb{R}^s$, and $L = \Lambda x$, with $\Lambda \in \mathbb{C}^{n_w \times s}$. Then

$$A_0 = \frac{4\boldsymbol{i}}{n} \sum_{k=1}^{n_w} \Theta^-(\Lambda_k^{\#}) \Lambda_k^T, \quad (13a)$$

$$A = -2\Theta^-(\alpha) + \sum_{k=1}^{n_w} (R_k - \boldsymbol{i}Q_k), \quad (13b)$$

$$B_{1k} = \Theta^-\left(\boldsymbol{i}(\Lambda_k^{\#} - \Lambda_k)\right), \quad (13c)$$

$$B_{2k} = -\Theta^-(\Lambda_k + \Lambda_k^{\#}), \quad (13d)$$

$$C_1 = \Lambda + \Lambda^{\#}, \quad (13e)$$

$$C_2 = \boldsymbol{i}\left(\Lambda^{\#} - \Lambda\right), \quad (13f)$$

where

$$R_k \triangleq \Theta^-(\Lambda_k)\Theta^-(\Lambda_k^{\#}) + \Theta^-(\Lambda_k^{\#})\Theta^-(\Lambda_k),$$
$$Q_k \triangleq \Theta^-(\Lambda_k)\Theta^+(\Lambda_k^{\#}) - \Theta^-(\Lambda_k^{\#})\Theta^+(\Lambda_k).$$

The next theorem present conditions for physical realizability in terms of $(A_0, A, B_1, B_2, C_1, C_2)$, in (11) and (12).

Theorem 2: [3] System (11) with output equation (12) is physically realizable if and only if

$$i.\ A_0 = \frac{1}{n} \sum_{k=1}^{n_w} (\boldsymbol{i}B_{1k} + B_{2k}) \left((C_1)_k + \boldsymbol{i}(C_2)_k\right)^T,$$

$$ii.\ B_{1k} = \Theta^-((C_2)_k),$$

$$iii.\ B_{2k} = \Theta^-((C_1)_k),$$

$$iv.\ A + A^T + \sum_{i,k=1}^{2,n_w} B_{ik} {B_{ik}}^T = \frac{n}{2} \Theta^+(A_0),$$

where $(C_i)_k$ indicates the k-th row of C_i. In which case, the coupling matrix can be identified to be

$$\Lambda = \frac{1}{2}(C_1 + \boldsymbol{i}C_2),$$

and α, defining the system Hamiltonian, is

$$\alpha = \frac{1}{4n} \mathrm{vec}\left(A^T - A + \frac{1}{2} \sum_{k=1}^{n_w} \left([B_{2k}, \Theta^+((C_2)_k)] - [B_{1k}, \Theta^+((C_1)_k)]\right)\right)^T F. \quad (14)$$

IV. Preservation of Commutation and Anti-Commutation Relations

In this section, the main results of the paper are provided. It is shown first under what conditions a state space model, described by the QSDE (11), preserves the commutation and anticommutation relations of $SU(n)$. The employed procedure for this task is independent of the physical realizability conditions given in Theorem 2. Also, recall from Section III that $G \in \mathbb{C}^{s\times s}$ can be always regarded as $G \in \mathfrak{T}(\mathfrak{H})^{s\times s}$ since the identity operator in $\mathfrak{T}(\mathfrak{H})$ can be considered attached to each component of G. The following Lemma plays a key role in the forthcoming theorems.

Lemma 3: Let $G \in \mathfrak{T}(\mathfrak{H})^{s\times s}$ be antisymmetric and x be a vector comprised by elements of a complete linearly independent set of operators spanning $\mathfrak{T}(\mathfrak{H})$. If G satisfies

$$G\Theta^-(x) + \Theta^-(x)G^T - \Theta^-(Gx) = 0, \tag{15}$$

then there exist $g \in \mathfrak{T}(\mathfrak{H})$ such that

$$G = \Theta^-(g), \tag{16}$$

where the unique g is given by

$$g \triangleq -\frac{1}{n}\begin{pmatrix}\mathrm{Tr}(F_1 G)\\ \vdots \\ \mathrm{Tr}(F_s G)\end{pmatrix}. \tag{17}$$

Conversely, if (16) holds, then (15) is true for any $x \in \mathfrak{T}(\mathfrak{H})^s$.

In order to be considered a quantum system, the system variables of (11) must preserve the product rule (1) for all times. This amounts to preserve the commutation and anticommutation relations of $SU(n)$. Thus, the conditions that (11) must satisfy are

$$d[x, x^T] - 2\boldsymbol{i}\Theta^-(dx) = 0, \tag{18a}$$

$$d\{x, x^T\} - 2\Theta^+(dx) = 0. \tag{18b}$$

Note by the linearity of Θ^- and Θ^+ that

$$\begin{aligned}\Theta^-(dx) &= \Theta^-(A_0)dt + \Theta^-(Ax)dt \\ &\quad + \Theta^-(B_1x)d\bar{W}_1 + \Theta^-(B_2x)d\bar{W}_2,\\ \Theta^+(dx) &= \Theta^+(A_0)dt + \Theta^+(Ax)dt \\ &\quad + \Theta^+(B_1x)d\bar{W}_1 + \Theta^+(B_2x)d\bar{W}_2.\end{aligned}$$

A condition for system (11) to satisfy (18a) and (18b) is given in the next theorem.

Theorem 3: Let $x(0) \in \mathfrak{T}(\mathfrak{H})^s$ be comprised by the generators of $SU(n)$. Then, system (11) implies

$$[x(t), x(t)] = 2\boldsymbol{i}\Theta^-(x(t)),$$
$$\{x(t), x(t)\} = \frac{4}{n}I + 2\Theta^+(x(t))$$

for all $t \geq 0$, if and only if

$$B_i = \Theta^-(b_i) \tag{19a}$$

$$\sum_{k=1}^{n_w} B_{1k}B_{2k}^T - B_{2k}B_{1k}^T = \frac{n}{2}\Theta^-(A_0) \tag{19b}$$

$$A = \Theta^-(a) - \frac{1}{2}\sum_{k=1}^{n_w}\left(B_{1k}B_{1k}^T + B_{2k}B_{2k}^T\right) + \frac{1}{2}\sum_{k=1}^{n_w}\left(B_{2k}\Theta^+(b_{1k}) - B_{1k}\Theta^+(b_{2k})\right). \tag{19c}$$

where b_{ik} and a are s-dimensional vectors as in (17) for $i = 1, \ldots, s$ and $k = 1, \ldots, n_w$.

Theorem 4: A physically realizable system satisfies the conditions of Theorem 3.

The implication of this theorem is that a state space model, as given by (11) and (12), describes an open n-level quantum system when it satisfies the physical realizability conditions since they also ensure preservation of commutation and anticommutation relations of $SU(n)$. In addition, it is known that the physical realizability conditions can be employed to obtain the (S, L, H) parametrization of that quantum system.

V. Conclusions

Conditions for preserving commutations and anticommutation relations have been provided for QSDE's of the form (11). These results used explicitly the algebra of $SU(n)$, and in particular algebraic expressions involving the anomaly tensor d had to be employed. Moreover, it was shown that the physical realizability implies the preservation of those relations, and therefore under physical realizability conditions the system given by (11) and (12) describes an open n-level quantum system.

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